Master's Thesis

Quantitative Aspects of Polytopes and Spectrahedra

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August 18, 2019

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Acknowledgements

At this point I would like to thank Mr. Univ.-Prof. Dr. Tim Netzer for his constructive and always helpful ideas and his expert guidance in search of basic closed descriptions of polytopes and spectrahedra with few polynomials.

Contents

Introduction

A *basic closed (semialgebraic) set* $\mathcal{W} \subseteq \mathbb{R}^n$ is defined as the set of solutions of a system of *non-strict polynomial inequalities*. If such a set is given by a huge number of polynomials, it could sometimes be useful to find a handier basic closed description of the set which is given by less polynomials. If a basic closed set is defined by as little polynomials as possible, the corresponding polynomials are said to be a *minimal description* of the set. One impulse that animated mathematicians to deal with minimal descriptions of basic closed sets comes from optimization theory. If one does linear or semidefinite programming, the feasible regions of the respective optimization problems are *polyhedra* or *spectrahedra*, which are important classes of basic closed semialgebraic sets. If a polyhedron or a spectrahedron is given by a huge number of inequalities, one could try to find a minimal description of the set. If the degrees of the corresponding polynomials are not too high, the new description of the feasible region could be easier to handle. Of course, the transition from the usual description of a feasible region to another representation creates some problems since the common optimization techniques do not work anymore. Nevertheless, there exist solution techniques for optimization problems where the feasible region is represented by non-strict polynomial inequalities. This is one reason why trying to find minimal descriptions of basic closed semialgebraic sets could be useful.

About thirty years ago, *Ludwig Bröcker* and *Claus Scheiderer* found out that it is possible to represent every basic closed subset of $\mathbb{R}^n$ – no matter how complicated it looks like – by at most $\frac{n(n+1)}{2}$ polynomials (for instance, one can find a proof of this fact in [**AnBrRu96**]). *Scheiderer* was even able to construct basic closed sets for which it is impossible to find a polynomial description with less than $\frac{n(n+1)}{2}$ polynomials. This shows that the number $\frac{n(n+1)}{2}$, which serves as an upper bound for the number of polynomials needed in a minimal description of a basic closed semialgebraic set, cannot be improved in general.

The only possible way to still find better bounds is to tighten the conditions on the basic closed semialgebraic set. *Scheiderer's* examples, which cannot be described by less than $\frac{n(n+1)}{2}$ polynomials, are not convex. Therefore, one could suggest that every full-dimensional *convex* basic closed semialgebraic set can be represented by less than $\frac{n(n+1)}{2}$ polynomials. This conjecture has not yet been proven. It should just demonstrate that *Scheiderer's* and *Bröcker's* discoveries were just the beginning of the story and served other mathematicians as a motivation to find descriptions of special kinds of basic closed semialgebraic sets with as little polynomials as possible.

About twenty years later, *Gennadiy Averkov* and *Bröcker* drew their attention to an important class of basic closed sets, namely *polyhedra*. They showed that it is possible to improve *Scheiderer's* and *Bröcker's* bound for n-dimensional polyhedra, which are basic closed semialgebraic sets given by *linear* polynomials.

After all the theoretical work that was done so far, people focused on the question how the polynomials which pertain to the minimal description of a basic closed semialgebraic set look like. *Scheiderer's* and *Bröcker's* proofs were not constructive. Therefore, the new aim concerning this topic was to explicitly construct the polynomials. *Andreas Bernig* [**Be97**] was able to show that the interior of a 2-dimensional polygon $P \subseteq \mathbb{R}^2$ can be described by 2 polynomials $p_1, p_2 \in \mathbb{R}[x, y]$. Remarkably, he was even able to construct these polynomials. Several years after *Bernig's* construction *Averkov* and *Martin Henk* constructed n polynomials which can be used for the description of a *simple* n-dimensional polytope $P \subseteq \mathbb{R}^n$. At this point, *simple* just means that each vertex arises as the intersection of exactly n facets. *Averkov* was even able to construct these polynomials for more general basic closed semialgebraic sets.

The main purpose of the present master's thesis is trying to find descriptions of *spectrahedra* with few polynomials. The starting point of all considerations are some results which were discovered by *Averkov*, *Bröcker*, *Grötschel*, *Henk*, and *Bernig* ([**Av08**], [**AvBr10**], [**AvHe07**], [**Be97**], [**GrHe03**]). *Averkov's* paper *Representing Elementary Semi-Algebraic Sets by a Few Polynomial Inequalities: A constructive Approach* [**Av08**] is of particular interest for this thesis. Given a basic closed semialgebraic set of a very special form, *Averkov* was able to find a basic closed description of the set with few polynomials. In this thesis some ideas of his discoveries will be applied to spectrahedra.

The master's thesis will be divided into two main chapters. The first chapter deals with polytopes. Apart from a short introduction concerning some basic properties of polytopes the work gives an overview of *Bernig's* construction in the 2-dimensional case. Afterwards, *Averkov's* main results will be stated.

The center of attention of the second chapter are *spectrahedra*. Some basic properties of spectrahedra will be stated. Afterwards, using a main result of *Averkov's* work, it will be shown that one can find basic closed descriptions of spectrahedra with two polynomials if they fulfill some smoothness conditions. This description neither depends on the dimension of the space nor on the size of the corresponding matrices.

Preliminaries

At the beginning of the thesis some basic notations will be introduced and some well known facts concerning basic closed semialgebraic sets will be stated.

Definitions and Notations

During the whole work the multivariate polynomial ring in n variables over the real numbers is denoted by $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$. A *basic closed semialgebraic set* defined by polynomials $p_1, \ldots, p_k \in \mathbb{R}[\underline{x}]$ will be abbreviated as follows:

$$\mathcal{W}(p_1, \ldots, p_k) := \{a \in \mathbb{R}^n : \ p_1(a) \geq 0, \ldots, \ p_k(a) \geq 0\} \tag{0.1}$$

Moreover, a *basic open semialgebraic set* is defined by

$$\mathcal{O}(p_1, \ldots, p_k) := \{a \in \mathbb{R}^n : \ p_1(a) > 0, \ldots, \ p_k(a) > 0\}. \tag{0.2}$$

Sometimes, the shorthand notation $\mathcal{W}$ (respectively $\mathcal{O}$) instead of $\mathcal{W}(p_1, \ldots, p_k)$ (respectively $\mathcal{O}(p_1, \ldots, p_k)$) will be used.

For a given basic closed semialgebraic set $\mathcal{W}$ the minimal number of polynomials which are necessary for a basic closed description of the set is denoted by

$$m(\mathcal{W}) := \min\{l \in \mathbb{N} : \exists p_1, \ldots, p_l \in \mathbb{R}[\underline{x}] : \mathcal{W} = \mathcal{W}(p_1, \ldots, p_l)\}. \tag{0.3}$$

The corresponding set of polynomials $\{p_1, \ldots, p_l\}$ is called a *minimal description* of $\mathcal{W}$. Analogously, for a basic open semialgebraic set $\mathcal{O}$

$$m(\mathcal{O}) := \min\{l \in \mathbb{N} : \exists p_1, \ldots, p_l \in \mathbb{R}[\underline{x}] : \mathcal{O} = \mathcal{O}(p_1, \ldots, p_l)\}. \tag{0.4}$$

An *algebraic set* in $\mathbb{R}^n$ is defined as the set of real solutions of system of polynomial equations. It is denoted by

$$\mathcal{V}(p_1, \ldots, p_k) := \{a \in \mathbb{R}^n : p_1(a) = \ldots = p_k(a) = 0\}. \tag{0.5}$$

It can easily be seen that an algebraic set can be defined by just one polynomial. If one sets $p := p_1^2 + \ldots + p_k^2$, it holds that

$$\mathcal{V}(p_1, \ldots, p_k) = \mathcal{V}(p).$$

This shows that it is not very interesting to search for minimal descriptions of algebraic sets.

A set $S \subseteq \mathbb{R}^n$ is called *semialgebraic* if it is a *Boolean combination* of basic open semialgebraic sets. This means that S arises from finite unions, intersections and complements of sets of the form (0.2). To be more precisely, a semialgebraic set is given by polynomial equations and inequalities of the form "$p(a) = 0$", "$p(a) \geq 0$", "$p(a) \leq 0$", "$p(a) > 0$", "$p(a) < 0$" and *Boolean combinations* of them.

Let $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$ be semialgebraic sets. A function $f : S \to T$ is called *semialgebraic* if its graph

$$\Gamma(f) := \{(a, f(a)) \in \mathbb{R}^n \times \mathbb{R}^m : a \in S\}$$

is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$.

0.1. First-Order Formulas. The concept of *first-order formulas* is sometimes a very useful tool for describing semialgebraic sets. It will be used at a later time. One can find detailed information about this concept in [**PrDe01**]. Let's start with the definition of first-order formulas:

DEFINITION 0.1: *A $\mathbb{R}$-prime formula is a formula of the form*

$$"p(x) > 0"$$

for a polynomial $p \in \mathbb{R}[\underline{x}]$. With the help of prime formulas one can iteratively define (first-order) $\mathbb{R}$-formulas as follows:

(i) *Every $\mathbb{R}$-prime formula is a $\mathbb{R}$ -formula.*
(ii) *If ϕ, ψ are $\mathbb{R}$-formulas, then also $\phi \wedge \psi$, $\neg\phi$ and $\exists x_i \phi$.*

A variable x_i of ϕ which is not quantified is called free variable.

EXAMPLE 0.2: With the help of the above definition it can be seen that the following expressions are also $\mathbb{R}$-formulas:

$$\begin{aligned} \phi \vee \psi &= \neg(\neg\phi \wedge \neg\psi) \\ \phi \to \psi &= (\neg\phi) \vee \psi \\ \forall x_i \phi &= \neg(\exists x_i(\neg\phi)) \end{aligned}$$

♦

PROPOSITION 0.3: Let ϕ be a $\mathbb{R}$-formula with free variables $x_1, \ldots, x_l$. Then

$$\phi(\mathbb{R}) := \{(x_1, \ldots, x_l) \in \mathbb{R}^l : \phi(x_1, \ldots, x_l) \text{ is true in } \mathbb{R}^l\} \tag{0.6}$$

is a semialgebraic set.

PROOF: The statement follows from the famous projection theorem, which states that projections of semialgebraic sets are again semialgebraic (for instance, see [**PrDe01**]). Let ϕ be a formula which is free of quantifiers. Then

$$\{x \in \mathbb{R}^n : \phi(x) \text{ is true in } \mathbb{R}\} \tag{0.7}$$

is semialgebraic. This is due to the fact that one can receive every form of polynomial inequality by linking formulas of the form $\phi \wedge \psi$, $\neg\phi$ or $\phi \vee \psi$, where ϕ and ψ are prime formulas. For example,

$$p(x) = 0 \quad \text{corresponds to the formula} \quad \neg(p(x) > 0) \wedge \neg(-p(x) > 0).$$

The expression (0.6) is not of the form (0.7) since not every variable is free. But a set of the form $\exists x_i \phi(\mathbb{R})$ corresponds to the projection of $\phi(\mathbb{R})$ onto the orthogonal complement of the x_i-axis. If $\phi(\mathbb{R})$ is semialgebraic, $\exists x_i \phi(\mathbb{R})$ is also semialgebraic by the projection theorem. This proves the above proposition. ■

Main Results

In this subsection the two most important results concerning minimal descriptions of basic closed semialgebraic sets will be summarized. The main result is the following:

THEOREM 0.4: (*Theorem of Bröcker and Scheiderer*)
Let $\mathcal{W} \subseteq \mathbb{R}^n$ be a basic closed semialgebraic set. Moreover, let $\mathcal{O} \subseteq \mathbb{R}^n$ be basic

open. Then it holds that

$$m(\mathcal{W}) \leq \frac{n(n+1)}{2},$$
$$m(\mathcal{O}) \leq n.$$

One can find detailed information about this theorem in [**AnBrRu96**].

Another important theorem concerning minimal descriptions of *polyhedra* was shown by *Averkov* and *Bröcker*.

THEOREM 0.5: (*Averkov and Bröcker* [**AvBr10**])
Let $P \subseteq \mathbb{R}^n$ be a n-dimensional polyhedron and k be the maximal dimension of an affine subspace contained in P. Then:

$$\exists\, p_1, \ldots, p_{n-k} \in \mathbb{R}[\underline{x}] : P = \{a \in \mathbb{R}^n : \ p_1(a) \geq 0, \ldots, \ p_{n-k}(a) \geq 0\}$$

Moreover, $m(P) = n - k$, which means that P cannot be represented by less polynomials.

REMARK 0.6: If one just focuses on *bounded* polyhedra, which are also known as *polytopes*, the maximal dimension of an affine subspace contained in a polytope is 0. Therefore, every n-dimensional polytope in $\mathbb{R}^n$ can be represented by n polynomial inequalities.

REMARK 0.7: Even though THEOREM 0.4 and THEOREM 0.5 are well-known and already proven results, the present master's thesis is orientating on a weaker result which has been proven by *Averkov*. This result only shows that every *simple* n-dimensional polytope in $\mathbb{R}^n$ can be represented by n polynomials. This weaker result is fully sufficient for the purpose of this thesis since it can be extended to spectrahedra. *Averkov's* main ideas will be presented later on.

CHAPTER 1

Polytopes

The following introductory chapter states some basic properties of polytopes. All of them are well-known. Therefore, they won't be proven. One can find detailed information about the properties including their proofs in [**Web94**] or [**Zie07**]. All of the plots below were created by the author of the thesis using *GeoGebra* or *Mathematica*.

1. Some basic properties

Let's start with the definition of a polytope, followed by an easy example:

DEFINITION 1.1: *A polytope $P \subseteq \mathbb{R}^n$ is defined as the convex hull of a finite set of points $v_1, \ldots, v_r \in \mathbb{R}^n$:*

$$P := \operatorname{conv}\{v_1, \ldots, v_r\} = \left\{ \sum_{i=1}^{r} \lambda_i v_i : \lambda_1 \geq 0, \ldots, \lambda_r \geq 0, \sum_{i=1}^{r} \lambda_i = 1 \right\} \tag{1.1}$$

EXAMPLE 1.2: Let

$$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \; v_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \; v_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \; v_4 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

and $P := \operatorname{conv}\{v_1, \ldots v_4\} \subseteq \mathbb{R}^2$. P has the following form:

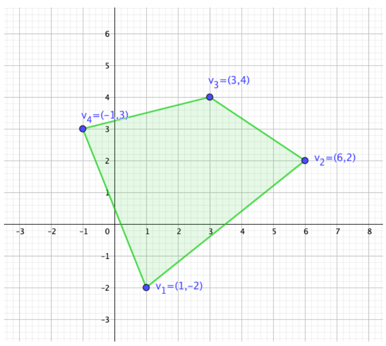


FIGURE 1.1. P is the convex hull of a finite set of points in $\mathbb{R}^2$

Since the main aim of the present master's thesis is to find minimal descriptions of basic closed semialgebraic sets, one has to show that polytopes are indeed basic closed semialgebraic. If one just takes a look at DEFINITION 1.1, it is not clear how the defining polynomial inequalities of P look like. But it turns out that P can also be represented as the set of solutions of a system of non-strict linear inequalities. Indeed, let

$$\begin{aligned} l_1 &:= -4x + 5y + 14 \\ l_2 &:= -2x - 3y + 18 \\ l_3 &:= x - 4y + 13 \\ l_4 &:= 5x + 2y - 1. \end{aligned}$$

Then in holds that $P = \{a \in \mathbb{R}^2 :\ l_1(a) \geq 0, \ldots,\ l_4(a) \geq 0\}$ as one can easily verify.

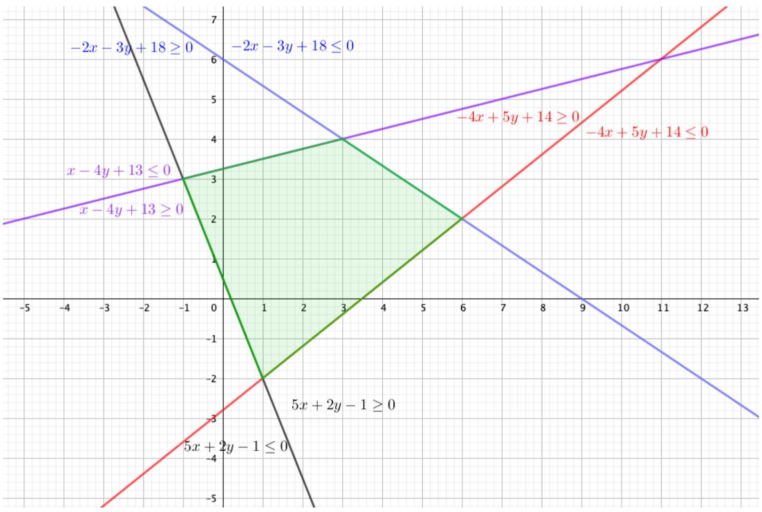


FIGURE 1.2. P ist the set of solutions of a system of non-strict linear inequalities

♦

The fact that the polytope P from EXAMPLE 1.2 can be described by linear polynomial inequalities is not a lucky coincidence. Indeed, every polytope possesses this remarkable property:

PROPOSITION 1.3: Let $P \subseteq \mathbb{R}^n$. Then the following are equivalent:

(i) P is a polytope

(ii) P is bounded and

$$P = \{a \in \mathbb{R}^n :\ l_1(a) \geq 0, \ldots,\ l_k(a) \geq 0\} \tag{1.2}$$

for linear polynomials $l_1, \ldots, l_k \in \mathbb{R}[\underline{x}]_{\leq 1}$. A set of the form (1.2) is called a *polyhedron*.

PROOF: One proof of this fact, which is not that trivial, can be found in [Web94], p.114. ■

Now let $P \subseteq \mathbb{R}^n$ be a polytope. According to PROPOSITION 1.3, there exist linear polynomials $l_1, \ldots, l_k \in \mathbb{R}[\underline{x}]_{\leq 1}$ such that

$$P = \{a \in \mathbb{R}^n : \; l_1(a) \geq 0, \ldots, \; l_k(a) \geq 0\}.$$

This shows that a polytope is a basic closed semialgebraic set and, therefore, one can try to find minimal descriptions of polytopes. The fact that polytopes are given by **linear** inequalities entails some very useful advantages, which will be stated in the next section.

2. Faces of Polytopes

DEFINITION 1.4: *Let $C \subseteq \mathbb{R}^n$ be a convex set. A subset $F \subseteq C$ is called a face of C*

$$:\Leftrightarrow \forall x, y \in C \; \forall \lambda \in (0,1) : (\lambda x + (1-\lambda)y \in F \Rightarrow x, y \in F) \tag{1.3}$$

An element $v \in C$ is called an extremal point of C if the set $\{v\}$ is a face of C, more precisely:

$$\forall x, y \in C \; \forall \lambda \in (0,1) : (v = \lambda x + (1-\lambda)y \Rightarrow x = v = y)$$

$\mathrm{ex}(C) := \{v \in C : v$ *is an extremal point*$\}$ *denotes the set of extremal points of C, $\dim(F)$ is the dimension of the affine hull of a face F. Since C and $\emptyset$ are also special kinds of faces, a face F with $\emptyset \neq F \neq C$ is called a proper face.*

Extremal points play an important role in describing convex and compact sets. This will be stated in the next theorem:

THEOREM 1.5: (*Krein-Milman*):
Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be convex and compact. Then the following holds:

(i) $\mathrm{ex}(C) \neq \emptyset$
(ii) $C = \mathrm{conv}(\mathrm{ex(C)})$

PROOF: See [Pla11], p. 19, or [Wer05], p. 418-419, for a more general version of the statement. ■

DEFINITION 1.6: *Let* $C \subseteq \mathbb{R}^n$ *be convex and* $l \in \mathbb{R}[\underline{x}]_{\leq 1}$ *such that* $l(c) \leq 0$ *for all* $c \in C$*. Then the following set is a face of* C *as one can easily verify:*

$$F = C \cap \{l = 0\} = \{c \in C : l(c) = 0\}$$

A proper face of this form is called exposed. It just arises as an intersection of the set with a hyperplane. The hyperplane $\{l = 0\}$ *is also called a supporting hyperplane.*

EXAMPLE 1.7: Let $C \subseteq \mathbb{R}^2$ be the convex and compact set which is shown in the figure below. The extremal points of the set are drawn in red. By taking a close look at the graphic one believes that $\mathrm{ex}(C) \neq \emptyset$ and $C = \mathrm{conv}(\mathrm{ex}(C))$ holds. Besides the already mentioned extremal points there exist some other proper faces, namely $[v_1, v_2]$, $[v_1, v_3]$ and $[v_3, v_4]$. Some of the faces are exposed, some of them are not. For example, $[v_3, v_4]$ and v are exposed. To see this, the two supporting hyperplanes are plotted in the graphic. On the other hand, the two extremal points v_2 and v_4 are non-exposed faces. It is impossible to find a line passing through the points with the required properties.

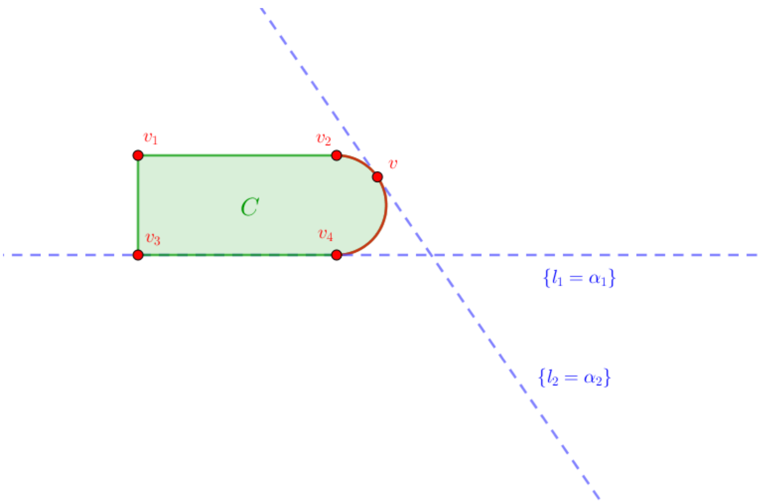


FIGURE 1.3. Faces of the set C

◆

By definition, polytopes are convex sets. Therefore, one can define *faces of polytopes*. The extremal points of polytopes are better known as *vertices*, the one-dimensional faces are called *edges* and the $(\dim(P)-1)$-dimensional faces are also known as *facets*. To avoid confusion, the set of all vertices of a polytope $P \subseteq \mathbb{R}^n$ will be denoted by $\mathrm{vert}(P)$ instead of $\mathrm{ex}(P)$. The faces of polytopes possess some interesting properties, which will be stated in the next proposition.

PROPOSITION 1.8: Let $P \subseteq \mathbb{R}^n$ be a n-dimensional polytope. Then the following holds:

(i) The set $\{F \subseteq P : F$ is a face of $P\}$ is finite.
(ii) Let F be a face of P. Then: $F = \mathrm{aff}(F) \cap P$
(iii) Every face of P is exposed.
(iv) An intersection of faces of P is again a face of P.
(v) Every proper face F arises as the intersection of all facets of P containing F.
(vi) Assume that $G_1 = P \cap \{l_1 = 0\}, \ldots, G_s = P \cap \{l_s = 0\}$ with $l_i|_P \leq 0$ for $i \in \{1, \ldots, s\}$ are all the facets of P. The polyhedral description of P is given by

$$P = \{a \in \mathbb{R}^n : l_1(a) \leq 0, \ldots, l_s(a) \leq 0\} = \{a \in \mathbb{R}^n : -l_1(a) \geq 0, \ldots, -l_s(a) \geq 0\}.$$

(vii) $\mathrm{bd}(P) = G_1 \cup \ldots \cup G_s$.
$(viii)$ If $F \subseteq P$ is a face of P with $\dim(F) = k$, then P has faces of all dimensions from k to $n = \dim(P)$. For each dimension from k to n one can find a face containing F.
(ix) Every face F of P is a polytope with

$$\exists V \subseteq \mathrm{vert}(P) : F = \mathrm{conv}(V).$$

(x) Let $F \subseteq P$ be a face of P. The faces of F are exactly the faces of P that are contained in F.

PROOF: This is a collection of results from [Web94] and [Zie07]. ■

With all the information from PROPOSITION 1.8 one can easily see that a k-dimensional face F of a n-dimensional polytope P is contained in at least $n-k$ facets. Like above, assume that $G_1 = P \cap \{l_1 = 0\}, \ldots, G_s = P \cap \{l_s = 0\}$ with $l_i|_P \leq 0$ for $i \in \{1, \ldots, s\}$ are all the facets of P. By part (v) of the above proposition one has

$$F = \bigcap_{i \in I} (P \cap \{l_i = 0\}) = P \cap \bigcap_{i \in I} \{l_i = 0\} \quad \text{for} \quad I \subseteq \{1, \ldots, s\}. \tag{1.4}$$

Note that P is a full-dimensional polytope. Intersecting $\operatorname{aff}(P) = \mathbb{R}^n$ with $|I|$ hyperplanes will reduce the dimension of the new set by at most $|I|$. In other words,

$$k = \dim(F) \geq n - |I| \quad \Leftrightarrow \quad |I| \geq n - k.$$

This shows that one needs at least $n - k$ facets in (1.4) to obtain a k-dimensional face. The same holds true if the polytope P is given by polynomial inequalities. This was proven by *Grötschel* and *Henk*. Their results will be stated in the next proposition.

PROPOSITION 1.9: (derived from *Grötschel and Henk*, [**GrHe03**], Proposition 2.1) Let $P \subseteq \mathbb{R}^n$ be a n-dimensional polytope. Moreover, for $i \in \{1, \ldots, s\}$ let $l_i \in \mathbb{R}[\underline{x}]_{\leq 1}$ with $l_i|_P \leq 0$ and

$$F_1 = P \cap \{l_1 = 0\}, \ldots, F_s = P \cap \{l_s = 0\}$$

be the facets of P. By PROPOSITION 1.8, part (vi), P has the following form:

$$P = \{a \in \mathbb{R}^n : l_1(a) \leq 0, \ldots, l_k(a) \leq 0\}$$

Assume that there exist (not necessarily linear) polynomials $p_1, \ldots, p_l \in \mathbb{R}[\underline{x}]$ such that P has another basic closed description of the form

$$P = \mathcal{W}(p_1, \ldots, p_l) = \{a \in \mathbb{R}^n : \ p_1(a) \geq 0, \ldots, \ p_l(a) \geq 0\}.$$

Then the following holds:

(i) $\forall i \in 1, \ldots, s \ \exists j \in 1, \ldots, l : l_i | p_j$

(ii) Let F be a k-dimensional face of P. Then:

$$\exists I \subseteq \{1, \ldots, l\}, |I| = n - k : \operatorname{aff}(F) \subseteq \{a \in \mathbb{R}^n : p_i(a) = 0 \ \forall i \in I\}$$

This shows that at least $n - k$ polynomials vanish on F, which has been demonstrated for the linear case above.

In the following, a short sketch of the proof of this proposition is given, which can be found in [**GrHe03**] (Proposition 2.1.):

PROOF: For $i \in \{1, \dots s\}$ let $F_i = P \cap \{l_i = 0\}$ be a facet of P and $y \in F_i$. Assume that $\forall j = 1, \dots, l : p_j(y) \neq 0$. Since $y \in F_i \subseteq P$, it holds that $\forall j = 1, \dots, l : p_j(y) > 0$. Therefore, one can find some $\epsilon > 0$ such that

$$\forall j = 1, \dots, l : p_j|_{B_\epsilon(y)} > 0$$

This implies that $y \notin \mathrm{bd}(P)$. By part (vii) of PROPOSITION 1.8 all facets of P are contained in the boundary of the polytope. Therefore, $y \notin \mathrm{bd}(P)$ yields a contradiction. Hence $\forall y \in F_i\ \exists j \in \{1, \dots, l\} : p_j(y) = 0$. Now define

$$f := \prod_{j=1}^{l} p_j \in \mathbb{R}[\underline{x}].$$

The above computation shows that $f|_{F_i} = 0$. Let $a \in \mathrm{relint}(F_i)$, where $\mathrm{relint}(F_i)$ denotes the *relative interior* of the face F_i. By definition of the relative interior one gets that

$$\begin{aligned} \exists \epsilon > 0 :& (B_\epsilon(a) \cap \mathrm{aff}(F_i)) \subseteq F_i \Rightarrow f|_{B_\epsilon(a) \cap \mathrm{aff}(F_i)} = 0 \\ & \Rightarrow f \equiv 0 \text{ on } \mathrm{aff}(F_i) = \{a \in \mathbb{R}^n : l_i(a) = 0\} = \mathcal{V}(l_i) \\ & \overset{(\star)}{\Rightarrow} l_i | f \end{aligned} \tag{1.5}$$

The implication $(\star)$ needs a more precise explanation. One important topic of real algebraic geometry is trying to formulate so called *Positivstellensätze* for polynomials. Such theorems describe how polynomials which vanish on certain basic closed semialgebraic or algebraic sets look like. One important theorem is called *Real Nullstellensatz* and states the following:

THEOREM 1.10: (*Real Nullstellensatz*)
Let $p, p_1, \dots, p_r \in \mathbb{R}[\underline{x}]$. Then:

$$p \equiv 0 \text{ on } \mathcal{V}(p_1, \dots, p_r) \Leftrightarrow p \in \mathrm{rrad}((p_1, \dots, p_r)), \tag{1.6}$$

where $(p_1, \dots, p_r)$ denotes the ideal generated by the polynomials $p_1, \dots, p_r$ and $\mathrm{rrad}((p_1, \dots, p_r)) := \{g \in \mathbb{R}[\underline{x}] : \exists m \in \mathbb{N}, \sigma \in \sum \mathbb{R}[\underline{x}]^2, g^{2m} + \sigma \in (p_1, \dots, p_r)\}$ is the real radical of $(p_1, \dots, p_r)$.

PROOF: One proof of this important theorem can be found in [**PrDe01**]. ■

Let us return to the primary proof. In (1.5) it was shown that

$$f \equiv 0 \text{ on } \mathcal{V}(l_i).$$

Hence by the *Real Nullstellensatz*

$$f \in \operatorname{rrad}((l_i)) \overset{(\triangle)}{=} (l_i). \tag{1.7}$$

The equality in $(\triangle)$ holds true since the real radical of an ideal I is just the intersection of all real prime ideals containing I. Since l_i is irreducible, (l_i) is a prime ideal. Without going into too much detail, it should be emphasized that (l_i) is also a real ideal. One can find information about real rings and real ideals in [**PrDe01**]. Hence the intersection of all real ideals containing (l_i) is just (l_i). Therefore, $f \in (l_i) := \{g \cdot l_i : g \in \mathbb{R}[\underline{x}]\}$, which implies that $l_i | f$.

Finally, notice that the polynomial l_i is irreducible and hence prime. Therefore,

$$l_i \Big| \prod_{j=1}^{l} p_j \Rightarrow \exists j \in \{1, \ldots, l\} : l_i | p_j \Rightarrow p_j\big|_{\operatorname{aff}(F_i)} = p_j\big|_{\{l_i = 0\}} = 0.$$

<u>ad (ii)</u>: The second part will be proven by induction over $k = \dim(F)$.

BASE CASE: Let $k = n-1$. Then F is a facet of P and by part (i) of the proof there exists $j \in \{1, \ldots, l\} : p_j|_{\operatorname{aff}(F)} = 0$.

STEP CASE: Let $k < n-1$. By PROPOSITION 1.8, part $(viii)$, there exists a face $G \subseteq P$ with $\dim(G) = k+1$ and $F \subseteq G$. By induction hypothesis

$$\exists I \subseteq \{1, \ldots, l\}, |I| = n-k-1 : \operatorname{aff}(G) \subseteq \{a \in \mathbb{R}^n : p_i(a) = 0 \ \forall i \in I\} \tag{1.8}$$

Applying part (ii) of PROPOSITION 1.8 yields

$$\begin{aligned} G = \operatorname{aff}(G) \cap P &= \{a \in \operatorname{aff}(G) : p_1(a) \geq 0, \ldots, p_l(a) \geq 0\} \\ &= [p_i(a) = 0 \ \forall a \in \operatorname{aff}(G), \ \forall i \in I] \\ &= \{a \in \operatorname{aff}(G) : p_j(a) \geq 0 \ \forall j \in \{1, \ldots, l\} \backslash I\} \end{aligned}$$

Finally, note that $F \subseteq G$ is also a face of G with $\dim(F) = \dim(G) - 1$ (by part (x) of PROPOSITION 1.8). This means that F is a facet of G (which is a polytope in the space $\operatorname{aff}(G)$) and by part (i) of the proof one can choose $j \in \{1, \ldots, l\} \backslash I$ such that

$p_j|_{\mathrm{aff}(F)} = 0$. Therefore, it holds that

$$\mathrm{aff}(F) \subseteq \{p_j = 0\} \cap \mathrm{aff}(G) \overset{(1.8)}{\subseteq} \{a \in \mathbb{R}^n : p_i(a) = 0 \ \forall i \in I, p_j(a) = 0\}.$$

Since $|I| = n - k - 1$, this shows that at least $n - k$ polynomials vanish on $\mathrm{aff}(F)$.

■

REMARK 1.11: The last proposition demonstrates a first important fact concerning minimal descriptions of polytopes. For a n-dimensional polytope $\emptyset \neq P \subseteq \mathbb{R}^n$ one has $\mathrm{vert}(P) \neq \emptyset$ by the *Theorem of Krein-Milman*. Therefore, there exists a vertex $v \in P$. Let $P = \mathcal{W}(p_1, \ldots, p_l)$ be any basic closed description of the polytope. Since vertices are 0-dimensional faces, it follows from PROPOSITION 1.9 that at least n polynomials $p_{i_1}, \ldots, p_{i_n} \in \{p_1, \ldots, p_l\}$ vanish on v. This shows that it is impossible to find basic closed descriptions of n-dimensional polytopes $P \subseteq \mathbb{R}^n$ with less than n polynomials. This important observation should be emphasized:

OBSERVATION 1: Let $\emptyset \neq P \subseteq \mathbb{R}^n$ be a n-dimensional polytope. Then the following holds:

$$m(P) \geq n$$

REMARK 1.12: It should be mentioned that n-dimensional polytopes cannot be described by less than n polynomials due to the fact that they are not *smooth* enough. The vertices cause some problems, which are responsible for the lower bound n of $m(P)$. If one takes a look at spectrahedra, there are no lower bounds concerning the number of polynomials which are necessary for a minimal description of them. For example, the unit ball $\{x \in \mathbb{R}^n : 1 - x_1^2 - x_2^2 - \ldots - x_n^2 \geq 0\}$ is a spectrahedron in $\mathbb{R}^n$, which can be described by just one polynomial (independent of the space-dimension). But these observations will be dealt with in the second chapter. The next aim of the thesis is showing that $m(P) = 2$ for polygons $P \subseteq \mathbb{R}^2$.

3. ***Bernig's*** construction

In [**Be97**] *Andreas Bernig* was able to show that each basic open semialgebraic set in $\mathbb{R}^2$ given by linear polynomials possesses a basic open description with two polynomials. He was even able to construct these polynomials. The purpose of this chapter is to give an overview of his construction. His discoveries can easily be extended to the basic closed case. This is formulated in the next theorem.

THEOREM 1.13: (derived from *Bernig* [**Be97**], p.18-21)
Let $P \subseteq \mathbb{R}^2$ be a 2-dimensional polytope. There exist polynomials $p_1, p_2 \in \mathbb{R}[x, y]$ such that

$$P = \mathcal{W}(p_1, p_2).$$

PROOF: The whole proof for basic open sets can be found in [**Be97**], p.18-21. In the present thesis the proof is slightly modified to be able to extend the statement to the basic closed case. This is done by a little additional consideration, namely by showing that one of *Bernig's* defining polynomials is *strictly concave*, which will be done in REMARK 1.16. Sometimes, *Bernig's* proof is slightly modified, but – on the whole – the following pages are derived from [**Be97**].

To be able to understand the main idea of the proof, one can consider the following example: Let

$$\begin{aligned} l_1 &:= -2x - 3y + 9 \in \mathbb{R}[x, y], \\ l_2 &:= -x + 5y - 2 \in \mathbb{R}[x, y], \\ l_3 &:= 3x - 2y + 6 \in \mathbb{R}[x, y] \end{aligned}$$

be linear polynomials and $P := \mathcal{W}(l_1, l_2, l_3) \subseteq \mathbb{R}^2$. The polytope defined by these polynomials is a triangle. This can be seen in FIGURE 1.4.

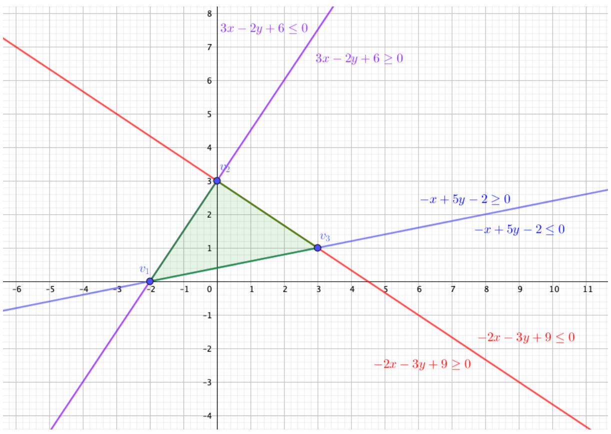


FIGURE 1.4. P is a triangle

In a first step, Bernig constructed a polynomial p_1 such that $\mathcal{W}(p_1)$ contains the set P, namely the product of all linear polynomials which pertain to the polyhedral description of the set. Without loss of generality one can assume that none of the defining linear polynomials is redundant.

$$p_1 := \prod_{k=1}^{s} l_k$$

For $a \in P$ it holds that $\forall k \in \{1, \dots, s\} : \ l_k(a) \geq 0$ and, therefore, $p_1(a) \geq 0$. Hence $P \subseteq \mathcal{W}(p_1)$. The problem is that $\mathcal{W}(p_1) \subseteq P$ does not hold. For example, let $a \in \mathbb{R}^2 \backslash P$ such that

$$\exists i, j \in \{1, \dots, s\} : l_i(a) < 0, \ l_j(a) < 0 \text{ and } l_k(a) \geq 0 \ \forall k \in \{1, \dots, s\} \backslash \{i, j\}.$$

Then it holds that

$$p_1(a) = \prod_{i=1}^{s} l_i(a) = \underbrace{\underbrace{l_i(a)}_{<0} \cdot \underbrace{l_j(a)}_{<0}}_{>0} \cdot \underbrace{\prod_{k \in \{1,\dots,l\} \backslash \{i,j\}} l_k(a)}_{\geq 0} \geq 0.$$

As long as an even number of polynomials is less than 0 on a, it holds that $a \in \mathcal{W}(p_1)$. Moreover, if just one linear polynomial l_i vanishes on a, the product of all linear polynomials evaluated at a is 0 – independent of the value $l_j(a)$ of the other linear polynomials $l_j, j \in \{1, \dots, s\} \backslash \{i\}$. Hence $p_1(a) = 0$ and therefore $a \in \mathcal{W}(p_1)$. This fact will be illustrated with the above example. If one defines $p_1 := l_1 \cdot l_2 \cdot l_3$, the green and orange areas in FIGURE 1.5 mark $\mathcal{W}(p_1)$.

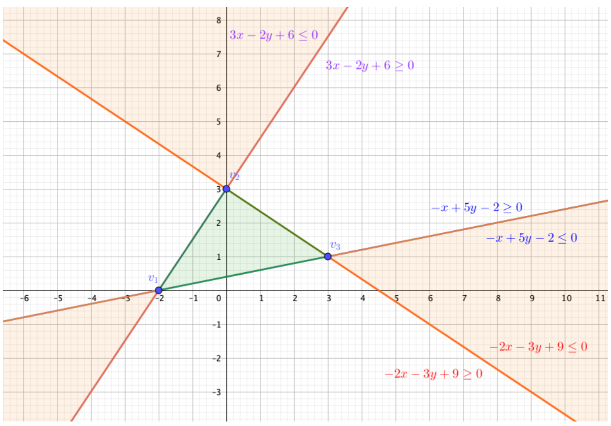


FIGURE 1.5. Visualisation of $\mathcal{W}(p_1)$

Since the polytope P is just the green set in FIGURE 1.5., one needs to construct a polynomial which, casually speaking, cuts off the orange area. If one just focuses on the present example, this can easily be done by taking

$$p_2 := \frac{13}{2} - \left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2.$$

The zero set of this polynomial is just the circumcircle of the triangle. The red area in FIGURE 1.6. represents $\mathcal{W}(p_2)$. By taking a close look at the picture, one could suggest that $P = \mathcal{W}(p_1, p_2)$. Indeed, the equality holds true since p_2 is a strictly concave polynomial. This will be proven below.

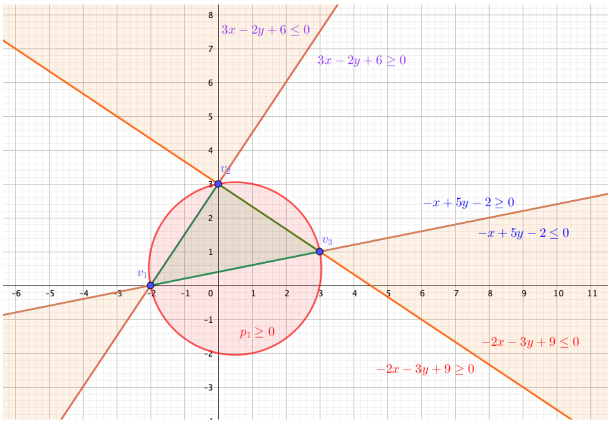


FIGURE 1.6. $P = \mathcal{W}(p_1, p_2)$

The tricky part of *Bernig's* proof was to explicitly construct a polynomial p_2 with the required property for any given polytope $P \subseteq \mathbb{R}^2$, namely a polynomial cutting off the surplus area arisen from $\mathcal{W}(p_1)$. He was able to show the following:

LEMMA 1.14: (derived from [**Be97**], Theorem 3.1.2)
With the same assumptions like in THEOREM 1.13 there exists a *concave* polynomial $p \in \mathbb{R}[x, y]$ such that

$$\forall v \in \operatorname{vert}(P) : p(v) = 0.$$

By the concavity of the function it follows that $\mathcal{W}(p)$ and $\mathcal{O}(p)$ are convex.

PROOF (SEE ALSO [**Be97**], P. 18-20):
Step 1: Choose $v \in \operatorname{vert}(P)$ and set $C := \operatorname{conv}(\operatorname{vert}(P)\backslash\{v\})$. At first, notice that $v \notin C$. This follows from the fact that v is a vertex. If $v \in C$, one could find

$v_1, \dots, v_k \in \operatorname{vert}(P)\backslash\{v\}$ such that $v = \sum_{i=1}^k \lambda_i v_i$ with $\sum_{i=1}^k \lambda_i = 1$. Hence

$$v = \lambda_1 v_1 + (1-\lambda_1) \underbrace{\sum_{i=2}^k \frac{\lambda_i}{1-\lambda_1}}_{=1} v_i. \tag{1.9}$$

Since $\sum_{i=2}^k \frac{\lambda_i}{1-\lambda_1} = 1$, $w := \sum_{i=2}^k \frac{\lambda_i}{1-\lambda_1} v_i$ is a convex combination of $v_2, \dots, v_k \in P$ and hence $w \in P$. Therefore, v can be written as a convex combination of two points $v_1, w \in P$, $v_1 \neq v$, $w \neq v$, which yields a contradiction since v is a vertex of P.

The set C is convex and closed. Therefore, one can apply a *Hahn-Banach Separation Theorem* and separate v from C. Indeed, there exists an affine-linear polynomial $l_v \in \mathbb{R}[x,y]_{\leq 1}$ such that

$$l_v(v) > 0 \quad \text{and} \quad l_v(c) \leq 0\, \forall c \in C \quad (\text{especially } l_v(c) \leq 0\, \forall c \in \operatorname{vert}(P)\backslash\{v\}).$$

After applying an appropriate linear transformation, one is even able to assume that

$$l_v(v) = 1 \quad \text{and} \quad -1 < l_v(c) < 1 \quad \forall\, c \in \operatorname{vert}(P)\backslash\{v\}.$$

Hence for a given $\epsilon > 0$ one can choose $s \in \mathbb{N}$ big enough and set $p_{v,\epsilon} := l_v^{2s}$, which yields a polynomial with

$$p_{v,\epsilon}(v) = 1 \quad \text{and} \quad |p_{v,\epsilon}(c)| < \epsilon \quad \forall c \in \operatorname{vert}(P)\backslash\{v\}. \tag{1.10}$$

$p_{v,\epsilon}$ is convex as a composition of convex functions (affine-linear functions are convex and $[x \mapsto x^{2l}]$ is convex).

<u>Step 2:</u> The idea of the proof is finding positive real numbers $\lambda_v \in \mathbb{R}_{>0}$ and $\epsilon > 0$ such that

$$p := 1 - \left(\sum_{v \in \operatorname{vert}(P)} \lambda_v p_{v,\epsilon} \right) \tag{1.11}$$

is the sought-after concave polynomial which is vanishing on all the vertices of P. Let $v_1, \dots, v_s \in \operatorname{vert}(P)$ be all the vertices of P. Remember that a polytope has only finitely many faces, hence also finitely many vertices. Since the polynomial has to vanish on each vertex, the following conditions have to be satisfied:

$$\begin{aligned} p(v_l) = 1 - \sum_{v \in \mathrm{vert}(P)} \lambda_v p_{v,\epsilon}(v_l) &\overset{!}{=} 0 \quad \forall l \in \{1,\dots,s\} \\ \Leftrightarrow \sum_{v \in \mathrm{vert}(P)} \lambda_v p_{v,\epsilon}(v_l) &= 1 \quad \forall l \in \{1,\dots,s\} \end{aligned}$$

Equivalently, one could write

$$\underbrace{\begin{pmatrix} p_{v_1,\epsilon}(v_1) & \dots & p_{v_s,\epsilon}(v_1) \\ & \ddots & \\ p_{v_1,\epsilon}(v_s) & \dots & p_{v_s,\epsilon}(v_s) \end{pmatrix}}_{=:B} \cdot \underbrace{\begin{pmatrix} \lambda_{v_1} \\ \vdots \\ \lambda_{v_s} \end{pmatrix}}_{=:\boldsymbol{\lambda}} = \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{:=\mathbf{1}}. \tag{1.12}$$

By construction of $p_{v_j,\epsilon}, j \in \{1,\dots,s\}$, $p_{v_j,\epsilon}(v_j) = 1$ and hence the diagonal of the above matrix contains only 1's. Therefore, one can write

$$B = \mathbb{I}_s + \underbrace{\begin{pmatrix} 0 & p_{v_2,\epsilon}(v_1) & \dots & p_{v_s,\epsilon}(v_1) \\ p_{v_1,\epsilon}(v_2) & 0 & \dots & p_{v_s,\epsilon}(v_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_{v_1,\epsilon}(v_s) & p_{v_2,\epsilon}(v_s) & \vdots & 0 \end{pmatrix}}_{=:A}, \tag{1.13}$$

where $\mathbb{I}_s$ denotes the unitary $(s \times s)$-matrix. Remember that the entries of the new matrix $A = (a_{ij})_{i,j} = (p_{v_j,\epsilon}(v_i))_{i,j}$, which are not in the diagonal, are of absolute value less than ϵ (see 1.10). By $\|\cdot\|_{\infty,\infty}$ let us denote the matrix norm induced by the vector norm $\|\cdot\|_\infty$, which is just the maximum row sum of the matrix. It holds that

$$\begin{aligned} \|A\|_{\infty,\infty} &= \max_{i=1,\dots s} \sum_{j=1}^{s} |a_{ij}| = \max_{i=1,\dots s} \left(\underbrace{|a_{ii}|}_{=0} + \sum_{j \neq i} \underbrace{|a_{ij}|}_{=|p_{v_j,\epsilon}(v_i)|} \right) \\ &= \max_{i=1,\dots s} \left(|0| + \sum_{j \neq i} \underbrace{|p_{v_j,\epsilon}(v_i)|}_{<\epsilon} \right) < (s-1)\epsilon. \end{aligned} \tag{1.14}$$

If one chooses $\epsilon < \frac{1}{2(s-1)}$, it follows from (1.14) that $\|A\|_{\infty,\infty} < \frac{1}{2}$.

To finish the proof, one needs the so called *Banach Lemma*. It states the following:

LEMMA 1.15: (*Banach Lemma*):
Let $\|\cdot\|$ be an operator norm in $\mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n\times n}$. If $\|M\| < 1$, then $\mathbb{I}_n - M$ is invertible. Moreover,

$$\|(\mathbb{I}_n - M)^{-1}\| \leq \frac{1}{1-\|M\|}. \tag{1.15}$$

PROOF: For instance, see [Sh07]. ∎

In the above situation for $\|\cdot\| := \|\cdot\|_{\infty,\infty}$ and $M := -A$ it has already been proven that $\|M\| = \|-A\| < \frac{1}{2} < 1$, which implies that $\mathbb{I}_s - M = \mathbb{I}_s + A = B$ is invertible. Hence (1.12) possesses a unique solution. This shows that one can find $(\lambda_v)_{v\in\mathrm{vert}(P)}$ as claimed in (1.11). It remains to show that $\lambda_v > 0$ for all $v \in \mathrm{vert}(P)$. This is necessary since $\lambda_v > 0$ guarantees that the polynomial p constructed in (1.11) is indeed concave.

It follows from (1.11) and (1.12) that

$$(I+A)\boldsymbol{\lambda} = \mathbf{1} \Leftrightarrow \boldsymbol{\lambda} + A\boldsymbol{\lambda} = \mathbf{1} \Leftrightarrow \boldsymbol{\lambda} - \mathbf{1} = -A\boldsymbol{\lambda}$$

Since $\lambda = (A+\mathbb{I})^{-1}\mathbf{1}$,

$$\begin{aligned}\|\boldsymbol{\lambda} - \mathbf{1}\|_\infty = \|-A\boldsymbol{\lambda}\|_\infty = \|A\boldsymbol{\lambda}\|_\infty &= \|A(A+\mathbb{I})^{-1}\mathbf{1}\|_\infty \leq \|A\|\|(A+\mathbb{I})^{-1}\| \underbrace{\|\mathbf{1}\|_\infty}_{=1} \\ &\overset{(1.15)}{\leq} \frac{\|A\|}{1-\|A\|} < \left[\|A\| < \frac{1}{2}\right] < 1.\end{aligned}$$

Note that

$$\|\boldsymbol{\lambda} - \mathbf{1}\|_\infty = \max_{v\in\mathrm{vert}(P)} |\lambda_v - 1|,$$

which implies $\forall v \in \mathrm{vert}(P) : |\lambda_v - 1| < 1$. This condition is not satisfied for $\lambda_v \leq 0$. Hence $\lambda_v > 0$ and this immediately finishes the proof.

By the convexity of the functions $p_{v,\epsilon}$ it follows that

$$h := \sum_{v\in\mathrm{vert}(P)} \underbrace{\lambda_v}_{>0} p_{v,\epsilon} \tag{1.16}$$

is a positive linear combination of convex functions and hence convex. Therefore, $p := 1 - h$ is concave. Moreover, for $x, y \in \mathcal{W}(p)$, $\tilde{x}, \tilde{y} \in \mathcal{O}(p)$, $\lambda \in [0,1]$, it holds that

$$p(\lambda x + (1-\lambda)y) \overset{p \text{ concave}}{\geq} \lambda \underbrace{p(x)}_{\geq 0} + (1-\lambda) \underbrace{p(y)}_{\geq 0} \geq 0$$

$$p(\lambda \tilde{x} + (1-\lambda)\tilde{y}) \overset{p \text{ convave}}{\geq} \lambda \underbrace{p(\tilde{x})}_{>0} + (1-\lambda) \underbrace{p(\tilde{y})}_{>0} > 0.$$

This shows that $\mathcal{W}(p)$ and $\mathcal{O}(p)$ are convex, which finishes the proof.

■

REMARK 1.16: If one takes a close look at the constructed polynomials $p_{v,\epsilon}$, it turns out that these polynomials are convex but not *strictly* convex. For given $x, y \in \mathbb{R}^2$, $x \neq y$, $\lambda \in (0,1)$ and a vertex $v \in \operatorname{vert}(P)$

$$\begin{aligned} p_{v,\epsilon}(\lambda x + (1-\lambda)y) = (l_v(\lambda x + (1-\lambda)y))^{2s} &\overset{l_v \text{ affine-linear}}{=} (\lambda l_v(x) + (1-\lambda) l_v(y))^{2s} \\ &< \lambda l_v(x)^{2s} + (1-\lambda) l_v(y)^{2s} = \lambda p_{v,\epsilon}(x) + (1-\lambda) p_{v,\epsilon}(y) \\ &\Leftrightarrow l_v(x) \neq l_v(y). \end{aligned} \tag{1.17}$$

This follows from the fact that the function $[x \mapsto x^{2s}]$ is strictly convex, hence $\forall x, y \in \mathbb{R}, x \neq y, \forall \lambda \in (0,1)$ it holds that $(\lambda x + (1-\lambda)y)^{2s} < \lambda x^{2s} + (1-\lambda)y^{2s}$. To get the strict inequality sign, one has to guarantee that $l_v(x)$ and $l_v(y)$ are not the same. This is not always the case. This consideration should just emphasize that one needs to be careful when using the strict inequality signs "$<$" or "$>$" instead of "$\leq$" or "$\geq$".

Nevertheless, the polynomial p defined by the polynomials $p_{v,\epsilon}, v \in \operatorname{vert}(P)$, is even strictly concave. To see this, let $x, y \in \mathbb{R}^2, x \neq y$, and $\lambda \in (0,1)$. One has to show that there exists at least one $v \in \operatorname{vert}(P)$ with $l_v(x) \neq l_v(y)$. If this was true, then

$$\begin{aligned} p(\lambda x + (1-\lambda)y) &= 1 \underbrace{-\lambda_v p_{v,\epsilon}(\lambda x + (1-\lambda)y)}_{> -\lambda_v(\lambda p_{v,\epsilon}(x) + (1-\lambda)p_{v,\epsilon}(y))} \underbrace{- \sum_{w \in \operatorname{vert}(P), w \neq v} \lambda_w p_{w,\epsilon}(\lambda x + (1-\lambda)y)}_{\geq -\sum_{w \in \operatorname{vert}(P), w \neq v} \lambda_w(\lambda p_{w,\epsilon}(x) + (1-\lambda)p_{w,\epsilon}(y))} \\ &> \underbrace{\lambda + (1-\lambda)}_{=1} - \lambda \sum_{v \in \operatorname{vert}(P)} \lambda_v p_{v,\epsilon}(x) - (1-\lambda) \sum_{v \in \operatorname{vert}(P)} \lambda_v p_{v,\epsilon}(y) \\ &= \lambda p(x) + (1-\lambda)p(y). \end{aligned}$$

Assume that $\forall v \in \mathrm{vert}(P) : l_v(x) = l_v(y) = c_v$, $c_v \in \mathbb{R}$. By the convexity of $\{l_v = c_v\}$ it follows that $[x, y] \subseteq \{l_v = c_v\}$ for all $v \in \mathrm{vert}(P)$. This is only possible if the sets $\{l_v = c_v\}, v \in \mathrm{vert}(P)$, coincide. Hence $\{l_v = 1\}, v \in \mathrm{vert}(P)$, are parallel lines.

Notice that $|\mathrm{vert}(P)| \geq 3$, since $\dim(P) = 2$. Hence pick v_1, v_2 and v_3 in $\mathrm{vert}(P)$. By the construction of the affine-linear functions l_v each vertex v_i is an element of $\{l_{v_i} = 1\}$. One ought to consider that the parallel lines $\{l_{v_i} = 1\}, i \in \{1, 2, 3\}$, do not coincide since $l_{v_i}(v_j) < 1$ for all $i, j \in \{1, 2, 3\}, i \neq j$. Since they do not coincide, one can assume without loss of generality that $\{l_{v_2} = 1\}$ separates the sets $\{l_{v_1} = 1\}$ and $\{l_{v_3} = 1\}$ (compare with FIGURE 1.7). But this must not happen since

$$l_{v_2}(v_1) < 1 \quad \text{and} \quad l_{v_2}(v_3) < 1.$$

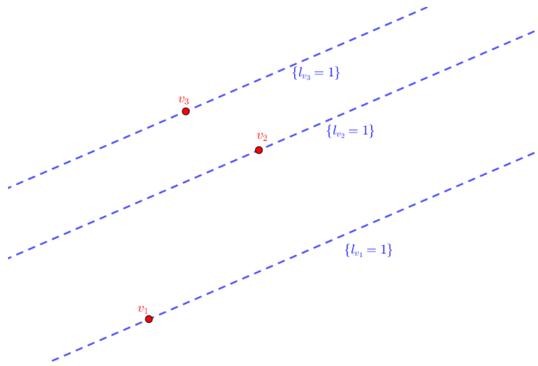


FIGURE 1.7. Three parallel lines

This computation shows that p is even strictly concave which will turn out to be a great advantage.

Now let us finish the proof of THEOREM 1.13:
Let $p_2 := p$. One has to show that $P = \mathcal{W}(p_1, p_2)$ indeed holds. Therefore, let $a \in P$. Then $p_1(a) \geq 0$ trivially holds. To show that $p_2(a) \geq 0$ is also true, again let $v_1, \ldots, v_s$ be all vertices of P. Using the *Theorem of Krein-Milman*, one can write $a = \sum_{i=1}^{s} \lambda_i v_i$ with $\sum_{i=1}^{s} \lambda_i = 1$. By the concavity of the polynomial p_2 one gets

$$p_2(a) = p_2(\sum_{i=1}^{s} \lambda_i v_i) \overset{p_2 \text{ concave}}{\geq} \sum_{i=1}^{s} \lambda_i \underbrace{p_2(v_i)}_{=0} = 0.$$

The fact $p_2(v_i) = 0$ holds true since p_2 vanishes on all vertices of P. For $a \in P \backslash \mathrm{vert(P)}$ it even holds that $p_2(a) > 0$ by the strict concavity of the polynomial.

Therefore,

$$P \subseteq \mathcal{W}(p_1, p_2).$$

For the other inclusion, let $a \in \mathbb{R}^2 \backslash P$. Then one has to show that $a \in \mathbb{R}^2 \backslash \mathcal{W}(p_1, p_2)$. We distinguish three cases:

Case 1: $p_1(a) < 0$: Then there is nothing to show.

Case 2: $p_1(a) > 0$: One has to show that $a \notin \mathcal{W}(p_1, p_2)$, i.e. $p_2(a) < 0$. Since $a \notin P$ and $p_1(a) > 0$, there exists an index set $J \subseteq \{1, \ldots, s\}, |J| = 2n, n \in \mathbb{N}$, such that

$$\forall j \in J : l_j(a) < 0, \qquad \forall i \in \{1, \ldots, s\} \backslash J : l_i(a) > 0.$$

Now choose a point $b \in P$ such that

$$\forall i, j \in \{1, \ldots, s\}, i \neq j : \;\; [a, b] \cap (\{l_i = 0\} \cap \{l_j = 0\}) = \emptyset. \tag{1.18}$$

This is possible since the number of the above intersection points is finite. Note that

$$|[a, b] \cap \{y \in \mathbb{R}^2 : \exists j \in J : l_j(y) = 0\}| = |J| = 2n.$$

This holds true since

$$\forall j \in J : \;\; l_j(a) < 0 \;\wedge\; l_j(b) > 0$$

and hence for all $j \in J$ there exists exactly one $v_j \in [a, b]$ such that $l_j(v_j) = 0$. (1.18) guarantees that $v_i \neq v_j$ for $i, j \in J, i \neq j$. Now choose $v \in \{v_j : j \in J\}$ such that

$$\|v - a\| = \min\{\|v_j - a\| : j \in J\},$$

where $\| \cdot \|$ denotes the Euclidian norm. Moreover, let $l \in \{l_j : j \in J\}$ be the corresponding linear polynomial with $l(v) = 0$. One can find an illustration of this procedure in the next graphic.

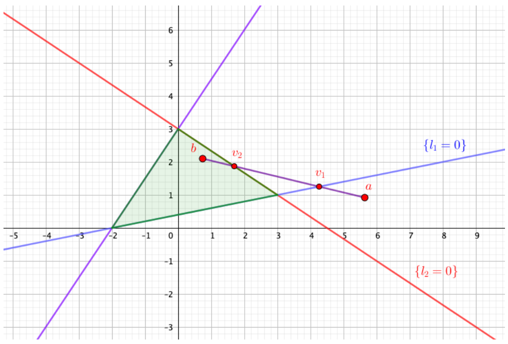


FIGURE 1.8. Correct construction

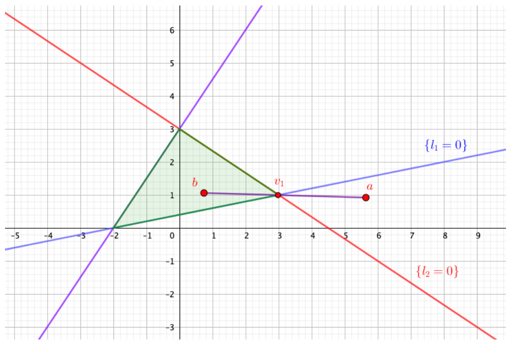


FIGURE 1.9. Incorrect construction

In the example on the left-hand side one can see that the set $[a, b]$ intersects exactly two times a zero set of $l_i, i = 1, 2$. According to the above procedure one has to choose $v := v_1$ and $l := l_1$. The picture on the right-hand side does not yield the desired result since (1.18) is violated.

Note that

$$\forall j \in J : (v_j \neq v \Rightarrow l_j(v) < 0).$$

Since $|J|$ is even, this computation shows that there exists at least one $j \in J$ with $l_j(v) < 0$ and hence $v \notin P$.

Assumption: $p_2(a) \geq 0$. Choose $\lambda \in (0, 1)$ such that

$$\begin{aligned} p_2(v) &= p_2(\lambda a + (1-\lambda)b) \\ &\geq \lambda \underbrace{p_2(a)}_{\geq 0 \text{ by assumption}} + (1-\lambda) \underbrace{p_2(b)}_{>0, \text{ since } b \notin \operatorname{vert}(P)} > 0 \end{aligned}$$

From $v \in \{l = 0\} \backslash P$ and the fact that $\{l = 0\}$ is a facet-defining equation it follows that there exist $e_1, e_2 \in \{l = 0\} \cap \operatorname{vert}(P)$. Without loss of generality one may assume that $e_2 \in [e_1, v]$. Since $v \notin P$ it holds that $v \neq e_1$, $v \neq e_2$, and, therefore, one can use the strict concavity of p to cause a contradiction:

$$\exists \lambda \in (0,1) : 0 = p_2(e_2) = p_2(\lambda e_1 + (1-\lambda)v) > \lambda \underbrace{p_2(e_1)}_{=0} + (1-\lambda) \underbrace{p_2(v)}_{>0} > 0 \quad \lightning \tag{1.19}$$

Case 3: $p_1(a) = 0$: By definition of p_1 there exists an $i \in \{1, \ldots, s\}$ such that $l_i(a) = 0$. Again, since l_i is a facet-defining linear polynomial, one can find $e_1, e_2 \in \operatorname{vert}(P)$

such that

$$\{l_i = 0\} \cap P = [e_1, e_2].$$

Since $a \notin P$, again without loss of generality one may assume that $e_2 \in [e_1, a]$, $e_1 \neq e_2 \neq a$. Hence there exists $\lambda \in (0,1)$ such that $e_2 = \lambda e_1 + (1-\lambda)a$. With a similar argumentation like above one can finish the proof:

$$0 = p_2(e_2) = p_2(\lambda e_1 + (1-\lambda)x) > \lambda \underbrace{p_2(e_1)}_{=0} + (1-\lambda)p_2(a). \tag{1.20}$$

Since $(1-\lambda) > 0$, this implies that $p_2(a) < 0$, hence $a \notin \mathcal{W}(p_1, p_2)$. ■

REMARK 1.17: As mentioned above, in [**Be97**] *Bernig* was able to show that THEOREM 1.13 holds for basic open semialgebraic sets. The slight difference when only concerning basic open sets is that one does not have to assume that p_2 is *strictly* concave. The strict concavity was needed to show inequality (1.20). If one focuses on basic open semialgebraic sets, the third case in the above proof is trivial. If $p_1(a) = 0$, there is nothing to show since it implies that $a \notin \mathcal{O}(p_1, p_2) = \{a \in \mathbb{R}^2 : p_1(a) > 0, p_2(a) > 0\}$. Hence one does not need the inequality in (1.20).

REMARK 1.18: One has to mention that *Bernig's* proof in the two-dimensional case, which was done about twenty years ago, was a great innovation in that times. Even though the *Theorem of Bröcker and Scheiderer* was well-known, nobody was able to explicitly construct polynomials which belong to a minimal description of a polytope. Hence *Bernig's* discovery turned out to be a great advance and, therefore, mathematicians tried to generalize his ideas. At first, *Henk* and *Averkov* managed to extend his idea on *simple polytopes* in $\mathbb{R}^n$ ([**AvHe07**]). With the help of the preparations from *Henk* and himself, *Averkov* was even able to find descriptions of more general basic closed semialgebraic sets with few polynomials ([**Av08**]). His ideas will be the topic of the next section. He did the following:

At first he defined a set of polynomials such that the corresponding basic closed semialgebraic set contains the original basic closed set. For this purpose *Bernig* just used the product of all the facet-defining linear polynomials. To achieve the same effect in $\mathbb{R}^n$, *Averkov* used *elementary-symmetric functions*, which can be seen as a generalization of *Bernig's* polynomial. Afterwards, he constructed a further polynomial which behaves like the strictly concave polynomial in the above proof. It cuts off the surplus area such that the basic closed semialgebraic set given by the *elementary-symmetric functions* and the further polynomial is a new basic closed description of the set.

Finally, let us bring the ideas of the present section to a conclusion:

OBSERVATION 2: Let $\emptyset \neq P \subseteq \mathbb{R}^2$ be a 2-dimensional polytope.

Then the following holds:

$$m(P) = 2$$

4. Simple Polytopes

The main purpose of the present section is precisely explaining an approach for finding basic closed descriptions of special kinds of basic closed sets developed by *Averkov*. Most of the theorems from this section can be found in [**Av08**]. On the following pages they will be worked out in detail and emphasized by some examples. Let us start with the main theorems:

THEOREM 1.19: ([**Av08**], Theorem 1.1.)
Let $p_1, \dots, p_s \in \mathbb{R}[\underline{x}]$ and $\emptyset \neq \mathcal{W}(p_1, \dots, p_s) \subseteq \mathbb{R}^n$ be the corresponding basic closed semialgebraic set. Moreover, assume that $\mathcal{W}$ is bounded. Define

$$d := \max\{|i \in \{1, \dots, s\} : p_i(a) = 0| : a \in \mathcal{W}\}. \tag{1.21}$$

d denotes the maximal number of polynomials which vanish on a point $a \in \mathcal{W}$. The following holds:

$$m(\mathcal{W}) \leq d + 1 \tag{1.22}$$

In other words:

$$\exists\, q_1, \dots, q_{d+1} \in \mathbb{R}[\underline{x}] : \mathcal{W} = \mathcal{W}(q_1, \dots, q_{d+1}) \tag{1.23}$$

THEOREM 1.20: ([**Av08**], Theorem 1.2.)
Let the assumptions from THEOREM 1.19 hold. If one additionally assumes that

$$|\{a \in S : |i \in \{1, \dots, s\} : p_i(a) = 0| = d\}| < \infty, \tag{1.24}$$

it even holds that

$$m(\mathcal{W}) \leq d, \tag{1.25}$$

or equivalentely

$$\exists q_1, \dots, q_d \in \mathbb{R}[\underline{x}] : \mathcal{W} = \mathcal{W}(q_1, \dots, q_d). \tag{1.26}$$

REMARK 1.21: If one takes a close look at the above theorems, it turns out that the basic closed semialgebraic set $\mathcal{W}$ does not have to be convex. Remember that the convexity of the set was an important assumption in *Bernig's* proof since it guaranteed that the set $\mathcal{W}(p_2)$ defined by the strictly concave polynomial p_2 indeed contains the polytope. As mentioned before, the idea of *Averkov's* proof is similar to *Bernig's* idea, namely cutting off the surplus area arisen from the basic closed set defined by *elementary-symmetric functions*. Nevertheless, this cannot be managed by just constructing a strictly concave polynomial, since with this approach one would need the convexity of $\mathcal{W}$. Hence *Averkov* constructed a different polynomial p with the property that $\mathcal{W}(p)$ approximates $\mathcal{W}$ sufficiently well.

REMARK 1.22: With the statement of the above theorem one is easily able to show that each simple polytope in $\mathbb{R}^n$ can be defined by n non-strict polynomial inequalities. This fact will be demonstrated at the end of the present section.

The next aim of the present work is to prove THEOREM 1.19 and THEOREM 1.20. To be able to understand the idea of the proof, one has to introduce the concept of *elementary-symmetric functions*:

DEFINITION 1.23: *(Elementary-Symmetric Functions)*
Let $y_1, \dots, y_s \in \mathbb{R}$. *For* $k \in \{1, \dots, s\}$

$$\sigma_k(y_1, \dots, y_s) := \sum_{\substack{I \subseteq \{1,\dots,s\}, \\ |I|=k}} \prod_{i \in I} y_i \tag{1.27}$$

is called the *k-th elementary-symmetric function* *in the variables* $y_1, \dots, y_s$.

The most important property of these functions concerning the proof of the above theorems is the following proposition, which was shown by *Bernig* in [**Be97**] and afterwards proven by *Averkov* using a slightly different idea:

PROPOSITION 1.24: For $y_1, \dots, y_s \in \mathbb{R}$ the following are equivalent:

(i) $y_1 \geq 0, \dots, y_s \geq 0$
(ii) $\sigma_1(y_1, \dots, y_s) \geq 0, \dots, \sigma_s(y_1, \dots, y_s) \geq 0$

The statement is also true if one replaces "$\geq$" by "$>$".

PROOF([**Av08**], P. 11):
Note that the inclusion $(i) \Rightarrow (ii)$ trivially holds. Hence one has to prove the other direction.
$(ii) \Rightarrow (i)$: Let $f := (t + y_1) \cdot \ldots \cdot (t + y_s) \in \mathbb{R}[t]$. Expanding the polynomial leads

to the following form:

$$\begin{aligned} f &= (t+y_1)\cdot\ldots\cdot(t+y_s) = t^s + (y_1+\ldots+y_s)t^{s-1} \\ &+ \left(\sum_{i<j} y_i y_j\right) t^{s-2} + \ldots + \left(\sum_{\substack{I\subseteq\{1,\ldots,s\},\\ |I|=s-1}} \prod_{i\in I} y_i\right) t^1 + \left(\prod_{i=1}^{s} y_i\right) t^0 \\ &= t^s + \sum_{i=1}^{s} \sigma_i(y_1,\ldots,y_s)t^{s-i} \end{aligned} \tag{1.28}$$

To be able to finish the proof of the theorem, let us state the following lemma:

LEMMA 1.25: Let

$$p = c_0 + c_1 t + \ldots + c_{k-1}t^{k-1} + t^k = (t-\alpha_1)\cdot\ldots\cdot(t-\alpha_s) \in \mathbb{R}[t]$$

be a monic polynomial with <u>real</u> roots $\alpha_1,\ldots,\alpha_s \in \mathbb{R}$. Then the following holds:

$$\alpha_1,\ldots,\alpha_s \leq 0 \Leftrightarrow \forall i \in \{0,\ldots,k-1\} : c_i \geq 0$$

Moreover,

$$\alpha_1,\ldots,\alpha_s \geq 0 \Leftrightarrow \forall i \in \{0,\ldots,k-1\} : (-1)^{k-i}c_i \geq 0,$$

which means that p has alternating coefficients.

PROOF: This is a well known fact and can easily be proven. ∎

By assumption, it holds that $\forall i \in \{1,\ldots,s\} : \sigma_i(y_1,\ldots,y_s) \geq 0$. Hence the coefficients of the polynomial f are greater or equal than 0. Therefore, using the results from LEMMA 1.25, it follows that

$$-y_1,\ldots,-y_s \leq 0 \Leftrightarrow y_1,\ldots,y_s \geq 0. \tag{1.29}$$

Moreover, assume that for $i \in \{1,\ldots,s\}$ the strict inequalities $\sigma_i(y_1,\ldots,y_s) > 0$ hold. By (1.28) it can be seen that

$$f(0) = \sigma_s(y_1,\ldots y_s) > 0.$$

Hence 0 is not contained in the zero set of f. This shows that $y_1,\ldots,y_s \neq 0$ and by (1.29) it follows that $y_1,\ldots,y_s > 0$. ∎

Before being able to prove THEOREM 1.19 and THEOREM 1.20, one needs some important approximation results, which will be stated in the next subsection.

4.1. Some Approximation Results.

ASSUMPTION 1.26: During the whole subsection let $p_1, \ldots, p_s \in \mathbb{R}[\underline{x}]$ be polynomials defining the non-empty and bounded set $\mathcal{W} := \mathcal{W}(p_1, \ldots, p_s)$.

REMARK 1.27: *Averkov's* preparations for the proofs of THEOREM 1.19 and THEOREM 1.20 can be divided into two main parts. In a first step he defined a set which depends on some $\epsilon > 0$, contains the original set $\mathcal{W}$ and approximates $\mathcal{W}$ sufficiently well for ϵ small enough. This set will later be called $\mathcal{W}_{M,\epsilon}$. At that moment it is not clear what this notation means in detail, but this neglicence will be caught up below. Afterwards, *Averkov* constructed two polynomials $g \in \mathbb{R}[\underline{x}]$ and $q \in \mathbb{R}[\underline{x}]$ such that $\mathcal{W} \subseteq \mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon}$ respectively $\mathcal{W} \subseteq \mathcal{W}(q) \subseteq \mathcal{W}_{M,2\epsilon}$ for some given $\epsilon > 0$. Once having accomplished this, one can prove THEOREM 1.19 and THEOREM 1.20 using the following idea: Since $\mathcal{W}_{M,\epsilon}$ approximates $\mathcal{W}$ sufficiently well, this is also true for $\mathcal{W}(g)$ just because of the fact that $\mathcal{W}(g)$ is pinched in between the two other sets. Hence g will be the sought-after polynomial, which – together with the *elementary-symmetric functions* – will provide the basic closed description of $\mathcal{W}$ in THEOREM 1.19. The same holds true for the polynomial q. It will be used for the proof of THEOREM 1.20.

PROOF-INGREDIENT 1 (CONSTRUCTION OF $\mathcal{W}_{M,\epsilon}$):
To be able to state some approximation results, first of all one has to define what *approximating* a set means in detail:

DEFINITION 1.28: *Let $A, B \subseteq \mathbb{R}^n$, $A \neq \emptyset \neq B$, be two compact sets. The number*

$$d_H(A,B) := \max \left\{ \max_{a \in A} \underbrace{\min_{b \in B} \|a-b\|}_{=d(a,B)}, \max_{b \in B} \underbrace{\min_{a \in A} \|a-b\|}_{=d(b,A)} \right\} \in \mathbb{R}$$

is called Hausdorff-distance between the two sets. Note that $\|\cdot\|$ denotes the Euclidian norm and $d(x,C)$ the Euclidian distance between a point x and a set C.

REMARK 1.29: Due to the compactness of the sets A and B it makes sense to define the *Hausdorff-distance* as above. Both sets are bounded and, therefore, the maxima and minima from the above definition are well defined.

DEFINITION 1.30: *Let $(A_n)_{n\in\mathbb{N}}$ and A be non-empty compact sets in $\mathbb{R}^n$. $(A_n)_{n\in\mathbb{N}}$ is called Hausdorff-convergent to A*

$$:\Leftrightarrow (A_n)_{n\in\mathbb{N}} \xrightarrow{H} A :\Leftrightarrow d_H(A_n, A) \xrightarrow{n\to\infty} 0,$$

where the latter convergence is just the usual convergence in $\mathbb{R}$. Note that one can define the limit of an arbitrary family $(A_t)_{t>0} \xrightarrow{H} A$ satisfying the above assumptions in a similar way. But this is not necessary since in $\mathbb{R}$ one can always pass to sequences.

EXAMPLE 1.31: Consider the following triangles A and B.

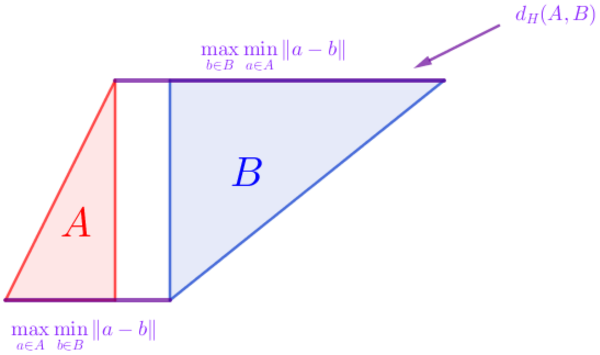


FIGURE 1.10. *Hausdorff-distance* between two sets

To be able to find the Hausdorff-distance between the two sets, in a first step one has to go through all elements $b \in B$ and compute the distance $\min_{a\in A} \|a-b\|$ between the point b and the set A. After doing this, one has to take the maximum of the arisen distances over all $b \in B$, which has been done in the above figure. Afterwards, one has to exchange the roles of the sets, namely passing through all the points $a \in A$, compute the distance between a and B and then taking the maximum of the distances over all $a \in A$, which was also done in FIGURE 1.10. The maximum over these two numbers is the sought-after *Hausdorff-distance*. ♦

With this knowledge one can state the first approximation result. Therefore, let $M \in \mathbb{N}$ and $\epsilon > 0$. Define

$$\mathcal{W}_{M,\epsilon} := \left\{a \in \mathbb{R}^n \middle| \; \forall i \in \{1,\ldots,s\} : (1+\|a\|^2)^M p_i(a) \geq -\epsilon\right\}. \tag{1.30}$$

THEOREM 1.32: ([Av08], Theorem 3.2)
Let $p_1, \dots, p_s \in \mathbb{R}[\underline{x}]$ be the polynomials from ASSUMPTION 1.26. There exists $M \in \mathbb{N}$ and $\epsilon_0 > 0$ such that

(i) $\mathcal{W}_{M,\epsilon_0}$ is bounded and
(ii) $\mathcal{W}_{M,\epsilon} \overset{\epsilon \to 0}{\longrightarrow} \mathcal{W}(p_1, \dots, p_s)$, where the convergence is given in the *Hausdorff-distance*.

REMARK 1.33: At first glance, the definition of the set (1.30) seems to be rather complicated. The main purpose of the set is approximating $\mathcal{W}$ sufficiently well if ϵ is small enough. Therefore, one could legitimately ask where the additional factor $(1 + \|a\|^2)^M$ comes from and why it is needed to achieve this aim. If one sets $M := 0$, *Averkov* was able to find an example such that the set $\mathcal{W}_{0,\epsilon}$ is unbounded for every $\epsilon > 0$ ([Av08], Remark 3.3.). This case must not happen since it implies that $d(\mathcal{W}, \mathcal{W}_{0,\epsilon})$ is not well-defined. To be able to prove part (ii) of the above theorem, one necessarily needs to ensure that the approximating set is bounded for some $\epsilon_0 > 0$. This is guaranteed by the additional factor $(1 + \|a\|^2)^M$.

To be able to prove the above theorem, one needs an important inequality from real algebraic geometry:

THEOREM 1.34: (*Łojasiewicz inequality*)
Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be a bounded and closed semialgebraic set. Moreover, let $f, g : A \to \mathbb{R}$ be continuous semialgebraic functions. Assume that the following holds: $\{x \in A : f(x) = 0\} \subseteq \{x \in A : g(x) = 0\}$. Then:

$$\exists M \in \mathbb{N}\ \exists \lambda \geq 0\ \forall x \in A : |g(x)|^M \leq \lambda |f(x)| \tag{1.31}$$

PROOF OF THEOREM 1.32 ([Av08], P. 5-6):
ad (i): By assumption, the set $\mathcal{W}$ is bounded and hence after applying an appropriate transformation one can assume that $\mathcal{W} \subseteq B_1(0)$, where $B_1(0) \subseteq \mathbb{R}^n$ denotes the open ball with midpoint 0 and radius 1. Define the function

$$f : \mathbb{R}^n \to \mathbb{R} : x \mapsto -\min_{1 \leq i \leq s} p_i(x).$$

f has an important property, which will will be needed below, namely:

$$\forall a \in \mathbb{R}^n : (\|a\| \geq 1 \Rightarrow f(a) > 0) \tag{1.32}$$

This is due to the assumption that $\mathcal{W} \subseteq B_1(0)$. Hence $\|a\| \geq 1$ implies that $a \notin \mathcal{W}$, which means that $\exists j \in \{1, \ldots, s\} : p_j(a) < 0$. This immediately implies that $\min_{1\leq i\leq s} p_i(a) < 0$ and, therefore,

$$f(a) = -\underbrace{\min_{1\leq i\leq s} p_i(a)}_{<0} > 0.$$

Moreover, with the help of f one can rewrite (1.30) into

$$\begin{aligned} \mathcal{W}_{M,\epsilon} &= \left\{a \in \mathbb{R}^n : \ \forall i \in \{1, \ldots, s\} : (1 + \|a\|^2)^M p_i(a) \geq -\epsilon\right\} \\ &= \left\{a \in \mathbb{R}^n : (1 + \|a\|^2)^M \min_{1\leq i\leq s} p_i(a) \geq -\epsilon\right\} \\ &= \left\{a \in \mathbb{R}^n : (1 + \|a\|^2)^M \left(-\min_{1\leq i\leq s} p_i(a)\right) \leq \epsilon\right\} \\ &= \left\{a \in \mathbb{R}^n : (1 + \|a\|^2)^M f(a) \leq \epsilon\right\}. \end{aligned} \tag{1.33}$$

In a next step, *Averkov* defined a function

$$\begin{aligned} a : [1, \infty) &\to \mathbb{R} \\ t &\mapsto \min\{f(c) : 1 \leq \|c\| \leq t\}. \end{aligned} \tag{1.34}$$

By (1.32), $a(t) > 0$ for every $t \geq 1$. Moreover, the function is decreasing since with increasing t there are more possible values for the minimum of $f(c)$. Let us state two more characteristic properties of a:

Property 1: a is *continuous*.

At first, let us notice that the function f is continuous since the polynomials $p_1, \ldots, p_s \in \mathbb{R}[\underline{x}]$ (viewed as functions $p_i : \mathbb{R}^n \to \mathbb{R}$) are continuous. The minimum of a finite set of continuous functions remains continuous. Assume that there exists $t \in [1, \infty)$ such that a is discontinuous at t. Since a is a decreasing function, the point of discontinuity fulfills one of the following two properties:

$$\lim_{s\to t^-} a(s) > a(t) \text{ for } t \in (1, \infty) \quad \text{or} \quad \lim_{s\to t^+} a(s) < a(t) \text{ for } t \in [1, \infty)$$

Let us consider the first case: For all $s \in [1, t)$ it holds that

$$a(s) \geq \lim_{s\to t^-} a(s) > a(t)$$

and hence

$$\forall c \in \mathbb{R}^n : (1 \leq \|c\| < t \Rightarrow f(c) \geq a(\|c\|) > a(t) = \min\{f(a) : 1 \leq \|a\| \leq t\})$$

The latter implies that there exists $b \in \mathbb{R}^n$ with $\|b\| = t$ such that $a(t) = f(b)$. Define $\epsilon := \lim_{s \to t^-} a(s) - a(t)$. Then:

$$\forall c \in \mathbb{R}^n : (1 \leq \|c\| < \|b\| \Rightarrow |f(c) - f(b)| \geq \epsilon),$$

which contradicts the continuity of f at b. Indeed, for every $\delta > 0$ there exists $c \in B_\delta(b) \cap [1, \infty)$ such that $|f(c) - f(b)| \geq \epsilon$. The second case works similar.

Observation 2: a is *semialgebraic*.

Let $p := x_1^2 + \ldots + x_n^2 \in \mathbb{R}[\underline{x}] \subseteq \mathbb{R}[t, s, \underline{x}]$. If one is able to represent $\Gamma(a)$ like in Proposition 0.3, it turns out that $\Gamma(a)$ is semialgebraic. We have

$$\Gamma(a) = \{(t, s) \in \mathbb{R}^2 : 1 \leq t < \infty \ \wedge \ s = \min\{f(a) : 1 \leq \|a\| \leq t\}\}.$$

Let us separately take a look at the above conditions:

(i) $1 \leq t < \infty$ corresponds to the following prime formula: $t - 1 \geq 0$ for $t - 1 \in \mathbb{R}[t] \subseteq \mathbb{R}[t, s, \underline{x}]$

(ii) $1 \leq \|x\| \leq t$ can be rewritten into $1 \leq x_1^2 + \ldots + x_n^2 \leq t^2$ and described with

$$\underbrace{p(\underline{x}) - 1}_{\in \mathbb{R}[t,s,\underline{x}]} \geq 0 \wedge \underbrace{t^2 - p(\underline{x})}_{\in \mathbb{R}[t,s,\underline{x}]} \geq 0.$$

With the help of the above formulas define the following $\mathbb{R}$-formulas:

$$\phi_1 : t - 1 \geq 0$$

$$\phi_2 : \exists x_1 \ldots \exists x_n : p(\underline{x}) - 1 \geq 0 \wedge t^2 - p(\underline{x}) \geq 0 \wedge \underbrace{f(\underline{x}) = s}_{(\star)}$$

$$\phi_3 : \forall x_1 \ldots \forall x_n : (p(\underline{x}) - 1 \geq 0 \wedge t^2 - p(\underline{x}) \geq 0 \to \underbrace{f(\underline{x}) \geq s}_{(\star)})$$

One has to emphasize that the expressions "$f(\underline{x}) = s$" and "$f(\underline{x}) \geq s$" in $(\star)$ are permitted in a first order formula since f is not a polynomial. But one can rewrite the expressions with the help of new polynomials $\overline{p_i}(\underline{x}, s) = p_i(\underline{x}) + s \in \mathbb{R}[\underline{x}, s] \subseteq$

$\mathbb{R}[t, s, \underline{x}]$:

$$f(\underline{x}) = s \Leftrightarrow -\min_{1\le i\le s} p_i(\underline{x}) = s \Leftrightarrow \min_{1\le i\le s} p_i(\underline{x}) + s = 0 \Leftrightarrow \min_{1\le i\le s} \overline{p_i}(\underline{x}, s) = 0$$
$$\Leftrightarrow (\overline{p_1}(\underline{x}, s) \ge 0 \wedge \ldots \wedge \overline{p_s}(\underline{x}, s) \ge 0) \wedge (\overline{p_1}(\underline{x}, s) = 0 \vee \ldots \vee \overline{p_s}(\underline{x}, s) = 0)$$

The new expression is a valid component in a first order formula. A similar argument can be applied for "$f(\underline{x}) \ge s$". Hence if one replaces "$f(\underline{x}) = s$" (respectively "$f(\underline{x}) \ge s$") by the above expression, $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$ turns out to be a first order formula. With the help of this formula one can write

$$\begin{aligned}\Gamma(a) &= \{(t, a(t)) : 1 \le t < \infty\}\\ &= \{(t, s) \in \mathbb{R}^2 : 1 \le t < \infty \ \wedge \ s = \min\{f(a) : 1 \le \|a\| \le t\}\}\\ &= \{(t, s) \in \mathbb{R}^2 : \phi(t, s) \text{ is true in } \mathbb{R}^2\}.\end{aligned}$$

Hence $\Gamma(a)$ is a semialgebraic set.

Since a is decreasing and $a(t) > 0$ for $t \ge 1$, one has two possibilities for the behavior of $\inf\{a(t) : t \ge 1\}$:

Case 1: $\inf\{a(t) : t \ge 1\} > 0$: Set $M := 0$ and $\epsilon_0 := \frac{1}{2}\inf\{a(t) : t \ge 1\}$. Now choose $c \in \mathcal{W}_{0,\epsilon_0}$.

Assumption: $\|c\| \ge 1$: Then:

$$a(\|c\|) = \min\{f(y) : 1 \le \|y\| \le \|c\|\} \le f(c) \overset{c\in\mathcal{W}_{0,\epsilon_0}}{\le} \epsilon_0 \tag{1.35}$$

Since the infimum of a set is its greatest lower bound, one has

$$2\epsilon_0 = \inf\{a(t) : t \ge 1\} \overset{\|c\|\ge 1}{\le} a(\|c\|) \tag{1.36}$$

Putting (1.35) and (1.36) together, one gets

$$2\epsilon_0 \le a(\|c\|) \le \epsilon_0. \quad \lightning$$

Hence $\forall c \in \mathcal{W}_{0,\epsilon_0} : \|c\| < 1$, which implies the boundedness of the set.

Case 2: $a(t) \overset{t\to\infty}{\longrightarrow} 0$. Then one can define

$$\begin{aligned} b : [0,1] &\to \mathbb{R} \\ t \mapsto b(t) &:= \begin{cases} a\left(\frac{1}{t}\right), & 0 < t \leq 1 \\ 0, & t = 0 \end{cases} \end{aligned}$$

Note that a is a continuous function. Therefore, b is also continuous due to the fact that $\lim_{t\to 0+} a\left(\frac{1}{t}\right) = \lim_{s\to\infty} a(s) = 0$ by case 2. Moreover, a is semialgebraic and hence also b. To see this, note that

$$\begin{aligned} c : [0,1] &\to \mathbb{R} \\ t &\mapsto \begin{cases} \frac{1}{t}, & 0 < t \leq 1 \\ 0, & t = 0 \end{cases} \end{aligned}$$

is semialgebraic. Indeed

$$\begin{aligned} \Gamma(c) &= \left\{(t,s) \in \mathbb{R}^2 : 0 < t \leq 1, s = \frac{1}{t}\right\} \cup \{(0,0)\} \\ &= \left\{(t,s) \in \mathbb{R}^2 : t > 0, 1 - t \geq 0, 1 - st = 0\right\} \cup \{(t,s) \in \mathbb{R}^2 : t = 0, s = 0\}, \end{aligned}$$

which is a semialgebraic set. $b = a \circ c$ is semialgebraic as a composition of semialgebraic functions. Note that

$$b(t) = 0 \Leftrightarrow \left(t = 0 \quad \vee \quad \left(0 < t \leq 1 \ \wedge \ a\left(\frac{1}{t}\right) = 0\right)\right).$$

$0 < t \leq 1$ implies $\frac{1}{t} \geq 1$ and, therefore, $a\left(\frac{1}{t}\right) > 0$. This shows that

$$\{0\} = \{b = 0\} \cap [0,1] = \{t = 0\} \cap [0,1] = \{t^2 = 0\} \cap [0,1].$$

Hence one can use the *Łojasiewicz-inequality*:

$$\exists M \in \mathbb{N}, \exists \lambda \geq 0 : t^{2M} \leq \lambda b(t) \quad \forall t \in [0,1] \tag{1.37}$$

The inequality is not true for $\lambda = 0$, hence $\lambda > 0$. After transforming the inequality for $t \in (0,1]$, one gets

$$t^{2M} \leq \lambda b(t) \overset{t>0}{\Rightarrow} \frac{1}{\lambda} \leq \left(\frac{1}{t}\right)^{2M} b(t) = \left(\frac{1}{t}\right)^{2M} a\left(\frac{1}{t}\right).$$

Substituting $s = \frac{1}{t}, s \geq 1,$ leads to the inequality

$$\frac{1}{\lambda} \leq s^{2M} a(s). \tag{1.38}$$

Now set $s := \sqrt{1 + \|c\|^2} \geq 1$ and assume that $\|c\| \geq 1$. Applying (1.38) leads to

$$\begin{aligned} \frac{1}{\lambda} &\leq (1 + \|c\|^2)^M a\left(\sqrt{1 + \|c\|^2}\right) = \\ &= (1 + \|c\|^2)^M \min\left\{f(y) : 1 \leq \|y\| \leq \sqrt{1 + \|c\|^2}\right\} \\ &\overset{(\star)}{\leq} (1 + \|c\|^2)^M f(c) \end{aligned} \tag{1.39}$$

$(\star)$ holds, since $1 \leq \|c\| \leq \sqrt{1 + \|c\|^2}$. Therefore, $f(c) \in \left\{f(y) : 1 \leq \|y\| \leq \sqrt{1 + \|c\|^2}\right\}$. Clearly, $f(c)$ is greater or equal than the minimum of the set.

Now choose M and λ like in (1.37) and set $\epsilon_0 := \frac{1}{2\lambda}$. Choose $c \in \mathcal{W}_{M,\epsilon_0}$.

Assumption: $\|c\| \geq 1$: (1.39) together with the fact that $(1 + \|c\|^2)^M f(c) \leq \epsilon_0$ causes a contradiction:

$$\frac{1}{\lambda} \leq (1 + \|c\|^2)^M f(c) \overset{c \in \mathcal{W}_{M,\epsilon_0}}{\leq} \frac{1}{2\lambda} \quad \lightning$$

Again, one gets $\forall c \in \mathcal{W}_{M,\epsilon_0} : \|c\| < 1$ and hence $\mathcal{W}_{M,\epsilon_0}$ is bounded.

ad (ii): It follows from part (i) of the proof that $W_{M,\epsilon}$ is bounded for $\epsilon \in (0, \epsilon_0]$. The boundedness of these sets implies that $d_H(\mathcal{W}, \mathcal{W}_{M,\epsilon})$ is well-defined. Since one has to compute a limit in $\mathbb{R}$, let us pass to sequences. Let $(c_n)_{n \in \mathbb{N}} \in (0, \epsilon_0]^{\mathbb{N}}$ be a sequence such that $c_n \overset{n\to\infty}{\longrightarrow} 0$. For better readability, let us write $\mathcal{W}_n := \mathcal{W}_{M,c_n}$. One has to show:

$$d_H(\mathcal{W}_n, \mathcal{W}) \overset{n\to\infty}{\longrightarrow} 0$$

Recall that $d_H(\mathcal{W}_n, \mathcal{W}) := \max\left\{\max_{a_n \in \mathcal{W}_n} d(a_n, \mathcal{W}), \max_{b \in \mathcal{W}} d(b, \mathcal{W}_n)\right\}$. Consider the two cases separately:

(i) $\max_{b \in \mathcal{W}} d(b, \mathcal{W}_n) \to 0$

(ii) $\max_{a_n \in \mathcal{W}_n} d(a_n, \mathcal{W}) \to 0$

ad (i): Note that this case is trivial. Let $b \in \mathcal{W}$. Then it holds that

$$\forall i \in \{1, \ldots, s\} : p_i(b) \geq 0 \Rightarrow \forall i \in \{1, \ldots, s\} : \underbrace{(1 + \|b\|^2)^M}_{\geq 0} p_i(b) \geq 0 \geq -c_n$$

Therefore, $b \in \mathcal{W}_n$ and hence $d(b, \mathcal{W}_n) = 0$. Since $b \in \mathcal{W}$ was arbitrary, $\max_{b \in \mathcal{W}} d(b, \mathcal{W}_n) = 0 \to 0$.

ad (ii): Note that

$$\begin{aligned}\max_{a_n \in \mathcal{W}_n} d(a_n, \mathcal{W}) \to 0 \Leftrightarrow \forall \epsilon > 0 \exists k \in \mathbb{N} : \forall n \geq k : \max_{a_n \in \mathcal{W}_n} d(a_n, \mathcal{W}) < \epsilon \\ \Leftrightarrow \forall \epsilon > 0 \exists k \in \mathbb{N} : \forall n \geq k : \forall a_n \in \mathcal{W}_n : d(a_n, \mathcal{W}) < \epsilon\end{aligned}$$

Assume that the converse is true, namely $\max_{a_n \in \mathcal{W}_n} d(a_n, W) \nrightarrow 0$. Hence

$$\begin{aligned}&\exists \epsilon > 0 : \forall k \in \mathbb{N}\ \exists n_k \geq k : \exists a_{n_k} \in \mathcal{W}_n : d(a_{n_k}, \mathcal{W}) > \epsilon \\ \Leftrightarrow\ &\exists \epsilon > 0\ \exists (a_{n_k})_{k \in \mathbb{N}} \in \mathcal{W}_{n_k}^{\mathbb{N}} : d(a_{n_k}, \mathcal{W}) > \epsilon,\end{aligned} \tag{1.40}$$

where $(n_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers. Note that $\bigcup_{k \in \mathbb{N}} \mathcal{W}_{n_k}$ is bounded by part (i) of the proof. Hence $(a_{n_k})_{k \in \mathbb{N}}$ is a bounded sequence and therefore, using the *Bolzano-Weierstrass theorem*, one can extract a further subsequence $(a_{n_{k_l}})_{l \in \mathbb{N}}$ converging to some $a \in \mathbb{R}^n$. In other words:

$$\exists N \in \mathbb{N} \forall l \geq N : \|a_{n_{k_l}} - a\| < \epsilon. \tag{1.41}$$

To finish the proof, one has to show that $a \in \mathcal{W}$. Clearly,

$$(1 + \|a_{n_{k_l}}\|^2)^M p_i(a_{n_{k_l}}) \geq -c_{n_{k_l}}.$$

Since $c_{n_{k_l}} \to 0$ for $l \to \infty$, taking the limit on both sides yields

$$(1 + \|a\|^2)^M p_i(a) \geq 0$$

and hence $a \in \mathcal{W}$. Now choose N like in (1.41). Then

$$\forall l \geq N : d(a_{n_{k_l}}, \mathcal{W}) = \min_{y \in \mathcal{W}} \|a_{n_{k_l}} - y\| \overset{a \in \mathcal{W}}{\leq} \|a_{n_{k_l}} - a\| \overset{(1.41)}{<} \epsilon,$$

which contradicts (1.40). ∎

PROOF-INGREDIENT 2 (CONSTRUCTION OF $g, q \in \mathbb{R}[\underline{x}]$ FROM REMARK 1.27:)

LEMMA 1.35: ([Av08], Lemma 3.4)
Let $\epsilon > 0, M \in \mathbb{N}_0$ and let $p_1, \ldots, p_s$ be the polynomials from ASSUMPTION 1.26. Moreover, choose $\gamma > 0$ and $k \in \mathbb{N}$ such that the following is satisfied:

$$\gamma \geq \max_{1 \leq i \leq s} \max_{a \in \mathcal{W}} (1 + \|a\|^2)^M p_i(a) \tag{1.42}$$

and

$$s \leq \left(1 + \frac{\epsilon}{\gamma}\right)^{2k} \tag{1.43}$$

Define the polynomial

$$g := 1 - \frac{1}{s} \sum_{i=1}^{s} \left(1 - \frac{1}{\gamma}(1 + \|a\|^2)^M p_i(a)\right)^{2k} \tag{1.44}$$

Then the following holds:

$$\mathcal{W} \subseteq \mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon} \tag{1.45}$$

PROOF:
The inclusion $\mathcal{W} \subseteq \mathcal{W}(g)$ can easily be shown: Let $a \in \mathcal{W}$. By (1.42) and the fact that $p_i(a) \geq 0$ for $i \in \{1, \ldots, s\}$:

$$\forall i \in \{1, \ldots, s\} : 0 \leq (1 + \|a\|^2)^M p_i(a) \leq \gamma$$

Therefore, $0 \leq \frac{1}{\gamma}(1 + \|a\|^2)^M p_i(a) \leq 1$ and hence $1 - \frac{1}{\gamma}(1 + \|a\|^2)^M p_i(a) \in [0, 1]$. This shows

$$\begin{aligned} g(a) &= 1 - \frac{1}{s} \sum_{i=1}^{s} \underbrace{\left(1 - \frac{1}{\gamma}(1 + \|a\|^2)^M p_i(a)\right)^{2k}}_{\in [0,1]} \\ &\geq 1 - \frac{1}{s} \sum_{i=1}^{s} 1 = 1 - 1 = 0. \end{aligned}$$

It remains to show $\mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon}$. Let $a \in \mathcal{W}(g)$:

$$\begin{aligned}
g(a) \geq 0 \Rightarrow \sum_{i=1}^{s} \left(1 - \frac{1}{\gamma}(1+\|a\|^2)^M p_i(a)\right)^{2k} &\leq s \\
\Rightarrow \max_{1\leq i\leq s} \left(1 - \frac{1}{\gamma}(1+\|a\|^2)^M p_i(a)\right)^{2k} &\leq s \overset{(1.43)}{\leq} \left(1+\frac{\epsilon}{\gamma}\right)^{2k} \\
\Rightarrow \underbrace{\max_{1\leq i\leq s} \left(1 - \frac{1}{\gamma}(1+\|a\|^2)^M p_i(a)\right)}_{=1+\frac{1}{\gamma}(1+\|a\|^2)^M \underbrace{\left(-\min_{1\leq i\leq s} p_i(a)\right)}_{=f(a)}} &\leq 1 + \frac{\epsilon}{\gamma} \\
\Rightarrow 1 + \frac{1}{\gamma}(1+\|a\|^2)^M f(a) &\leq 1 + \frac{1}{\gamma}\epsilon \\
\Rightarrow (1+\|a\|^2)^M f(a) \leq \epsilon &\overset{(1.33)}{\Rightarrow} a \in \mathcal{W}_{M,\epsilon}
\end{aligned}$$

■

REMARK 1.36: Note that one can find $\gamma > 0$ such that (1.42) is satisfied. This is just due to the fact that $\mathcal{W}$ is bounded and hence $\max_{a\in\mathcal{W}}(1+\|a\|^2)^M p_i(a)$ will be indeed achieved. Moreover, for given $s \in \mathbb{N}$ the condition in (1.43) is always satisfied if one chooses $k \in \mathbb{N}$ big enough. Hence the conditions (1.42) and (1.43) are no restrictions on the given basic closed semialgebraic set.

THEOREM 1.37: ([**Av08**], Theorem 3.5)
Let $p_1, \ldots, p_s \in \mathbb{R}[\underline{x}]$ be the polynomials from ASSUMPTION 1.26 and $\mathcal{W} = \mathcal{W}(p_1, \ldots, p_s)$ be the corresponding bounded basic closed semialgebraic set. Moreover, assume

$$d := \max\{|i \in \{1, \ldots, s\} : p_i(a) = 0| : a \in \mathcal{W}\} < s$$

and

$$A := |\{a \in \mathcal{W} : |i \in \{1, \ldots, s\} : p_i(a) = 0| = d\}| < \infty.$$

Let $M \in \mathbb{N}_0$ and $\epsilon > 0$. Then there exists a polynomial $q \in \mathbb{R}[\underline{x}]$ such that

$$\begin{aligned}
&\mathcal{W} \subseteq \mathcal{W}(q) \subseteq \mathcal{W}_{M,2\epsilon}, \\
&A \subseteq \mathcal{V}(q).
\end{aligned} \tag{1.46}$$

The polynomial is given by

$$q = \sigma_{s-d+1}(p_1, \dots, p_s) - \left(\frac{1}{s} \sum_{i=1}^{s} \left(1 - \frac{1}{\gamma}(1 + \|x\|^2)^M p_i \right)^{2k} \right)^l \left(\prod_{v \in A} \left(\frac{\|x - v\|}{\mu} \right)^2 \right)^m \tag{1.47}$$

for appropriate $k, l, m \in \mathbb{N}$, $\lambda > 0, \mu > 0$.

PROOF: Define

$$q_1 := p^l := \left(\underbrace{\frac{1}{s} \sum_{i=1}^{s} \left(1 - \frac{1}{\gamma}(1 + \|x\|^2)^M p_i \right)^{2k}}_{=p} \right)^l \tag{1.48}$$

and

$$q_2 := \left(\prod_{v \in A} \left(\frac{\|x - v\|}{\mu} \right)^2 \right)^m \tag{1.49}$$

$\underline{\mathcal{W} \subseteq \mathcal{W}(q)}$: Let $a \in \mathcal{W}$. This just means that $p_i(a) \geq 0$ for $i \in \{1, \dots, s\}$, which immediately implies $\sigma_{s-d+1}(p_1(a), \dots, p_s(a)) \geq 0$. If one chooses γ and k like in (1.42) and (1.43) (and that's what *Averkov* did), it holds that $p(a) \in [0, 1]$ for p defined in (1.48) (compare with the proof of LEMMA 1.35). It even holds that $0 \leq p(a) < 1$ since

$$\frac{1}{\gamma}(1 + \|a\|^2)^M p_i(a) = 0 \Leftrightarrow p_i(a) = 0.$$

Note that $p(a) = 1$ if and only if $\forall i \in \{1, \dots, s\} : p_i(a) = 0$. In this case $p(a)$ simplifies to $\frac{1}{s} \sum_{i=1}^{s} 1^{2k} = 1$. But this must not happen since $d < s$. Hence for every point in $\mathcal{W}$ there exists a polynomial which does not vanish at this point. This means that there always exists a summand which is less than 1 and hence $p(a) < 1$. Therefore,

$$\max_{a \in P} p(a) \leq \alpha < 1 \tag{1.50}$$

for appropriate $\alpha > 0$. This implies that $\forall a \in \mathcal{W} \exists \delta_a > 0 : p|_{B_{\delta_a}(a)} < 1$. Choose

$$\delta_1 := \frac{\min\{\|v-w\| : v, w \in A\}}{3}$$
$$\delta := \min\{\delta_1, \delta_{v_1}, \dots, \delta_{v_m}\}$$

with $A = \{v_1, \dots, v_m\}$. The definition of δ_1 and δ is reasonable since $|A| < \infty$. Then it holds that

$$\bigcup_{v \in A} \overline{B}_\delta(v) \subseteq \{a \in \mathbb{R}^n : p(a) \leq 1\} \quad \text{and} \quad \forall v, w \in A : \overline{B}_\delta(v) \cap \overline{B}_\delta(w) = \emptyset.$$

Now one has to consider the following two cases:

<u>Case 1:</u> Let $a \in \mathcal{W} \cap \left(\bigcup_{v \in A} \overline{B}_\delta(v)\right)$. Choose $w \in A$ with $\|a - w\| \leq \delta$. Moreover, choose

$$\mu \geq \max\{\|a - b\| : a, b \in \mathcal{W}\}.$$

The idea is to apply the *Łojasiewicz inequality* to the following two semialgebraic functions:

<u>Function 1:</u> $\sigma_{s-d+1}(p_1, \dots, p_s)$ restricted to $\overline{B}_\delta(w) \cap \mathcal{W}$. Note that

$$\sigma_{s-d+1}(p_1(a), \dots, p_s(a))|_{\overline{B}_\delta(w) \cap \mathcal{W}} = 0 \Leftrightarrow a = w.$$

This is due to the fact that w is the only point where exactly d polynomials vanish. Hence every summand of $\sigma_{s-d+1}(p_1, \dots, p_s)$ possesses at least one factor p_i which vanishes at w. This is not the case for $a \neq w$.

<u>Function 2:</u> $\left(\frac{\|x-w\|}{\mu}\right)^2$ restricted to $\overline{B}_\delta(w) \cap \mathcal{W}$. It also holds that

$$\left(\frac{\|w - w\|}{\mu}\right)^2 = 0.$$

Hence applying the *Łojasiewicz inequality* leads to

$$\exists m(w) \in \mathbb{N} \exists \lambda(w) \geq 0 : \left(\frac{\|a-w\|}{\mu}\right)^{2m(w)} \leq \lambda(w) \sigma_{s-d+1}(p_1(a), \dots, p_s(a))$$

If one defines $m := \max_{v\in A} m(v)$ and $\lambda := \max_{v\in A} \lambda(v)$, it holds that

$$\begin{aligned}\left(\frac{\|a-w\|}{\mu}\right)^{2m} \overset{(\star)}{\leq} \left(\frac{\|a-w\|}{\mu}\right)^{2m(w)} &\leq \underbrace{\lambda(w)}_{\leq\lambda}\sigma_{s-d+1}(p_1(a),\dots,p_s(a)) \\ &\leq \lambda\sigma_{s-d+1}(p_1(a),\dots,p_s(a))\end{aligned} \tag{1.51}$$

$(\star)$ holds since $\left(\frac{\|a-w\|}{\mu}\right) < 1$. Therefore,

$$\begin{aligned} q_1(a)q_2(a) = p(a)^l q_2(a) &\overset{(1.50)}{\leq} \alpha^l \left(\frac{\|a-w\|}{\mu}\right)^{2m} \underbrace{\prod_{v\in A\setminus\{w\}} \left(\frac{\|a-v\|}{\mu}\right)^{2m}}_{\leq 1 \text{ (by definition of } \mu)} \\ &\leq \alpha^l \left(\frac{\|a-w\|}{\mu}\right)^{2m} \overset{(1.51)}{\leq} \lambda\alpha^l \sigma_{s-d+1}(p_1(a),\dots,p_s(a)) = (\star)\end{aligned}$$

If one chooses $l \in \mathbb{N}$ big enough, one can always achieve $\lambda\alpha^l < 1$ and hence $(\star) \leq \sigma_{s-d+1}(p_1(a),\dots,p_s(a))$, which proves $q(a) \geq 0$.

Case 2: $a \in \mathcal{W}\setminus\left(\bigcup_{v\in A} \overline{B}_\delta(v)\right)$. If this was the case, one does not have to consider those points where exactly d polynomials vanish. Hence for $b \in \left(\mathcal{W}\setminus\bigcup_{v\in A} \overline{B}_\delta(v)\right)$ there exist at least $s-d+1$ polnomials which do not vanish on b. In other words:

$$\min\left\{\sigma_{s-d+1}(p_1(b),\dots,p_s(b)) : b \in \left(\mathcal{W}\setminus\bigcup_{v\in A} \overline{B}_\delta(v)\right)\right\} \geq \gamma > 0$$

for appropriate $\gamma > 0$. Therefore,

$$q_1(a)q_2(a) = p(a)^l q_2(a) \leq \alpha^l \underbrace{q_2(a)}_{\leq 1 \text{ (by definition of } \mu)} \leq \alpha^l \tag{1.52}$$

and

$$\gamma \leq \sigma_{s-d+1}(p_1(a),\dots,p_s(a)). \tag{1.53}$$

Again, if one chooses $l \in \mathbb{N}$ sufficiently large, $\alpha^l < \gamma$ and hence, by (1.52) and (1.53), it follows that $q_1(a)q_2(a) \leq \sigma_{s-d+1}(p_1(a),\dots,p_s(a))$, or equivalentely $q(a) \geq 0$.

$\underline{\mathcal{W}(q) \subseteq \mathcal{W}_{M,2\epsilon}}$: Assume $a \notin \mathcal{W}_{M,2\epsilon}$. In this case, using some appropriate estimations, *Averkov* was able to show that

$$|\sigma_{s-d+1}(p_1(a), \ldots, p_s(a))| \leq \frac{1}{2} q_1(a) q_2(a)$$

for suitable $l \in \mathbb{N}$ (his estimations are not complicated, but highly technical; see [**Av08**] for more details). Hence $q(a) < 0$.

$\underline{A \subseteq \mathcal{V}(q)}$: Note that for $a \in A$ it holds that $q_2(a) = 0$ and, therefore, $q(a)$ simplifies to

$$q(a) = \sigma_{s-d+1}(p_1(a), \ldots, p_s(a))$$

Every summand in $\sigma_{s-d+1}(p_1, \ldots, p_s)$ consists of a product of $s - d + 1$ different $p_i(a)$. $a \in A$ implies that exactly d polynomials vanish on a. Hence there exist only $s - d$ polynomials which do not vanish on a. This shows that every summand of $\sigma_{s-d+1}(p_1, \ldots, p_s)$ contains a factor $p_i(a)$ which equals 0. Therefore, $q(a) = 0$.

■

4.2. Proofs of THEOREM 1.19 and THEOREM 1.20. With all the preparations from above we are finally able to prove the main theorems of the chapter.

PROOF OF THEOREM 1.19, ([**Av08**], P.12):
Let us start with the main idea of the proof: In PROPOSITION 1.24 it was shown that *elementary-symmetric functions* possess a remarkable property. The proposition states that $y_1, \ldots, y_s \in \mathbb{R}$ are greater or equal than 0 if and only if the *elementary-symmetric functions* $\sigma_1, \ldots \sigma_s$ in the variables $y_1, \ldots, y_s$ have the same property. If one converts this statement to a basic closed semialgebraic set $\mathcal{W}(p_1, \ldots, p_s)$, it leads to another basic closed description of $\mathcal{W}$. For $a \in \mathbb{R}^n$ it holds that

$$\begin{aligned} &p_1(a) \geq 0, \ldots p_s(a) \geq 0 \\ &\qquad \Leftrightarrow \sigma_1(p_1(a), \ldots, p_s(a)) \geq 0, \ldots \sigma_s(p_1(a), \ldots, p_s(a)) \geq 0 \end{aligned} \tag{1.54}$$

This shows that one can also describe $\mathcal{W}$ by *elementary-symmetric functions*:

$$\mathcal{W} = \mathcal{W}(\sigma_1(p_1, \ldots, p_s), \ldots, \sigma_s(p_1, \ldots, p_s)) \tag{1.55}$$

As one can see, the representation (1.55) of $\mathcal{W}$ does not really provide benefits compared with the primary representation $\mathcal{W} = \mathcal{W}(p_1, \ldots, p_s)$. It is just another basic closed description of $\mathcal{W}$ with as many polynomials as before. To overcome this problem, *Averkov* used the following beautiful trick. Instead of looking at all *elementary-symmetric functions*, he just used some of them. With the chosen $\sigma_i(p_1, \dots, p_s)$ together with the polynomial g from above he defined a basic closed semialgebraic set $\mathcal{W}_{\text{new}}$ and verified $\mathcal{W} = \mathcal{W}_{\text{new}}$. To achieve this aim, he just showed that the remaining *elementary-symmetric functions*, which do not pertain to the basic closed description of $\mathcal{W}_{\text{new}}$, also fulfill property (1.54). To be able to do this, one needs some preparations.

Observation: We may assume $d < s$. If $d \geq s$, the statement of THEOREM 1.19 trivially holds. Note that d was the greatest possible number of polynomials which vanish at $a \in \mathcal{W}$. Hence $s - d$ is the smallest possible number of polynomials which do not vanish at $a \in \mathcal{W}$. Let $a \in \mathcal{W}$ and $p_1^a, \dots, p_{s-d}^a \in \{p_1, \dots, p_s\}$ such that $p_j^a(a) \neq 0$ for $j \in \{1, \dots, s-d\}$. Since $a \in \mathcal{W}$, it holds that $p_j^a(a) > 0$ and at least $p_i(a) \geq 0$ for $i \in \{1, \dots, s\}$. Let $1 \leq i \leq d - s$. Then:

$$\sigma_i(p_1(a), \dots, p_s(a)) = \underbrace{\sum_{\substack{I \subseteq \{1,\dots,s\} \\ |I| = i \\ I \not\subseteq \{p_1^a, \dots, p_{s-d}^a\}}} \prod_{i \in I} \underbrace{p_i(a)}_{\geq 0}}_{\geq 0} + \underbrace{\sum_{\substack{I \subseteq \{1,\dots,s\} \\ |I| = i \\ I \subseteq \{p_1^a, \dots, p_{s-d}^a\}}} \prod_{i \in I} \underbrace{p_i^a(a)}_{>0}}_{>0} > 0 \tag{1.56}$$

For better readability let us write $\sigma_i := \sigma_i(p_1, \dots, p_s)$ for $i \in \{1, \dots, s\}$. By (1.56) $\sigma_i(a) > 0$ for $i \in \{1, \dots, d - s\}$. Note that σ_i is continuous and therefore

$$\exists \delta_a > 0 \; \forall i \in \{1, \dots, s - d\} : \; \sigma_i|_{B_{\delta_a}(a)} > 0. \tag{1.57}$$

Let us cover the set $\mathcal{W}$ with the following open sets:

$$\mathcal{W} \subseteq \bigcup_{a \in \mathcal{W}} B_{\frac{\delta_a}{2}}(a) \tag{1.58}$$

Since $\mathcal{W}$ is bounded and (basic) closed by assumption, the set is compact. Hence (1.58) has a finite subcover:

$$\exists a_1, \dots, a_k \in \mathcal{W} : \mathcal{W} \subseteq \bigcup_{i=1}^{k} B_{\frac{\delta_{a_i}}{2}}(a_i) \tag{1.59}$$

Define $\delta := \min\{\delta_{a_1}, \dots, \delta_{a_k}\}$. Choose $M \in \mathbb{N}_0$ and $\epsilon_0 > 0$ such that $\mathcal{W}_{M,\epsilon_0}$ is bounded (compare with THEOREM 1.32). Moreover, choose $\epsilon \in (0, \epsilon_0]$ such that

$$d_H(\mathcal{W}, \mathcal{W}_{M,\epsilon}) < \frac{\delta}{2}. \tag{1.60}$$

This is also possible by THEOREM 1.32. Let $y \in \mathcal{W}_{M,\epsilon}$. Then, $\exists x \in \mathcal{W} : \|x-y\| < \frac{\delta}{2}$. By (1.59):

$$x \in B_{\frac{\delta_{a_i}}{2}}(a_i) \wedge \|x-y\| < \frac{\delta}{2} \tag{1.61}$$

Hence

$$\|y-a_i\| = \|y-x+x-a_i\| \leq \underbrace{\|y-x\|}_{\leq\frac{\delta}{2} \text{ by (1.61)}} + \underbrace{\|x-a_i\|}_{\leq\frac{\delta_{a_i}}{2} \text{ by (1.61)}} \leq \delta_{a_i} \tag{1.62}$$

The above computation shows that $y \in B_{\delta_{a_i}}(a_i)$. From (1.57) it immediately follows that $\sigma_i(y) > 0$ for $i \in \{1, \dots, s-d\}$. Since y was arbitrary,

$$\forall i \in \{1, \dots, d-s\} : \sigma_i|_{\mathcal{W}_{M,\epsilon}} > 0. \tag{1.63}$$

Now let us define the new set $\mathcal{W}_{\text{new}}$ with the following polynomials:

$q_i := \sigma_{s-d+i}(p_1, \dots, p_s)$ for $i = 1, \dots, d$

$q_{d+1} := g$ (from LEMMA 1.35 with $M \in \mathbb{N}_0$ and $\epsilon > 0$ chosen above such that (1.63) holds)

$\mathcal{W}_{\text{new}} = \mathcal{W}(q_1, \dots, q_{d+1})$

To finish the proof, one has to show that $\mathcal{W} = \mathcal{W}_{\text{new}}$ indeed holds.

$\underline{\mathcal{W} \subseteq \mathcal{W}_{\text{new}}}$: Let $a \in \mathcal{W}$. Then $p_i(a) \geq 0$ for every $i \in \{1, \dots, s\}$. Since $q_j(a)$, $j \in \{1, \dots, d\}$, is just defined as a sum of products of $p_i(a)$, $i \in \{1, \dots, s\}$, with $p_i(a) \geq 0$, it trivially holds that $q_i(a) \geq 0$ for all $i \in \{1, \dots, d\}$.

Moreover, $q_{d+1}(x) \geq 0$ due to the fact that $\mathcal{W} \subseteq \mathcal{W}(q_{d+1})$ (compare with LEMMA 1.35).

$\underline{\mathcal{W}_{\text{new}} \subseteq \mathcal{W}}$: Let $a \in \mathcal{W}_{\text{new}}$, thus

$$\sigma_{s-d+1}(p_1(a), \dots, p_s(a)) \geq 0, \dots, \sigma_s(p_1(a), \dots, p_s(a)) \geq 0, g(a) \geq 0 \tag{1.64}$$

If one is able to show

$$\sigma_1(p_1(a), \dots, p_s(a)) \geq 0, \dots, \sigma_{s-d}(p_1(a), \dots, p_s(a)) \geq 0, \tag{1.65}$$

it follows by (1.54) that $p_1(a) \geq 0, \dots, p_s(a) \geq 0$, or equivalentely $a \in \mathcal{W}$. In the above observation it was shown that $\sigma_i|_{\mathcal{W}_{M,\epsilon}} > 0$ (compare with (1.63)). By (1.35) $\mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon}$. Since $a \in \mathcal{W}_{\text{new}}$, $g(a) \geq 0$ and therefore

$$a \in \mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon}.$$

Hence $\mathcal{W}_{\text{new}} \subseteq \mathcal{W}_{M,\epsilon}$ and therefore $\sigma_i|_{\mathcal{W}_{\text{new}}} > 0$. More precisely,

$$\sigma_i(a) := \sigma_i(p_1(a), \dots, p_s(a)) > 0 \qquad \text{for } i \in \{1, \dots, d-s\}. \tag{1.66}$$

Altogether, from (1.64) and (1.66) it follows that

$$\sigma_i(p_1(a), \dots, p_s(a)) \geq 0 \qquad \forall i \in \{1, \dots, s\}$$

and by (1.54) this is equivalent to

$$p_1(a) \geq 0, \dots, p_s(a) \geq 0,$$

which shows $a \in \mathcal{W}$. ∎

PROOF OF THEOREM (1.20): The idea of this proof is similar. Choose $M \in \mathbb{N}_0$ and $\epsilon_0 > 0$ such that $\mathcal{W}_{M,\epsilon_0}$ is bounded. Moreover, with the same idea as before, choose $\epsilon \in (0, \frac{\epsilon_0}{2}]$ such that

$$\forall i \in \{1, \dots, s-d\} : \sigma_i(p_1, \dots, p_s)|_{\mathcal{W}_{M,2\epsilon}} > 0. \tag{1.67}$$

Let us again define a new basic closed semialgebraic set $\mathcal{W}_{\text{new}}$ and verify $\mathcal{W} = \mathcal{W}_{\text{new}}$:

$q_i := \sigma_{s-d+1+i}(p_1, \dots, p_s)$ for $i = 1, \dots, d-1$

$q_d := q$ (from THEOREM 1.37 with $M \in \mathbb{N}_0$ and $\epsilon > 0$ chosen above such that (1.67) holds)

$\mathcal{W}_{\text{new}} = \mathcal{W}(q_1, \dots, q_d)$

$\underline{\mathcal{W} \subseteq \mathcal{W}_{\text{new}}}$: Let $a \in \mathcal{W}$. Again, $q_i(a) \geq 0$ for $i \in \{1, \dots, d-1\}$ trivially holds. Further, by (1.46), $\mathcal{W} \subseteq \mathcal{W}(q_d)$ and hence $q_d(a) \geq 0$.

$\mathcal{W}_{\text{new}} \subseteq \mathcal{W}$: Let $a \in \mathbb{R}^n \backslash \mathcal{W}$. By contraposition one has to show that $a \in \mathbb{R}^n \backslash \mathcal{W}_{\text{new}}$. Let us consider two cases:

Case 1: $a \in \mathbb{R}^n \backslash \mathcal{W}_{M,2\epsilon}$. By (1.46), $\mathcal{W}(q_d) \subseteq \mathcal{W}_{M,2\epsilon}$ and, therefore, $a \in \mathbb{R}^n \backslash \mathcal{W}(q_d)$. Hence $q_d(a) < 0$ which implies $a \notin \mathcal{W}_{\text{new}}$.

Case 2: $a \in \mathcal{W}_{M,2\epsilon} \backslash \mathcal{W}$. Since $a \notin \mathcal{W}$, there exists $i \in \{1, \dots, s\} : p_i(a) < 0$. Using the result for *elementary-symmetric functions* from PROPOSITION 1.24, it follows that

$$\exists j \in \{1, \dots, s\} : \sigma_j(a) := \sigma_j(p_1(a), \dots, p_s(a)) < 0. \tag{1.68}$$

Let us restrict the $j \in \{1, \dots, s\}$ which possibly come into question for (1.68). Remember that $a \in \mathcal{W}_{M,2\epsilon}$. Hence by (1.67) $\sigma_j(a) > 0$ for $j \in \{1, \dots, s-d\}$. This shows that (1.68) is only possible if $j \in \{s-d+1, \dots, s\}$. Let us distinguish two cases:

(i) Let $j \in \{s-d+2, \dots, s\}$: Then $\sigma_j(a) = q_i(a)$ for appropriate $i \in \{1, \dots, d-1\}$. If

$$\sigma_j(a) = q_j(a) < 0,$$

it immediately follows that $a \notin \mathcal{W}_{\text{new}}$.

(ii) Let $j = s-d+1$. Assume that $\sigma_{s-d+1}(a) < 0$. Then

$$q_d(a) = \sigma_{s-d+1}(a) - q_1 q_2(a),$$

where $q_1 q_2$ is the subtrahend of the polynomial q defined in (1.47). Recall that $q_1 q_2$ is of the form

$$\left(\frac{1}{s} \sum_{i=1}^{s} \left(1 - \frac{1}{\gamma}(1 + \|x\|^2)^M p_i(x) \right)^{2k} \right)^l \prod_{v \in A} \left(\frac{\|x - v\|}{\mu} \right)^{2m}.$$

Note that by definition $q_1 q_2(a) \geq 0$. Hence

$$q_d(a) := \underbrace{\sigma_{s-d+1}(a)}_{<0} \underbrace{-q_1 q_2(a)}_{\leq 0} < 0.$$

This again implies $a \notin \mathcal{W}_{\text{new}}$ and finishes the proof.

■

4.3. Examples.

EXAMPLE 1.38: Let

$$P := \operatorname{conv}\left\{\begin{pmatrix}0\\4\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\4\end{pmatrix}, \begin{pmatrix}-4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-4\\0\end{pmatrix}\right\}$$

be a polytope. If one defines

$$\begin{aligned} l_1 &:= x + y - z + 4 \in \mathbb{R}[x,y,z], \\ l_2 &:= x - y - z + 4 \in \mathbb{R}[x,y,z], \\ l_3 &:= -x - y - z + 4 \in \mathbb{R}[x,y,z], \\ l_4 &:= -x + y - z + 4 \in \mathbb{R}[x,y,z] \\ l_5 &:= z \in \mathbb{R}[x,y,z], \end{aligned}$$

the basic closed description of P with linear polynomials looks as follows:

$$P := \{a \in \mathbb{R}^3 : l_1(a) \geq 0, \ldots, l_5(a) \geq 0\} \tag{1.69}$$

P has the following form:

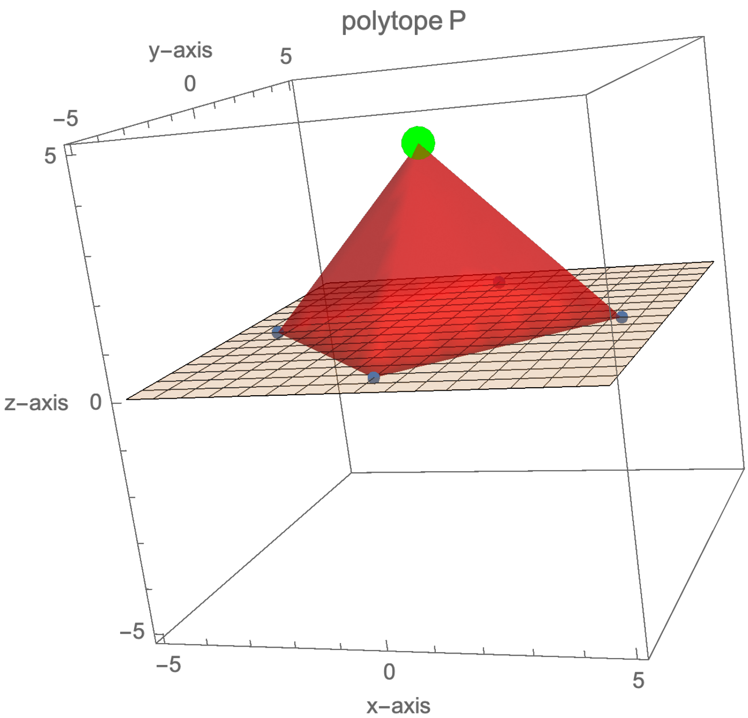


FIGURE 1.11. Visualization of the polytope P

The maximal number of polynomials which vanish on a point in P is 4. Moreover, there is only one point $a \in P$ where exactly 4 polynomials vanish. This point is drawn in green in the above graphic. To be able to distinguish it from the other vertices, it is drawn in greater size. On every other vertex (drawn in blue) only 3 of the defining polynomials vanish. The plotted plane should just represent the xy-plane.

Since $d = 4$, using THEOREM 1.20 one can find another basic closed description of P given by 4 polynomials. Following the proof of the just mentioned theorem, the first three polynomials are given by *elementary-symmetric functions*:

$$
\begin{aligned}
q_1 &:= \sigma_{s-d+1+1}(l_1, \ldots, l_5) \overset{s=5,d=4}{=} \sigma_3(l_1, \ldots, l_5) \\
q_2 &:= \sigma_4(l_1, \ldots, l_5) \\
q_3 &:= \sigma_5(l_1, \ldots, l_5)
\end{aligned}
$$

The following graphic visualizes $\mathcal{W}(q_i)$ for $i \in \{1, 2, 3\}$. The polytope, which is of course contained in the sets, is plotted in red.

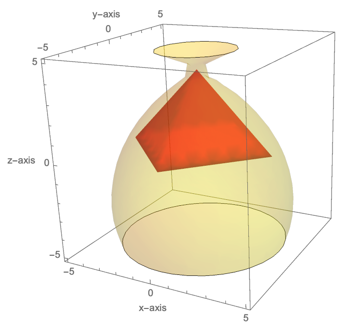


FIGURE 1.12. $\mathcal{W}(q_1)$

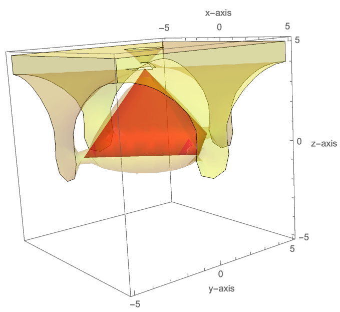


FIGURE 1.13. $\mathcal{W}(q_2)$

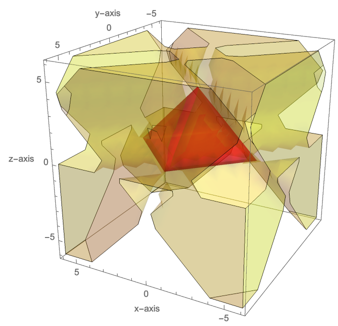


FIGURE 1.14. $\mathcal{W}(q_3)$

As one can easily see, the vertex $a = (0, 0, 4)^t$, which was emphasized by a huge green dot in FIGURE 1.10, causes some problems. This is the only point $a \in P$ where exactly 4 linear polynomials vanish. It is responsible for the necessity of q_1 in the sought-after new polynomial description of P. To see this, one can take a look at $\mathcal{W}(q_2, q_3)$:

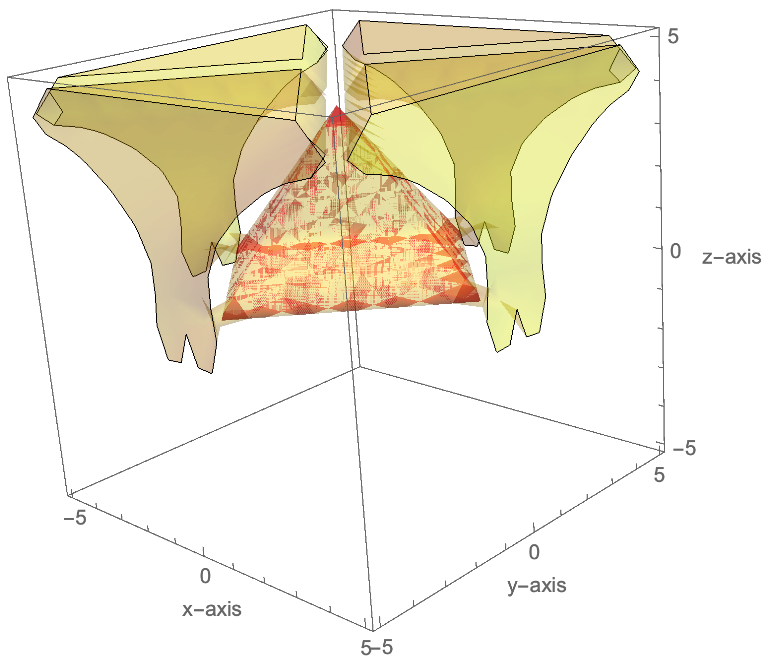


FIGURE 1.15. Visualization of $\mathcal{W}(q_2, q_3)$

If one just focuses on the vertices where only 3 defining polynomials vanish, it seems that one can manage to control the surplus area arisen from $\mathcal{W}(q_2, q_3)$. Nevertheless, the fact that 4 linear polynomials vanish at a is responsible for the quite complicated appearance of $\mathcal{W}(q_2, q_3)$. Hence one urgently needs the third polynomial q_1 to get the surplus area under control:

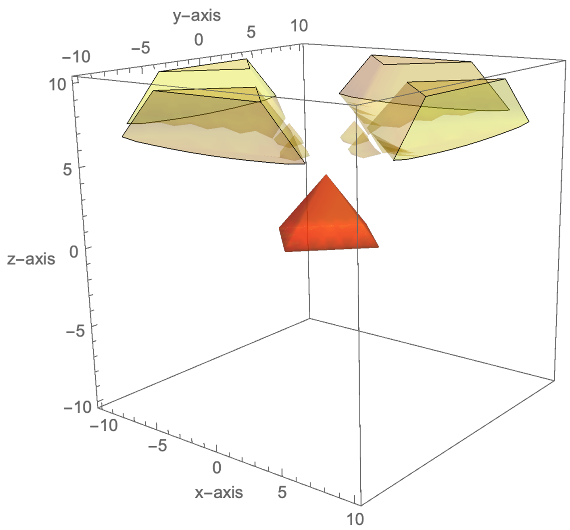


FIGURE 1.16. Visualization of $\mathcal{W}(q_1, q_2, q_3)$

Now the surplus area, which is drawn in yellow, can easily be cut off by a forth polynomial. Therefore, one can use the polynomial q from the proof of THEOREM 1.20. Since the present example is relatively simple, one can also consider another polynomial, namely

$$q_4 := 16 - x^2 - y^2 - z^2,$$

which is a circle with center 0 and radius 4 passing through all vertices of the polytope. This leads to another basic closed description of the polytope:

$$P = \mathcal{W}(q_1, q_2, q_3, q_4).$$

♦

REMARK 1.39: The above example should demonstrate the importance of *elementary-symmetric functions* in search of a new basic closed description of a polytope P. Therefore, the polynomial q_4 from THEOREM 1.20 was replaced by a polynomial which is easier to handle. Nevertheless, one has to emphasize that $q_4 := 16 - x^2 - y^2 - z^2$ is indeed a good choice and fulfills the required properties. Therefore, let $\mathcal{W} = \mathcal{W}(q_1, q_2, q_3, q_4)$. One has to show that $P = \mathcal{W}$. Since q_4 is a strictly concave polynomial containing all vertices of P, the implication $P \subseteq \mathcal{W}$ easily follows. For the other direction, taking the main idea of the proof of THEOREM 1.20 into account, let us show that $\sigma_1|_{\mathcal{W}(q_4)} \geq 0$ and $\sigma_2|_{\mathcal{W}(q_4)} \geq 0$, where $\sigma_i := \sigma_i(l_1, \ldots, l_5)$ for $i \in \{1, 2\}$. Let us compute σ_1 and σ_2:

$$\begin{aligned}\sigma_1 &= 16 - 3z \\ \sigma_2 &= 96 - 2x^2 - 2y^2 - 32z + 2z^2\end{aligned}$$

Notice that $\sigma_1 \geq 0 \Leftrightarrow z \leq \frac{16}{3}$. Hence $\sigma_1|_{\mathcal{W}(q_4)} \geq 0$. Moreover, note that $\mathcal{W}(q_4)$ is a compact set and σ_2 is a continuous function. Applying the MIN-MAX-*Theorem*, $\sigma_2|_{\mathcal{W}(q_4)}$ attains a global minimum on $\mathcal{W}(q_4)$. If we are able to show that $\min_{a \in \mathcal{W}(q_4)} \sigma_2(a) \geq 0$, it immediately follows that $\sigma_2|_{\mathcal{W}(q_4)} \geq 0$. Hence let us compute the gradient of σ_2:

$$\nabla \sigma_2(x, y, z) = \begin{bmatrix} -4x \\ -4y \\ -4z + 32 \end{bmatrix}$$

It holds that $\sigma_2(x,y,z) = 0 \Leftrightarrow (x,y,z) = (0,0,8)$. This point is not part of $\mathcal{W}(q_4)$. Moreover, by computing the Hessian matrix it turns out that $(0,0,8)$ is not an extremum. Hence the minimum will be attained at the boundary of $\mathcal{W}(q_4)$. This leads to the constraint "$q_4 = 0$". Applying the *method of Lagrange multipliers* leads to

$$\nabla\sigma_2(x,y,z) + \lambda\nabla q_4(x,y,z) = 0$$

or equivalently

$$\begin{bmatrix} -4x \\ -4y \\ -4z+32 \end{bmatrix} + \lambda \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix} = 0.$$

This is only possible if $x = 0$, $y = 0$ and $z \neq 0$. Since the minimum is attained at the boundary of $\mathcal{W}(q_4)$, the following points come into question: $(0,0,4)$ and $(0,0,-4)$. We have

$$\begin{aligned} \sigma_2(0,0,4) &= 96 - 32\cdot 4 + 2\cdot 4^2 = 0 \\ \sigma_2(0,0,-4) &= 96 - 32\cdot(-4) + 2\cdot(-4)^2 = 256. \end{aligned}$$

Hence $\min_{a\in\mathcal{W}(q_4)} \sigma_2(a) = 0$ and $\max_{a\in\mathcal{W}(q_4)} \sigma_2(a) = 256$. This immediately implies $\sigma_2|_{\mathcal{W}(q_4)} \geq 0$. The following graphic shows $\mathcal{W}(\sigma_1)$ and $\mathcal{W}(\sigma_2)$. Both sets are drawn in purple. Moreover, $\mathcal{W}(q_4)$ is drawn in green. One can see that $\mathcal{W}(q_4) \subseteq \mathcal{W}(\sigma_1)$ respectively $\mathcal{W}(q_4) \subseteq \mathcal{W}(\sigma_2)$.

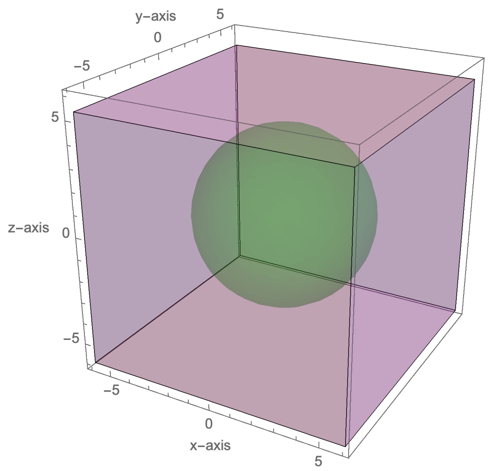


FIGURE 1.17. $\mathcal{W}(q_4) \subseteq \mathcal{W}(\sigma_1)$

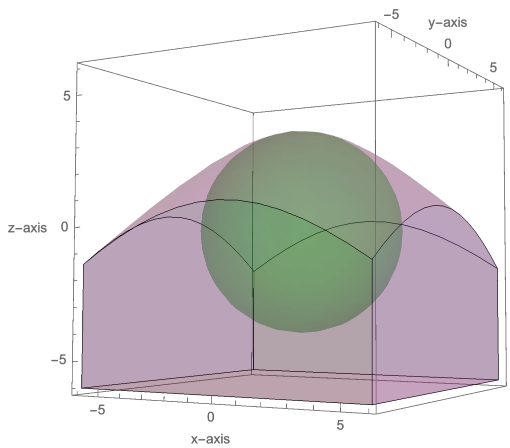


FIGURE 1.18. $\mathcal{W}(q_4) \subseteq \mathcal{W}(\sigma_2)$

REMARK 1.40: With the help of THEOREM 1.20 it was possible to show that the polytope from the above example possesses a basic closed description with 4 polynomials. Compared with the primary description (1.69) the new polynomial description does not really entail a great benefit. It just leads to a description with 4 instead of 5 polynomials. Hence the number of polynomials needed in the new basic closed description of P is only improved by 1. Moreover, the new polynomials are not linear anymore. Hence it is debatable whether the new description leads to a better representation of the polytope.

Moreover, as mentioned in the preliminaries of the present thesis, *Averkov* and *Bröcker* proved the equality

$$m(P) = n$$

for every n-dimensional polytope $P \subseteq \mathbb{R}^n$ (**[AvBr10]**). Hence there exists another basic closed description of the polytope P from EXAMPLE 1.38 with 3 polynomials. Nevertheless, with the help of THEOREM 1.20 one can at least show that $m(P) = n$ holds for *simple* n-dimensional polytopes.

DEFINITION 1.41: *Let $P = \mathcal{W}(l_1, \dots, l_s) \subseteq \mathbb{R}^n$ be a n-dimensional polytope given by the facet-defining linear polynomials $l_1, \dots, l_s \in \mathbb{R}[\underline{x}]_{\leq 1}$. P is called simple*

$$:\Leftrightarrow \forall v \in \mathrm{vert}(P) : |\{i \in \{1, \dots, s\} : l_i(v) = 0\}| = n$$

In other words, every vertex of P arises as an intersection of exactly n facets.

REMARK 1.42: The polytope from EXAMPLE 1.38 is not simple. The vertex $a = (0, 0, 4)$ - which caused some troubles before - occurred from the intersection of $4 > 3$ facets.

Now let $P \subseteq \mathbb{R}^n$ be a simple polytope. Again, let

$$d := \max\{|i \in \{1, \dots, s\} : p_i(a) = 0| : a \in \mathcal{W}\}$$

By the fact that P is simple, $d = n$. Moreover,

$$|\{a \in P : |i \in \{1, \dots, s\} : p_i(a) = 0| = n\}| = |\mathrm{vert}(P)| < \infty.$$

Hence one can apply THEOREM 1.20, which – together with the fact $m(P) \geq n$ from OBSERVATION 1 – yields

$$m(P) = n.$$

Let us emphasize this important fact:

OBSERVATION 3: Let $\emptyset \neq P \subseteq \mathbb{R}^n$ be a *simple* n-dimensional polytope. Then the following holds:

$$m(P) = n$$

Let us finish the section with another example:

EXAMPLE 1.43: We just modify EXAMPLE 1.38 a little bit. Let

$$P := \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

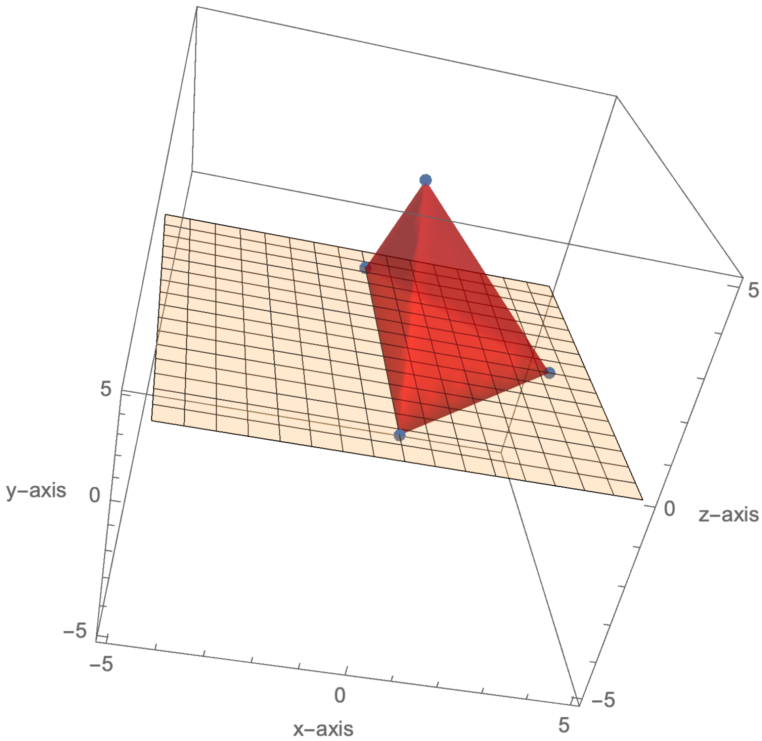
With the help of the facet-defining linear polynomials one gets the following basic closed description of P.

$$P = \{x \in \mathbb{R}^3 : l_1(x) \geq 0, l_2(x) \geq 0, l_3(x) \geq 0, l_4(x) \geq 0\},$$

where

$$\begin{aligned} l_1 &:= -x - y - z + 4 \in \mathbb{R}[x, y, z], \\ l_2 &:= -x + y - z + 4 \in \mathbb{R}[x, y, z], \\ l_3 &:= x \in \mathbb{R}[x, y, z], \\ l_4 &:= z \in \mathbb{R}[x, y, z]. \end{aligned}$$

The polytope is shown in the next graphic. As one can see, it is simple.



Indeed, with the notations from THEOREM 1.20 $d = n = 3$ and

$$|\{a \in P : |i \in \{1, \dots, s\} : p_i(a) = 0| = n\}| = |\mathrm{vert}(P)|$$
$$= \left| \left\{ \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \right\} \right| = 4 < \infty$$

By THEOREM 1.20 there exists another basic closed description of P given by the polynomials $(s = 4, d = 3)$

$$\begin{aligned} q_1 &:= \sigma_{s-d+1+1}(l_1, \dots, l_4) = \sigma_3(l_1, \dots, l_4) \\ q_2 &:= \sigma_4(l_1, \dots, l_4) \\ q_3 &:= 16 - x^2 - y^2 - z^2. \end{aligned}$$

Again, since the present example is quite simple, one can choose q_3 as above. To be able to show that the above choice for q_3 yields the desired result, one needs further computations which won't be carried out in detail.

Let us visualize the basic closed sets $\mathcal{W}(q_1)$, $\mathcal{W}(q_2)$ and $\mathcal{W}(q_1, q_2)$. Again, the polytope P can also be seen in the graphic below (drawn in red).

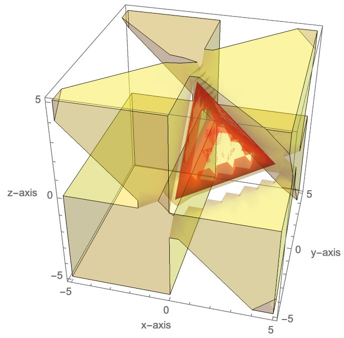


FIGURE 1.19. $\mathcal{W}(q_1)$

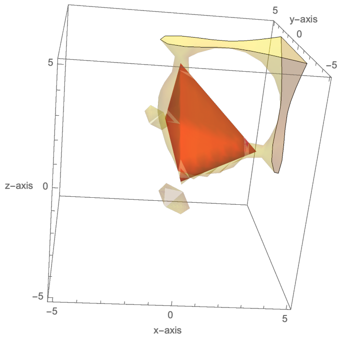


FIGURE 1.20. $\mathcal{W}(q_2)$

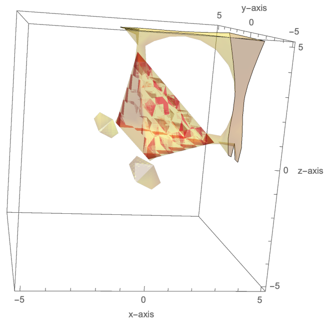


FIGURE 1.21. $\mathcal{W}(q_1, q_2)$

Cutting off the surplus area arisen by $\mathcal{W}(q_1, q_2)$ with the help of the polynomial q_3 leads to the desired result:

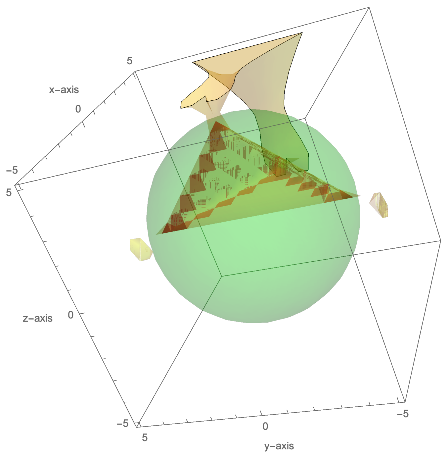


FIGURE 1.22. Cutting off the surplus area with $\mathcal{W}(q_3)$

♦

5. Polytopes

For the sake of completeness let us again state the most important result concerning minimal descriptions of polyhedra and polytopes. As mentioned before, it was shown by *Averkov* and *Bröcker* and unfortunately is quite technical. Since it is not needed for the next chapter, it won't be proven in detail. Interested readers can find the statement including its proof in [**AvBr10**]. Nevertheless, let us at least state the main idea for finding minimal descriptions of polytopes. Therefore, one needs the following definition:

DEFINITION 1.44: *For a n-dimensional polytope P define the set of k-extremal points of P by*

$$\mathrm{ex}_k P := \bigcup_{F \in \mathcal{F}_k(P)} F,$$

where $\mathcal{F}_k(P)$ denotes the set of all k-dimensional faces of P. Since faces of polytopes are exposed, one can write $F = \{l_F = 0\} \cap P$ with $l_F \in \mathbb{R}[\underline{x}]_{\leq 1}$ such that $l_F|_P \leq 0$. With the help of this notation define the k-support of P by

$$D_k(P) := \bigcap_{F \in \mathcal{F}_k(P)} \{l_F \leq 0\}.$$

REMARK 1.45: Note that for $k \in \{0, \ldots, n-1\}$ it holds that $\mathcal{F}_k(P) \neq \emptyset$. This follows from the *Theorem of Krein-Milman* and PROPOSITION 1.8. Moreover, again using the results from PROPOSITION 1.8, for every k-dimensional face of P there exists a $(k+1)$-dimensional face of P which contains the lower dimensional face. Hence

$$\emptyset = \mathrm{ex}_{-1}(P) \subseteq \mathrm{vert}(P) = \mathrm{ex}_0(P) \subseteq \mathrm{ex}_1(P) \subseteq \ldots \subseteq \mathrm{ex}_{n-1}(P).$$

More precisely,

$$\bigcup_{F \in \mathcal{F}_k(P)} F \subseteq \bigcup_{F \in \mathcal{F}_{k+1}(P)} F.$$

Moreover, a similar statement for k-supports holds true:

$$P = D_{n-1}(P) \subseteq D_{n-2}(P) \subseteq \ldots \subseteq D_0(P) \subseteq D_{-1}(P) = \mathbb{R}^n \tag{1.70}$$

EXAMPLE 1.46: Consider the polytope $P = \mathrm{conv}\{v_1, \ldots, v_5\}$, which can be seen in the next graphic:

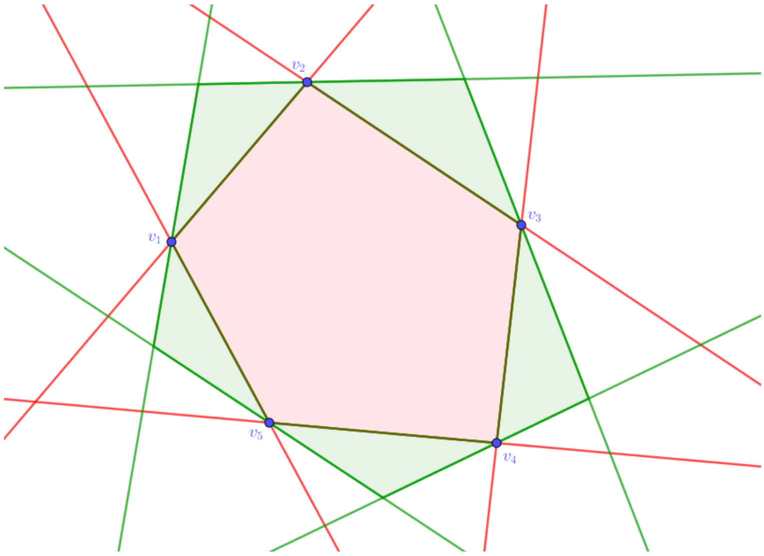


FIGURE 1.23. Supports of P

It holds that $D_1(P) = P$. The corresponding hyperplanes $\{l_F = 0\}$ for $F \in \mathcal{F}_1(P)$ are drawn in red. Moreover, possible choices for $\{l_{\tilde{F}} = 0\}$ with $\tilde{F} \in \mathcal{F}_0(P)$ are drawn in green. It can be seen that $D_1(P) \subseteq D_0(P) \subseteq \mathbb{R}^2$. ⧫

PROPOSITION 1.47: ([**AvBr10**], p. 6-7)
For $k = 0, \dots, n-1$ there exists a polynomial p_k such that

$$(i)\ p_k \geq 0 \text{ on } P$$
$$(ii)\ p_k \leq 0 \text{ on } D_{k-1}(P) \backslash D_k(P)$$
$$(iii)\ \{p_k = 0\} \cap (D_{k-1}(P) \backslash D_k(P)) \subseteq P$$

PROOF: We refer to [**AvBr10**], p. 7-9. ∎

With the help of this proposition it immediately follows that

OBSERVATION 4: Let $\emptyset \neq P \subseteq \mathbb{R}^n$ be a n-dimensional polytope.
Then the following holds:

$$m(P) = n$$

PROOF: Define $\mathcal{W} := \mathcal{W}(p_0, \dots, p_{n-1})$ with p_k from PROPOSITION 1.47. By the above proposition it immediately follows that $P \subseteq \mathcal{W}$ (compare with (i)). For the other inclusion let $a \notin P$. It remains to show $a \notin \mathcal{W}$. Since $a \notin P$, there exists

$k \in \{0, \ldots, n-1\}$ such that

$$a \in D_{k-1}(P) \backslash D_k(P).$$

This immediately follows from the fact

$$D_{k-1}(P) \backslash D_k(P) \cup D_k(P) \backslash D_{k+1}(P) = D_{k-1}(P) \backslash D_{k+1}(P),$$

and hence

$$\begin{aligned} \bigcup_{k \in \{0, \ldots, n-1\}} D_{k-1}(P) \backslash D_k(P) &= D_{-1}(P) \backslash D_0(P) \cup \ldots \cup D_{n-2}(P) \backslash D_{n-1}(P) \\ &= D_{-1}(P) \backslash D_{n-1}(P) = \mathbb{R}^n \backslash P. \end{aligned}$$

Recall that

$$D_{n-1}(P) = \bigcap_{F \in \mathcal{F}_{n-1}(P)} \{l_F \leq 0\} = P,$$

since the $n-1$-dimensional faces of P are just the facets of P. From PROPOSITION 1.8 it follows that they provide a polyhedral description of P.

By PROPOSITION 1.47, part (ii), $p_k(a) \leq 0$. Since $a \notin P$ it even holds that $p_k(a) < 0$ by part (iii) of the proposition. This shows $a \notin \mathcal{W}$. ∎

REMARK 1.48: The similar result for polyhedra stated in THEOREM 0.5 can be found in [**AvBr10**].

REMARK 1.49: It should be emphasized that the particular properties of (faces of) polytopes are responsible for the validity of the above proof. On the contrary, THEOREM 1.19 and THEOREM 1.20 from Chapter 1.5 hold true for more general basic closed semialgebraic sets. Therefore, for finding basic closed descriptions of spectrahedra with few polynomials, the present thesis uses an approach similar to the one described in Chapter 1.5. In the following chapter it will be shown that bounded *smooth spectrahedra* possess basic closed descriptions with two polynomials.

CHAPTER 2

Spectrahedra

The following pages can be seen as a short introduction concerning the most important properties of *spectrahedra*. The present pages are orientating on [**NePl+**]. For more detailed information about spectrahedra we also refer to this book.

Let $k \in \mathbb{N}$. During the whole chapter

$$\mathrm{Sym}_k(\mathbb{R}) := \{A \in \mathbb{R}^{k \times k} : A = A^t\}$$

denotes the set of all symmetric $k \times k$-matrices. Moreover

$$\mathrm{Sym}_k^+(\mathbb{R}) := \{A \in \mathrm{Sym}_k(\mathbb{R}) : A \succcurlyeq 0\}$$

is the set of all positive semidefinite matrices, which are defined as follows:

DEFINITION 2.1: *A matrix $A \in \mathrm{Sym}_k(\mathbb{R})$ is called* *<u>positive semidefinite</u>*

$$:\Leftrightarrow A \succcurlyeq 0 :\Leftrightarrow \forall v \in \mathbb{R}^k : v^t A v \geq 0.$$

A is called *<u>positive definite</u>*

$$:\Leftrightarrow A \succ 0 :\Leftrightarrow \forall v \in \mathbb{R}^k \backslash \{0\} : v^t A v > 0.$$

There are different possibilities to characterize positive semidefinite matrices. Since they are very useful for the following considerations, some of them are stated in the next lemma:

LEMMA 2.2: Let $A \in \mathrm{Sym}_k(\mathbb{R})$. The following are equivalent:

(i) $A \succcurlyeq 0$
(ii) $\forall \lambda \in \mathrm{Spec}(A) : \lambda \geq 0$, where $\mathrm{Spec}(A)$ denotes the set of all eigenvalues of A.
(iii) The *principal minors* of A are greater or equal than 0.

PROOF: The statement can easily be proven by using some well-known results from linear algebra. ■

With all the preparations from above one is able to define a *spectrahedron* in $\mathbb{R}^n$.

DEFINITION 2.3: *Let $k \in \mathbb{N}$ and $A_0, \ldots, A_n \in \operatorname{Sym}_k(\mathbb{R})$. The set*

$$\mathcal{S}(A_0, \ldots, A_n) := \{a \in \mathbb{R}^n : A_0 + a_1 A_1 + \ldots + a_n A_n \succcurlyeq 0\} \tag{2.1}$$

is called a spectrahedron in $\mathbb{R}^n$. It represents the set of solutions of a linear matrix inequality. The polynomial

$$A := A_0 + x_1 A_1 + \ldots + x_n A_n$$

is called a linear matrix polynomial. It differs from a classical polynomial since its coefficients are symmetric matrices.

EXAMPLE 2.4: Let $k = n = 2$. Define

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The set

$$\begin{aligned} \mathcal{S}(A_0, A_1, A_2) &= \left\{ (a_1, a_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \succcurlyeq 0 \right\} \\ &= \left\{ (a_1, a_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 + a_1 & a_2 \\ a_2 & 1 - a_1 \end{pmatrix} \succcurlyeq 0 \right\} \end{aligned}$$

is a spectrahedron. One can use LEMMA 2.2 to further specify $S(A_0, A_1, A_2)$. A matrix is positive semidefinite if its principal minors are greater or equal than 0. Therefore,

$$\mathcal{S}(A_0, A_1, A_2) = \left\{ (a_1, a_2) \in \mathbb{R}^2 : 1 + a_1 \geq 0, 1 - a_1 \geq 0, 1 - a_1^2 - a_2^2 \geq 0 \right\}.$$

$\mathcal{S}(A_0, A_1, A_2)$ describes the unit circle in $\mathbb{R}^2$, which is shown in the graphic below. One can see that the two inequalities $1 - a_1 \geq 0$ and $1 + a_1 \geq 0$ are redundant since they do not change the set $\{(a_1, a_2) \in \mathbb{R}^2 : 1 - a_1^2 - a_2^2 \geq 0\}$. Hence the spectrahedron from the present example is a basic closed semialgebraic set given by just one polynomial inequality. More precisely, $\mathcal{S}(A_0, A_1, A_2) = \mathcal{W}(1 - x^2 - y^2)$.

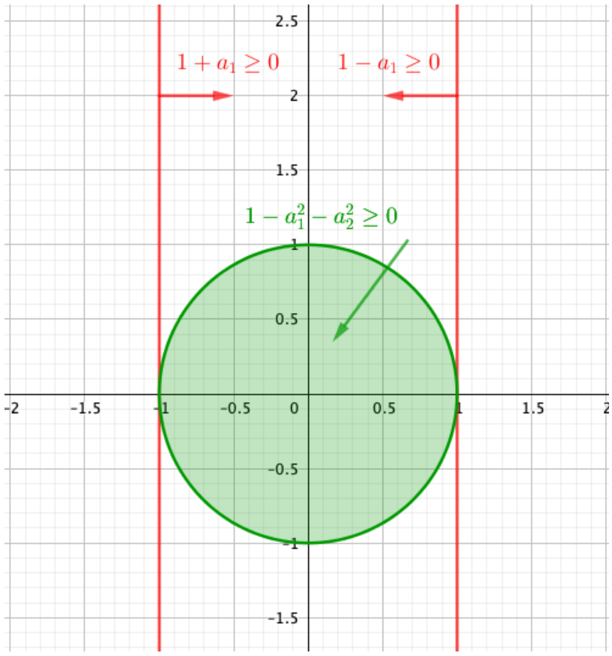


FIGURE 2.1. $\mathcal{S}(A_0, A_1, A_2)$ is the unit circle in $\mathbb{R}^2$

♦

EXAMPLE 2.5: Since the main objective of the first chapter was to find minimal basic closed descriptions of polytopes, it should be emphasized that spectrahedra can be seen as a generalization of polyhedra. In other words, every polyhedron is a spectrahedron. To see this, for $i \in \{1, \ldots, m\}$ let

$$l_i := a_0^{(i)} + a_1^{(i)} x_1 + \ldots + a_n^{(i)} x_n \in \mathbb{R}[\underline{x}]_{\leq 1}$$

and $P := \mathcal{W}(l_1, \ldots, l_m)$ be a polyhedron. Define

$$A_0 := \begin{pmatrix} a_0^{(1)} & & 0 \\ & \ddots & \\ 0 & & a_0^{(m)} \end{pmatrix}, \ \ldots, \ A_n := \begin{pmatrix} a_n^{(1)} & & 0 \\ & \ddots & \\ 0 & & a_n^{(m)} \end{pmatrix}.$$

Then

$$A_0 + x_1 A_1 + \ldots + x_n A_n = \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_m \end{pmatrix}.$$

Note that a diagonal matrix is positive semidefinite iff the entries in the diagonal are greater or equal than 0. This is due to the fact that these entries coincide with the eigenvalues of the matrix. Hence

$$\begin{aligned}\mathcal{S}(A_0,\dots,A_n) &= \left\{ a \in \mathbb{R}^n : \begin{pmatrix} l_1(a) & & 0 \\ & \ddots & \\ 0 & & l_m(a) \end{pmatrix} \succcurlyeq 0 \right\} \\ &= \{a \in \mathbb{R}^n : l_1(a) \geq 0, \dots, l_m(a) \geq 0\} = \mathcal{W}(l_1,\dots,l_m) = P,\end{aligned}$$

which shows that P is a spectrahedron. ⧫

1. Some basic properties

In the present section the most important properties of spectrahedra will be stated. Since this master's thesis deals with basic closed semialgebraic sets, let us start with the following proposition:

PROPOSITION 2.6: Let $k \in \mathbb{R}$ and $A_0,\dots,A_n \in \mathrm{Sym}_k(\mathbb{R})$. $\mathcal{S}(A_0,\dots,A_n)$ is a basic closed semialgebraic set.

PROOF:
Option 1: Let $a \in \mathbb{R}^n$. Then

$$\begin{aligned} a \in \mathcal{S}(A_0,\dots,A_n) &\Leftrightarrow A_0 + a_1A_1 + \ldots + a_nA_n \succcurlyeq 0 \\ &\Leftrightarrow \text{ the principal minors of } A_0 + a_1A_1 + \ldots + a_nA_n \text{ are } \geq 0. \end{aligned} \tag{2.2}$$

Set $A := A_0 + x_1A_1 + \ldots + x_nA_n$. The entry A_{ij} looks as follows:

$$A_{ij} = \underbrace{(A_0)_{ij}}_{\in\mathbb{R}} + x_1 \underbrace{(A_1)_{ij}}_{\in\mathbb{R}} + \ldots + x_n \underbrace{(A_n)_{ij}}_{\in\mathbb{R}} =: l_{ij} \in \mathbb{R}[\underline{x}]$$

This shows that the entries of the matrix A are just linear polynomials: $A = (l_{ij})_{i,j}$. The computation of the principal minors of A – which are just determinants of special submatrices – just leads to other polynomials in $\mathbb{R}[\underline{x}]$ (which are in general not linear anymore). A $(k \times k)$-matrix possesses $2^k - 1$ principal minors. Let us denote the polynomials arisen from the computation of all principal minors by

$$p_1,\dots,p_{2^k-1} \in \mathbb{R}[\underline{x}].$$

From (2.2) it finally follows

$$\mathcal{S}(A_0,\dots,A_n) = \{a \in \mathbb{R}^n : p_1(a) \geq 0, \dots p_{2^k-1}(a) \geq 0\},$$

which is a basic closed description of $\mathcal{S}(A_0, \ldots, A_n)$.

Option 2: Again, let $a \in \mathbb{R}^n$. Let us use another characterization of positive semidefiniteness:

$$\begin{aligned} a \in \mathcal{S}(A_0, \ldots, A_n) \Leftrightarrow \forall \lambda \in \operatorname{Spec}(A_0 + a_1 A_1 + \ldots + a_n A_n) : \lambda \geq 0 \\ \forall \lambda \in \mathbb{R} : (\det(\lambda \mathbb{I}_k - A_0 - a_1 A_1 - \ldots - a_n A_n) = 0 \Rightarrow \lambda \geq 0) \end{aligned} \tag{2.3}$$

Let us again define $A := A_0 + x_1 A_1 + \ldots + x_n A_n \in \operatorname{Sym}_k(\mathbb{R})$ and $A(a) := A_0 + a_1 A_1 + \ldots + a_n A_n$. Condition (2.3) requires that the zero set of the characteristic polynomial $\det(t\mathbb{I}_k - A(a)) \in \mathbb{R}[t]$ is a subset of $\mathbb{R}_{\geq 0}$. Let $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ be the roots of $\det(t\mathbb{I}_k - A(a))$. The condition $A \in \operatorname{Sym}_k(\mathbb{R})$ guarantees that the roots are real. Using the same trick like in LEMMA 1.25,

$$\alpha_1 \geq 0, \ldots, \alpha_s \geq 0 \Leftrightarrow \text{ the polynomial } \det(t\mathbb{I}_k - A(a)) \text{ has alternating coefficients.} \tag{2.4}$$

Expanding the polynomial $\det(t\mathbb{I}_k - A(a))$ leads to

$$\det(t\mathbb{I}_k - A(a)) = t^k + p_1(a)t^{k-1} + \ldots + p_{k-1}(a)t^1 + p_k(a),$$

where $p_j, j = 1, \ldots, k$, are polynomials in $\mathbb{R}[\underline{x}]$. By (2.4)

$$\alpha_1 \geq 0, \ldots, \alpha_s \geq \Leftrightarrow \forall i \in \{1, \ldots, k\} : (-1)^i p_i(a) \geq 0$$

This leads to the following basic closed description of $\mathcal{S}(A_0, \ldots, A_n)$:

$$\mathcal{S}(A_0, \ldots, A_n) = \{a \in \mathbb{R}^n : \forall i \in \{1, \ldots, k\} : (-1)^i p_i(a) \geq 0\}.$$

■

EXAMPLE 2.7: Let us demonstrate the second approach with EXAMPLE 2.4. Define $A := A_0 + xA_1 + yA_2$ with A_0, A_1, A_2 from the above example. The characteristic polynomial $\det(t\mathbb{I}_2 - A)$ is given by

$$\begin{aligned}
\det(t\mathbb{I}_2 - A) &= \det\left(t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1+x & y\\ y & 1-x\end{pmatrix}\right)\\
&= \det\begin{pmatrix}t-1-x & -y\\ -y & t-1+x\end{pmatrix}\\
&= (t-1-x)(t-1+x) - y^2 = (t-1)^2 - x^2 - y^2\\
&= t^2 \underbrace{-2}_{=:p_1} t + \underbrace{1 - x^2 - y^2}_{=:p_2}.
\end{aligned}$$

This leads to the following basic closed description of the spectrahedron:

$$\mathcal{S}(A_0, A_1, A_2) = \{(a_1, a_2) \in \mathbb{R}^2 : 2 \geq 0, 1 - a_1^2 - a_2^2 \geq 0\}$$

Since $2 \geq 0$ trivially holds, the present approach leads to same basic closed description of $\mathcal{S}(A_0, A_1, A_2)$ like before. ♦

REMARK 2.8: Following the proof of PROPOSITION 2.6 one can find a first upper bound for the number of polynomials which are necessary in a basic closed description of a spectrahedron. If $A_0, \ldots, A_n \in \mathrm{Sym}_k(\mathbb{R})$ and $\mathcal{S} := \mathcal{S}(A_0, \ldots, A_n)$, one can estimate $m(\mathcal{S})$ from above:

$$m(\mathcal{S}) \leq k \tag{2.5}$$

In EXAMPLE 2.7 it was shown that this bound is quite pessimistic. Moreover, the bound is not very useful since k does not depend on the space dimension n. For example, with the help of some matrices $A_0, A_1, A_2 \in \mathrm{Sym}_{100}(\mathbb{R})$ one can define a spectrahedron in $\mathbb{R}^2$:

$$\mathcal{S}(A_0, A_1, A_2) := \{(a_1, a_2) \in \mathbb{R}^2 : \underbrace{A_0 + a_1 A_1 + a_2 A_2}_{\in \mathrm{Sym}_{100}(\mathbb{R})} \succcurlyeq 0\}$$

In this case $\mathcal{S}(A_0, A_1, A_2)$ can be represented by at most 100 polynomials. Recall that *Bröcker* and *Scheiderer* were able to show that

$$m(\mathcal{S}) \leq \frac{n(n+1)}{2} \overset{n=2}{=} \frac{2 \cdot 3}{2} = 3.$$

This bound is independent of the complexity of $\mathcal{S}$. Hence the estimation from (2.5) turns out to be a terrible upper bound.

Just to anticipate one thing: One can find much better bounds than demonstrated in the above remark. This will be the main purpose of the present chapter. To be able to find such bounds, one needs some more preparations.

Let us close this chapter with another useful property of spectrahedra, which will be needed at a later time.

DEFINITION 2.9: *A linear matrix polynomial $A_0 + x_1 A_1 + \ldots + x_n A_n$ with coefficients in $\mathrm{Sym}_k(\mathbb{R})$ is called monic if $A_0 = I_k$.*

PROPOSITION 2.10: Let A be a monic linear matrix polynomial. Then the following holds:

$$\mathrm{int}(\mathcal{S}(A)) = \{a \in \mathbb{R}^n : A(a) \succ 0\}$$

Moreover, if $0 \in \mathrm{int}(\mathcal{S}(A))$, there exists a monic linear matrix polynomial B such that

$$\mathcal{S}(A) = \mathcal{S}(B).$$

PROOF: For a proof of this statement we refer to [**NePl+**]. ∎

2. Basic closed descriptions of smooth spectrahedra with two polynomials

The main aim of the present section is to find basic closed descriptions of smooth spectrahedra with two polynomials. There exists a class of polynomials which will play an important role for the following considerations:

DEFINITION 2.11: *(Real zero polynomial)*
A polynomial $p \in \mathbb{R}[\underline{x}]$ is called a real zero polynomial if $p(0) > 0$ and

$$\forall a \in \mathbb{R}^n \backslash \{0\} : p_a := p(ta) \in \mathbb{R}[t] \text{ has only real roots.}$$

It is a common practice to use the abbreviation RZ-polynomial when talking about real zero polynomials.

REMARK 2.12: If a set $\mathcal{S}(A)$ defines a spectrahedron, there exists a RZ-polynomial which can be characterized with the help of the defining linear matrix polynomial A. This polynomial will play an important role in search of new basic closed descriptions of spectrahedra. To be more specific, the following proposition holds:

PROPOSITION 2.13: Let $k \in \mathbb{N}, A_1, \ldots, A_n \in \mathrm{Sym}_k(\mathbb{R})$. Moreover, let $A_0 := I_k$ and $A := I_k + x_1A_1 + \ldots + x_nA_n$. Define

$$p := \det(A) \in \mathbb{R}[\underline{x}]_{\leq k}.$$

Then p is a RZ-polynomial.

PROOF: At first notice that

$$p(0) = \det(I_k) = 1 > 0. \tag{2.6}$$

Let $a \in \mathbb{R}^n \backslash \{0\}$. One has to show that $p(ta)$ has only real roots.

$$\begin{aligned} p(ta) &= \det(A(ta)) = \det(I_k + ta_1A_1 + \ldots + ta_nA_n) \\ &\overset{(\star)}{=} t^k \det\left(\frac{1}{t}I_k + a_1A_1 + \ldots + a_nA_n\right) \\ &= t^k \mathrm{charpol}_{(-a_1A_1-\ldots-a_nA_n)}\left(\frac{1}{t}\right) \end{aligned} \tag{2.7}$$

The equality $(\star)$ holds since $\det(cM) = c^k \det(M)$ for a matrix $M \in \mathbb{R}^{k\times k}$ and $c \in \mathbb{R}$. By (2.6) $\lambda = 0$ is not a root of p_a. Hence the roots of p_a satisfy $\lambda \neq 0$. Let λ be a root of p_a. (2.7) and the fact that $\lambda \neq 0$ lead to the condition

$$\mathrm{charpol}_{(-a_1A_1-\ldots-a_nA_n)}\left(\frac{1}{\lambda}\right) \overset{!}{=} 0. \tag{2.8}$$

By (2.8) and the fact that $\lambda \neq 0$, λ is a root of p if and only if $\frac{1}{\lambda}$ is an eigenvalue of $-a_1A_1 - \ldots - a_nA_n$. Since $A_1, \ldots, A_n \in \mathrm{Sym}_k(\mathbb{R})$, the matrix $-a_1A_1 - \ldots - a_nA_n$ is symmetric too and hence one can immediately conclude that $\frac{1}{\lambda}$ is real. This implies $\lambda \in \mathbb{R}$ and hence p is a RZ-polynomial. ■

REMARK 2.14: The assumption $A_0 = I_k$ from the above proposition does not really restrict the set of spectrahedra we are looking at. Without loss of generality one can always assume that a spectrahedron is given by a monic linear matrix polynomial. Indeed, as stated in PROPOSITION 2.10, it is possible to show that every spectrahedron $\mathcal{S} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathcal{S})$ can be represented by a monic linear matrix polynomial. The condition $0 \in \mathrm{int}(\mathcal{S})$ can easily be satisfied. If $\mathrm{int}(\mathcal{S}) = \emptyset$, one can pass to $\mathrm{relint}(\mathcal{S})$ and replace $\mathbb{R}^n$ by $\mathrm{aff}(\mathcal{S})$. Additionally, after applying a linear transformation, one can always assume $0 \in \mathrm{int}(\mathcal{S})$.

EXAMPLE 2.15: Consider the following famous example of a spectrahedron (see also [NePlSc09]), which is given by

$$A_0 := \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} -2 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As mentioned before, even though A_0 is not the identity matrix, one can find a monic linear matrix polynomial B such that $\mathcal{S}(A) = \mathcal{S}(B)$. Using the eigenvalue criterion (the characteristic polynomial has alternating coefficients) one can show that

$$\mathcal{S}(A_0, A_1, A_2) := \{(a_1, a_2) \in \mathbb{R}^2 : a_1^3 - a_1^2 - a_1 - a_2^2 + 1 \geq 0, \\ a_1^2 - 5a_1 - a_2^2 + 4 \geq 0, 4 - 3a_1 \geq 0\}.$$

$\mathcal{S}(A_0, A_1, A_2)$ has the following form:

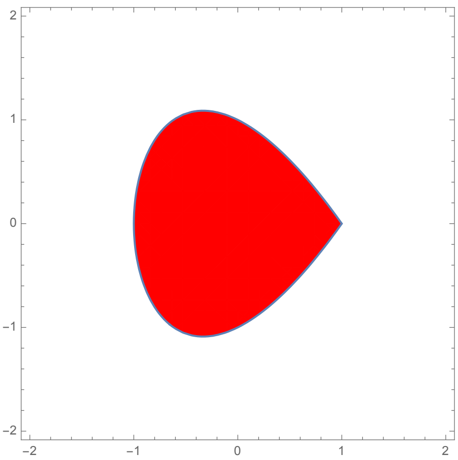


FIGURE 2.2. Visualization of $\mathcal{S}(A_0, A_1, A_2)$

Computing $\det(A)$ leads to the RZ-polynomial

$$p := \det(A) = x^3 - x^2 - x - y^2 + 1.$$

In the following graphic the zero set of p is drawn in red. To illustrate that p is indeed RZ, there is also a blue line in the graphic. It is a line passing through $0 = (0,0)^t$ and $a = (2,1)^t$. There are three real intersection points. The fact that the degree of p is three shows that p_a has only real zeros with multiplicity 1.

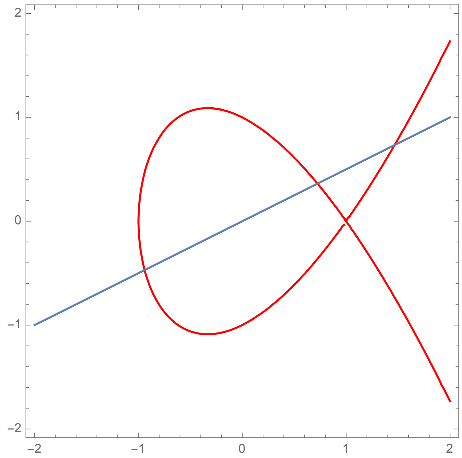


FIGURE 2.3. Visualization of $\mathcal{V}(p)$.

♦

REMARK 2.16: Let us take a close look at the above graphics. As one can suggest, it is possible to describe the spectrahedron shown in FIGURE 2.2 with the help of its corresponding RZ-polynomial (which is shown in FIGURE 2.3). Take a point $a = (a_1, a_2) \in \mathbb{R}^2$. To check if the point lies in the spectrahedron, one can do the following: Restrict p to the connecting line $[0, a]$. Check if

$$\mathcal{V}(p|_{[0,a]}) \subseteq \{a\}.$$

If $\mathcal{V}(p|_{[0,a]}) = \emptyset$, the RZ-polynomial p does not have any roots restricted to the set $[0, a]$. In other words, one can move from 0 along the line in direction a and will not pass a (real) root of p. This implies that $a \in \operatorname{int}(\mathcal{S})$ as one might think taking a look at the above pictures. If $\mathcal{V}(p|_{[0,a]}) = \{a\}$, the point a is the first root of the polynomial p if one moves from 0 along the line in direction a. If this is the case, one can show $a \in \operatorname{bd}(\mathcal{S})$. Let us precisely formulate these observations in the following proposition:

PROPOSITION 2.17: Let $k \in \mathbb{N}$, $A_0 = I_k, A_1, \ldots A_n \in \operatorname{Sym}_k(\mathbb{R})$, $A := I_k + x_1 A_1 + \ldots + x_n A_n$ be a monic linear matrix polynomial and $\mathcal{S}(A) \subseteq \mathbb{R}^n$ be a spectrahedron. Moreover, let $p := \det(A)$ be the corresponding RZ-polynomial and $p_a := p(ta)$ for any $a \in \mathbb{R}^n \backslash \{0\}$. Then the following holds:

$$a \in \mathcal{S}(A) \Leftrightarrow \mathcal{V}(p|_{[0,a)}) = \emptyset \Leftrightarrow \mathcal{V}(p_a) \cap [0, 1) = \emptyset \tag{2.9}$$

In addition, we have the following characterization of the interior and the boundary of $\mathcal{S}(A)$:

$$\begin{aligned} a \in \operatorname{int}(\mathcal{S}(A)) &\Leftrightarrow \mathcal{V}(p_a) \cap [0,1] = \emptyset \\ a \in \operatorname{bd}(\mathcal{S}(A)) &\Leftrightarrow \mathcal{V}(p_a) \cap [0,1] = \{1\} \end{aligned} \tag{2.10}$$

PROOF: Let $a \in \mathbb{R}^n$. Note that $a = 0$ always lies in $\mathcal{S}(A)$ since $A(0) = I_k$ and the identity matrix is positive definite. Moreover, $p_0 \equiv \det(I_k) = 1 \neq 0$ and hence one does not have to consider the case $a = 0$. Therefore, let $a \in \mathbb{R}^n \setminus \{0\}$.

$$\begin{aligned} a \in \mathcal{S}(A) &\Leftrightarrow \text{ all eigenvalues of } I_k + a_1A_1 + \ldots + a_nA_n \text{ are } \geq 0 \\ &\Leftrightarrow \text{ all eigenvalues of } -a_1A_1 - \ldots - a_nA_n \text{ are } \leq 1 \\ &\Leftrightarrow \text{ all roots of } \underbrace{\det(tI + a_1A_1 + \ldots + a_nA_n)}_{=t^k p_a\left(\frac{1}{t}\right)} \text{ are } \leq 1 \\ &\Leftrightarrow \text{ all roots of } p_a\left(\frac{1}{t}\right) \text{ are } \leq 1 \\ &\Leftrightarrow \text{ all roots of } p_a \text{ are } \geq 1 \\ &\Leftrightarrow \mathcal{V}(p_a) \cap [0,1) = \emptyset \end{aligned}$$

Moreover, by LEMMA 2.10, $a \in \operatorname{int}(\mathcal{S})$ if and only if $A(a) > 0$. Hence the above computations can be executed by using the strict inequality sign. This shows that $\mathcal{V}(p_a) \cap [0,1] = \emptyset$. On the other hand, if $a \in \operatorname{bd}(S)$, 0 is an eigenvalue of $A(a)$. Hence 1 is a root of p_a, which implies $\mathcal{V}(p_a) \cap [0,1] = \{1\}$. ■

EXAMPLE 2.18: Let us again consider EXAMPLE 2.15. Let

$$a_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad a_2 := \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad a_3 := \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

In the following graphic $\mathcal{V}(p)$ is drawn in red. The blue lines represent the lines passing through 0 and either a_1, a_2 or a_3. The green dots represent the points a_1, a_2 and a_3. With the help of the above proposition one can test if the three points lie in the spectrahedron. Since $\mathcal{V}(p_{a_1}) \cap [0,1) \neq \emptyset$, it immediately follows that $a_1 \notin \mathcal{S}$. Moreover, $a_2 \in \operatorname{bd}(\mathcal{S})$ due to the fact that $\mathcal{V}(p_{a_2}) \cap [0,1] = \{1\}$. The point a_2 lies in the interior of the spectrahedron since $\mathcal{V}(p_{a_3}) \cap [0,1] = \emptyset$.

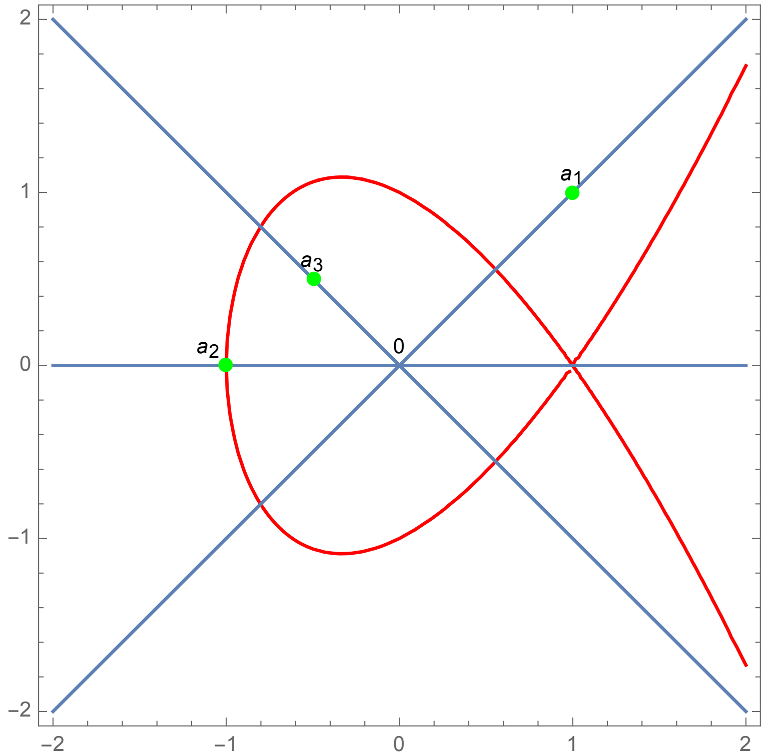


FIGURE 2.4. $a_1 \notin \mathcal{S}, a_2 \in \operatorname{bd}(\mathcal{S}), a_3 \in \operatorname{int}(\mathcal{S})$

♦

REMARK 2.19: Let $S \subseteq \mathbb{R}^n$ be a given subset of $\mathbb{R}^n$. If one wants to solve an optimization problem over this set, it could be an interesting question to check if S is a spectrahedron. This is due to the fact that spectrahedra serve as feasible regions of semidefinite programs. Indeed, there exist efficient algorithms to solve optimization problems over spectrahedra. In general, it is not that easy to verify if a set is a spectrahedron. How might the corresponding symmetric matrices look like? It seems almost impossible to reconstruct the matrices which serve as coefficients of the linear matrix polynomial if one does not have more information about the set S.

To overcome this problem, people tried to characterize spectrahedra with the help of real zero polynomials. As shown above, the determinant of the matrix polynomial of a spectrahedron is RZ and the following holds:

$$\mathcal{S}(A) = \{a \in \mathbb{R}^n : p = \det(A) \text{ does not have any root in } [0, a)\} =: \overline{R(p)}$$

The set on the right hand side is called a *rigidly convex set*. Now let us try to apply our considerations the other way round. Let $p \in \mathbb{R}[\underline{x}]$ be a RZ-polynomial. Does it hold that there exists a linear matrix polynomial A such that

$$\overline{R(p)} \stackrel{?}{=} \mathcal{S}(A).$$

Unfortunately, this conjecture, which is called *Generalized Lax-Conjecture*, has not yet been proven. *Helton* and *Vinnikov* were able to show that the conjecture indeed holds for $n = 2$ (compare with [**NePl+**] for more details). Not much more is known about this problem.

Nevertheless, let us come back to the initial problem. If the conjecture was true, one could try to find a RZ-polynomial which at the first time vanishes on the boundary of S. It seems to be easier to find such a polynomial than searching for some symmetric matrices describing the spectrahedron.

3. The Main Theorem

As the above propositions and examples have shown, one is able to precisely describe spectrahedra with the help of RZ-polynomials. Hence it could be a good idea to use these polynomials in search of a useful basic closed description of spectrahedra. Therefore, again let $\mathcal{S}(A)$ be a spectrahedron given by a monic linear matrix polynomial A with coefficients in $\mathrm{Sym}_k(\mathbb{R})$ and let $p = \det(A)$ be the corresponding RZ-polynomial. Let us start with the first (very naive) attempt:

$$\mathcal{S} \stackrel{?}{=} \mathcal{W}(p) \tag{2.11}$$

One can easily convince oneself that the above equation is only true for very special kinds of spectrahedra. For instance, if we consider EXAMPLE 2.4, the above equation turns out to be true. But in general, the appearance of $\mathcal{W}(p)$ heavily depends on the RZ-polynomial and can be quite complicated. Nevertheless, since $0 \in \mathcal{S}(A)$ and $p(0) = 1 > 0$, one can at least guarantee that the following inclusion holds true:

$$\mathcal{S} \subseteq \mathcal{W}(p) \tag{2.12}$$

Indeed, let $a \in \mathcal{S}(A)$. Assume that $p(a) < 0$. If one restricts to the univariate polynomial $p_a \in \mathbb{R}[t]$, one has

$$p_a(0) > 0, \quad p_a(1) < 0.$$

Hence by the *intermediate value theorem* there exists $b \in (0,1)$ such that $p_a(b) = 0$. This contradicts $\mathcal{V}(p_a) \cap [0,1) = \emptyset$. Hence the inclusion (2.12) holds.

If one wants to find a handy basic closed description of $\mathcal{S}$, one could use the same approach like in the first chapter. Indeed, if one is able to find a polynomial $g \in \mathbb{R}[\underline{x}]$ such that $\mathcal{W}(g)$ "cuts away" the surplus area $\mathcal{W}(p)\backslash\mathcal{S}(A)$, this approach leads to another basic closed description of $\mathcal{S}(A)$:

$$\mathcal{S}(A) = \mathcal{W}(p, g)$$

In the following, we will try to find such a polynomial for special kinds of spectrahedra:

DEFINITION 2.20: *Let $\mathcal{S}(A)$ be a spectrahedron defined by a monic linear matrix polynomial A with coefficients in $\mathrm{Sym}_k(\mathbb{R})$. Moreover, let $a \in \mathrm{bd}(\mathcal{S}(A))$, $p := \det(A)$ and $p_a := p(ta) \in \mathbb{R}[t]$. As shown above, 1 is a root of p_a. Hence one can talk about the multiplicity of the root 1. Let us define the multiplicity of a point $a \in \mathrm{bd}(\mathcal{S}(A))$ as follows:*

$$\mathrm{mult}(a) := \textit{ multiplicity of } 1 \textit{ as a root of } p_a.$$

We call the spectrahedron $\mathcal{S}(A)$ a smooth spectrahedron

$$:\Leftrightarrow \forall a \in \mathrm{bd}(\mathcal{S}(A)) : \mathrm{mult}(a) = 1.$$

EXAMPLE 2.21: The spectrahedron from EXAMPLE 2.15 is not smooth. The multiplicity of $a = (1, 0)^t$ is 2. On the other hand, for every $n \in \mathbb{N}$, the unit circle in $\mathbb{R}^n$ is a smooth spectrahedron. ♦

THEOREM 2.22: (*Main Theorem*)
Let $\mathcal{S}(A)$ be a spectrahedron defined by a monic linear matrix polynomial A with coefficients in $\mathrm{Sym}_k(\mathbb{R})$. Let $p := \det(A)$ be the corresponding RZ-polynomial. Moreover, assume that $\mathcal{S}(A)$ is *bounded* and *smooth*. Then there exists another polynomial $g \in \mathbb{R}[\underline{x}]$ such that

$$\mathcal{S}(A) = \mathcal{W}(p, g).$$

Hence every smooth spectrahedron possesses a basic closed description with two polynomials.

PROOF:
Step 1: Let $a \in \mathrm{bd}(\mathcal{S}(A))$. By the fact that $\mathrm{mult}(a) = 1$ one can find some $\epsilon_a > 0$

such that

$$B_\epsilon(a) \cap \mathcal{S}(A) = B_\epsilon(a) \cap \mathcal{W}(p). \tag{2.13}$$

In other words, the spectrahedron is *locally* defined by the condition $p \geq 0$. This is due to the fact that $\operatorname{mult}(a) = 1$ implies $\nabla p(a) \neq 0$. Hence a change of sign takes place. Let us illustrate this fact with the following draft. The red area of the following set defines a spectrahedron $\mathcal{S}(A)$ (compare with EXAMPLE 2.24). At the boundary of $\mathcal{S}(A)$ it holds that $p = \det(A) = 0$. Nevertheless, the zero set of p looks more complicated. It is represented by the blue curves. The fact $\operatorname{mult}(a) = 1$ for $a \in \operatorname{bd}(\mathcal{S}(A))$ allows the following conclusion: One can find a ball with radius ϵ_a (drawn in green) such that

$$p \geq 0 \text{ on } B_{\epsilon_a}(a) \cap \mathcal{S}(A) \quad \text{ and } \quad p < 0 \text{ on } B_{\epsilon_a}(a) \backslash \mathcal{S}(A).$$

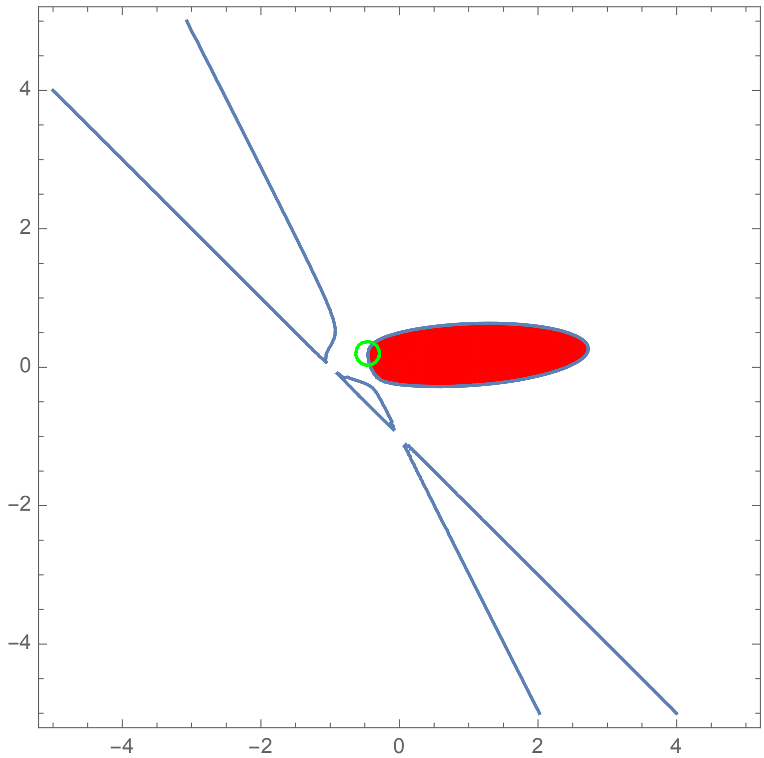


FIGURE 2.5. $\mathcal{W}(p)$ defines $\mathcal{S}(A)$ locally

Let us cover the boundary of $\mathcal{S}(A)$ by

$$\operatorname{bd}(\mathcal{S}(A)) \subseteq \bigcup_{a \in \operatorname{bd}(\mathcal{S}(A))} B_{\frac{\epsilon_a}{2}}(a) \tag{2.14}$$

Since $\mathcal{S}(A)$ is compact, the same holds true for its closed subset $\mathrm{bd}(\mathcal{S}(A))$ and hence one can find a finite subcover of (2.14). More precisely, let $a_1, \dots, a_s \in \mathrm{bd}(\mathcal{S}(A))$ such that

$$\mathrm{bd}(\mathcal{S}(A)) \subseteq \bigcup_{i=1,\dots,s} B_{\frac{\epsilon_{a_i}}{2}}(a_i). \tag{2.15}$$

Now define $\delta := \min\{\epsilon_{a_1}, \dots, \epsilon_{a_s}\}$.

Step 2: By assumption, $\mathcal{S}(A)$ is bounded and non-empty. We have already proven that $\mathcal{S}(A)$ is basic closed semialgebraic. Let $\mathcal{W}(p_1, \dots, p_k) = \mathcal{S}(A)$ be the basic closed description arisen from the coefficients of the characteristic polynomial $\det(tI_k - A)$. Moreover, let $\delta > 0$ be defined as above. With the help of THEOREM 1.32 choose $M \in \mathbb{N}$ and $\epsilon > 0$ such that

$$d_H(\mathcal{S}(A), \mathcal{W}_{M,\epsilon}) < \frac{\delta}{2}, \tag{2.16}$$

where $d_H(\cdot,\cdot)$ denotes the *Hausdorff-distance*. Moreover, with the same numbers $M \in \mathbb{N}$ and $\epsilon > 0$ and the polynomials $p_1, \dots, p_k \in \mathbb{R}[\underline{x}]$ define the polynomial g from LEMMA 1.35, which fulfills the following property:

$$\mathcal{S}(A) \subseteq \mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon} \tag{2.17}$$

Now choose $a \in \mathcal{W}_{M,\epsilon} \backslash \mathcal{S}(A)$. By (2.16) there exists a $y \in \mathrm{bd}(\mathcal{S}(A))$ such that

$$\|y - a\| < \frac{\delta}{2}.$$

Since $\mathrm{bd}(\mathcal{S}(A))$ is covered by (2.15), there exists $i \in \{1, \dots, s\}$ such that

$$\|y - a_i\| < \frac{\epsilon_{a_i}}{2}$$

This implies

$$\|a - a_i\| = \|a - y + y - a_i\| \leq \|a - y\| + \|y - a_i\| < \underbrace{\frac{\delta}{2}}_{\leq \frac{\epsilon_{a_i}}{2}} + \frac{\epsilon_{a_i}}{2} \leq \epsilon_{a_i}$$

Hence one can cover $\mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon}$ by

$$\mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon} \subseteq \bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i) \cup \mathcal{S}(A) \tag{2.18}$$

It is important to emphasize that

$$\mathcal{S}(A) \cap \bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i) = \mathcal{W}(p) \cap \bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i)$$

by (2.13). Notice that

$$\mathcal{W}(p) \cap \left(\bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i) \cup \mathcal{S}(A) \right) = \left(\mathcal{W}(p) \cap \bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i) \right) \cup \left(\underbrace{\mathcal{W}(p) \cap \mathcal{S}(A)}_{=\mathcal{S}(A)=\mathcal{S}(A)\cap\mathcal{S}(A)} \right)$$

$$= \left(\mathcal{S}(A) \cap \bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i) \right) \cup (\mathcal{S}(A) \cap \mathcal{S}(A)) = \mathcal{S}(A) \cap \left(\bigcup_{i=1,\dots,s} B_{\epsilon_{a_i}}(a_i) \cup \mathcal{S}(A) \right)$$

Clearly, the equality holds also true if one restricts to a smaller set (using (2.18)):

$$\mathcal{S}(A) \cap \mathcal{W}(g) = \mathcal{W}(p) \cap \mathcal{W}(g) \tag{2.19}$$

By (2.17),

$$\mathcal{S}(A) \cap \mathcal{W}(g) = \mathcal{S}(A).$$

Moreover,

$$\mathcal{W}(p) \cap \mathcal{W}(g) = \mathcal{W}(p, g)$$

Hence (2.19) leads to the desired result:

$$\mathcal{S}(A) = \mathcal{W}(p, g)$$

■

REMARK 2.23: The statement of the above theorem seems to be quite surprising. It holds for every smooth and bounded spectrahedron – independent of the space dimension and the size of the corresponding matrices. As one can guess taking a look at the first chapter, finding minimal descriptions of polytopes is hard work. This is due to the fact that the vertices of polytopes cause lots of problems. If one takes a look at smooth spectrahedra, it is easy to overcome this problems. Nevertheless, let us emphasize that the idea of the above proof is very similar to *Bernig's* and *Averkov's* approach. The product of all facet-defining linear polynomials of a polytope was an important ingredient for the sought-after basic closed descriptions of polytopes with less polynomials.

It should be clear that the RZ-polynomial $\det(A)$ of a spectrahedron $\mathcal{S}(A)$ is closely related to the just mentioned polynomial. Recall that polytopes are special forms of spectrahedra which arise from a diagonal linear matrix polynomial. Hence the determinant of this polynomial is just the product of all diagonal entries. The polytope $P = \{a \in \mathbb{R}^n : \ l_1(a) \geq 0, \ldots, \ l_m(a) \geq 0\}$ can be rewritten into a spectrahedron $\mathcal{S}(A)$ represented by the linear matrix polynomial

$$A = \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_m \end{pmatrix}.$$

This shows that $\det(A) = l_1 \cdot \ldots \cdot l_m$ leads to the same polynomial as in the first chapter.

Nevertheless, it is easier to deal with RZ-polynomials of smooth spectrahedra than with the product of the facet-defining linear polynomials of a polytope. In this case, the surplus area arisen from $\mathcal{W}(p)$, which does not belong to the spectrahedron, can be easily cut off by another polynomial.

EXAMPLE 2.24: Let

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ A_1 := \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ A_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\mathcal{S}(A) := \mathcal{S}(A_0 + xA_1 + yA_2) \subseteq \mathbb{R}^2$ be spectrahedron. Using the eigenvalue criterion, one can show that there exists a (rather complicated) basic closed description of $\mathcal{S}(A)$ given by 4 polynomials. Let us take a look at the corresponding RZ-polynomial:

$$p = \det(A) = 1{+}4x{+}4x^2{-}x^4{+}4y{+}12xy{+}8x^2y{-}3y^2{-}15xy^2{-}14x^2y^2{-}14y^3{-}23xy^3{-}8y^4$$

In the following graphic one can see the spectrahedron (drawn in red) and $\mathcal{V}(p)$ (drawn in blue).

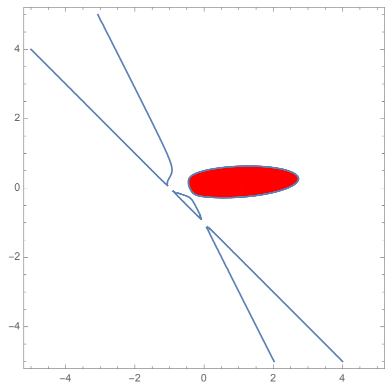


FIGURE 2.6. Visualization of $\mathcal{S}(A)$ and $\mathcal{V}(p)$

The fact that the present spectrahedron is smooth shows that one can find another basic closed description of $\mathcal{S}(A)$ with two polynomials. Let us outline the approach how to find such a description without going into too much detail. At first, one has to find $\epsilon > 0$ and $M \in \mathbb{N}$ such that $\mathcal{W}_{M,\epsilon}$ cuts off the surplus area arisen from the basic closed set $\mathcal{W}(p)$ of the RZ-polynomial. Afterwards, by choosing γ and k in an appropriate way (compare with LEMMA 1.35), one can find another polynomial g with $\mathcal{W}(g) \subseteq \mathcal{W}_{M,\epsilon}$. $\mathcal{W}(p,g)$ is another basic closed description of $\mathcal{S}(A)$.

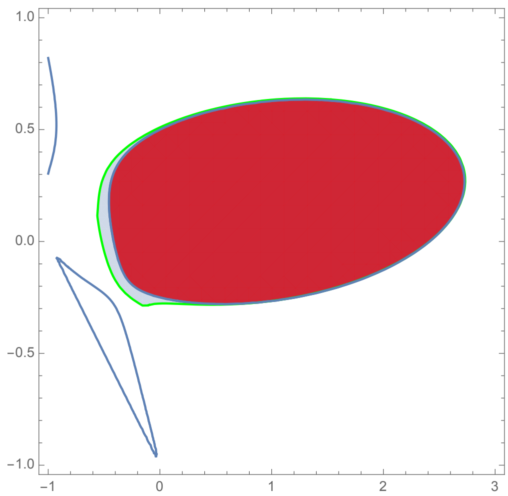


FIGURE 2.7. A possible choice of $W_{M,\epsilon}$ (green)

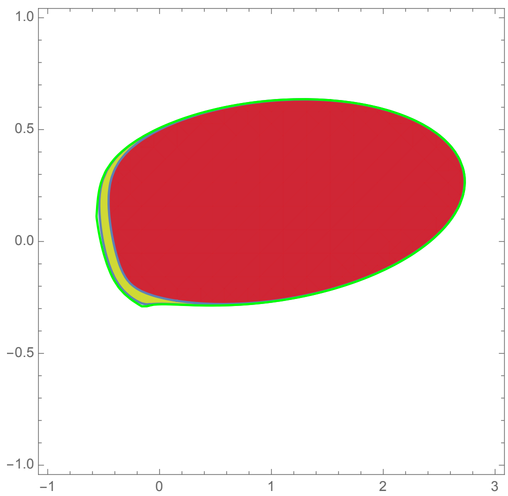


FIGURE 2.8. Construction of $\mathcal{W}(g)$ (yellow)

♦

Conclusion

The main intention of the present master's thesis was to find polynomial descriptions of important classes of basic closed semialgebraic sets with few polynomials. The thesis was orientating on an approach developed by *Averkov*. Given a basic closed semialgebraic set of a very special form, he was able to find a polynomial such that the basic closed set defined by this polynomial approximates the original set sufficiently well. This polynomial has played an important role in search of another basic closed description of the set with less polynomials. By applying his idea to smooth spectrahedra, it was possible to find basic closed descriptions of these sets with only two polynomials.

The search for minimal descriptions of special kinds of basic closed semialgebraic sets seems to be a never ending story, which does not lose its fascination for mathematicians who deal with this topic. Hence, one can look forward to new polynomial descriptions of basic closed sets, which will be certainly discovered in the near future.

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