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Bachelor's Thesis

Universal Embedding Theorems in Combinatorial Game Theory

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12th September, 2023

Contents

1 Introduction

It is probably safe to say that every human plays games from a young age on, many of which are of the combinatorial nature we focus on here. Games like *Chess* and *Go* are even played at world-renowned tournaments. The surprising discovery from a mathematical point of view is that playing games leads to a very intricate, transfinite algebraic structure. Games can be *added*, where the sum of two games is obtained by "playing both games at once". Asking which player has an advantage in a given game induces a *partial order* on games, where the *value* of a game describes which player can win the game at hand, depending on who moves first. Through this, we will see that the games exhibit the structure of a *partially ordered abelian group*, which we will denote by **Pg**.

But the fascinating thing is that **Pg** do not just form any group. The partially ordered abelian group of games (or rather game values) is actually *the biggest* such group, in the sense that every other such group can be found as a subgroup of **Pg**. This is roughly what is meant by saying that the games satisfy a *universal embedding property*.

The collection of games is however not the only structure from combinatorial game theory with such universality properties. Among the vast range of games are hidden some which very much behave like numbers. These *surreal numbers*, which we denote by **No**, include all the familiar real numbers, as well as all the transfinite ordinal numbers encountered in set theory. There is even a suitable definition of multiplication on these numbers, which turns the **No** into an *ordered field*. And again, this ordered field is universally embedding, meaning that every other ordered field is isomorphic to a subfield of **No**. In this sense, the surreals are said to encompass "all numbers great and small".

In order to arrive at these universal embedding theorems, we will first look at the construction and the basic properties of **Pg** and **No** in chapters 2 and 3, respectively. In chapter 4 we will then have a look at the universal embedding properties of **No**, since these are significantly easier to prove than those for **Pg**. The fact that **No** is a universally embedding ordered field was published by John H. Conway (who also discovered the surreals in the first place) in 1976, whereas the corresponding theorem for **Pg** stood as a conjecture until 2002, where a proof was published by Jacob Lurie. We describe this proof in chapter 5. Finally in chapter 6, we show that, under appropriate set theoretic assumptions, **Pg** and **No** are characterized up to isomorphism by their universal embedding properties. We will also use this last chapter as a place to discuss some more technical details omitted in earlier chapters.

Preliminaries

This thesis is aimed at readers who have finished an undergraduate degree in mathematics or are close to doing so. We will give references to suitable textbooks whenever using material that might not be covered in undergraduate courses and, in most cases, also briefly summarize the results used. This also means that, other than being familiar with Zorn's lemma, no deeper knowledge of set theory is necessary to understand everything but some technical details. In particular, we will define ordinal numbers within the surreal numbers and prove some basic properties within this setting without assuming previous knowledge of how ordinals are usually defined in set theory.

The term *proper class* will pop up quite often over the course of this thesis. A reader that is unfamiliar with that notion may think of proper classes simply as collections that are too big to be sets (like, for example, the collection of all sets). How this is formalized does not really concern us here. The terms *class*, or just simply *collection*, are then used to refer to something that is either a set or a proper class. We denote proper classes in boldface (e.g., **Pg**, **No**, **U**) and will from now on capitalize algebraic structures that are proper classes (e.g., **Pg** is a Group, rather than a group).

In the event that a reader really does not want to use proper classes, we give an alternative in section 6.1. We will however stick to proper classes for the main part of the thesis, since this constitutes the most general case.

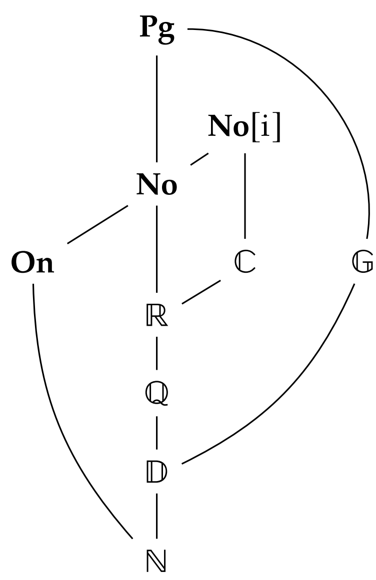


Figure 1.1: Overview of the structures of numbers and games that appear in this thesis.

Pg	game values (def. 2.4.9)
No	surreal numbers (def. 3.0.1)
No[i]	surcomplex numbers (p. 28)
On	ordinal numbers (def. 3.4.1)
$\mathbb{G}$	short game values (def. 6.1.1)
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{Q}$	rational numbers
$\mathbb{D}$	dyadic rationals (fig. 3.2)
$\mathbb{N}$	natural numbers

Combinatorial Game Theory 2

This chapter covers the preliminaries of combinatorial game theory necessary to understand the main topic of this thesis. This kind of game theory, alongside with the surreal numbers, was first introduced in [1]. In [2], the theory was developed further and applied to numerous example games. For a more modern textbook on the topic, see [3]. This chapter loosely follows those sources.

[1]: Conway (2001), *On Numbers and Games*
[2]: Berlekamp et al. (2001), *Winning Ways for Your Mathematical Plays*
[3]: Siegel (2013), *Combinatorial Game Theory*

Combinatorial game theory studies so-called *partisan games*, which are defined by having the following properties:

- Two players (usually called *Left* and *Right*) take alternating turns.
- A player who does not have a legal move loses.
- The game will always end in a finite number of moves, no matter what choices the players make.[1]
- Both players have perfect information about the game at all times. No chance is involved.

1: This restriction is dropped for so-called *loopy games*. However, in this thesis, we only consider loop-free games.

A key part of this definition is that, in contrast to the simpler *impartial games*, we do not demand that both players have the same moves available to them.

As we will see shortly, every partisan game can be described by a neat abstract definition of a game. However, it is often useful for intuition to have some concrete games in mind when studying this theory.

2.1 Domineering and Hackenbush

In *Domineering*, the two players Left and Right alternately place dominoes on a square grid, with the following restrictions:

- Each domino must exactly cover two adjacent cells of the grid.
- Dominoes are not allowed to overlap.
- Left is only allowed to place dominoes vertically, whereas Right is only allowed to place dominoes horizontally.

Another typical combinatorial game is *Hackenbush*. In this game, the playfield consists of one line, called the *ground* (here drawn as a dashed line), as well as several blue, red (here depicted in orange), and green line segments. Each of these segments is either directly connected to the ground, or indirectly connected to the ground by a chain of other segments.

Now, in each of their turns, Left and Right remove ("cut") one of the segments, with the following rules in place:

- Blue lines may only be cut by Left.
- Orange lines may only be cut by Right.
- Green lines may be cut by either player.

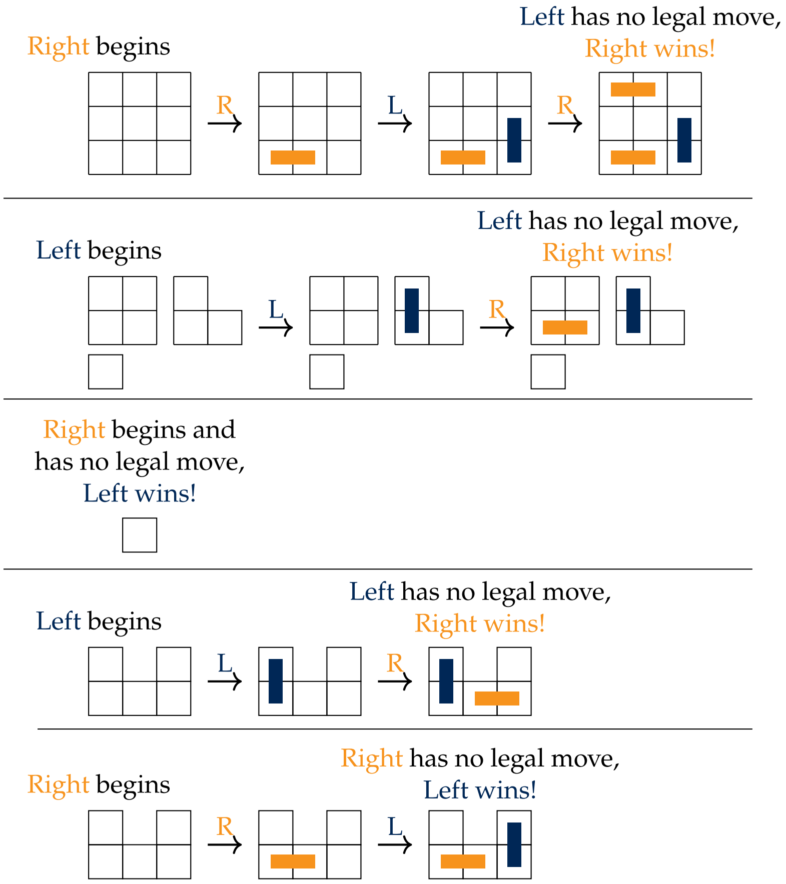


Figure 2.1: Examples of a few games of Domineering with different playfields and different starting players.

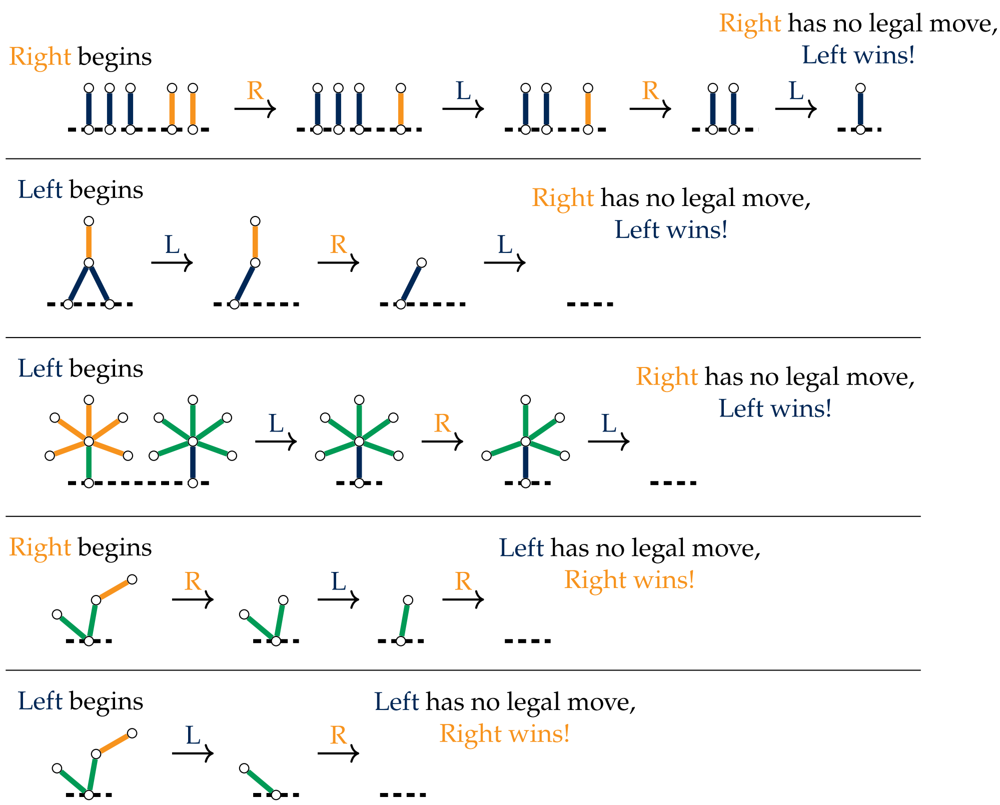
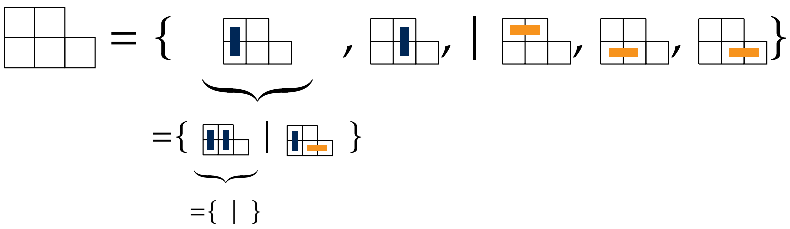


Figure 2.2: Examples of a few games of Hackenbush with different playfields and different starting players.

If some segments are no longer connected to the ground after a cut, they get removed as well (they "fall down").

Notice that in both of these games, making a move is essentially the same thing as making the playfield smaller. The game then continues on this subposition of the previous position. The key concept behind combinatorial game theory is that we mathematically model a game via those subpositions. We describe a game by a pair of two sets, the first being all the subpositions Left can choose, whereas the second set contains all of Right's options.

We illustrate this on the following Domineering position.



As depicted in the example, we can do this recursively and also describe all the subpositions by their subpositions, and so on. Since we demand that partisan games always end in finitely many moves, after finitely many steps, we have to reach $\{\ \mid\ \}$, the game where none of the players have any options to move to.

This concept motivates how we mathematically define a game.

2.2 Abstract Game Definition

Definition 2.2.1 (Game)
A game is a pair $\{L \mid R\}$, where L and R are sets of games.

While this is the shortest and most elegant way to define games, it might be confusing at first. One strange aspect might be that the definition appears to be circular: games are defined in terms of games. However, this turns out not to be a problem, because even by this definition, we have at least one game: $\{\emptyset \mid \emptyset\}$ is a game, since the empty set is a set of games (because every "for all"-statement quantifying over the empty set is trivially true). More briefly, we denote this game by $0 := \{\ \mid\ \}$. It is also called the *empty game* or the *endgame*.

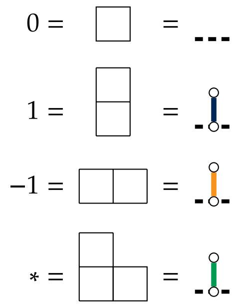


Figure 2.3: The simplest four games and their representation in Domineering and Hackenbush.

Once we have 0, the above definition turns out to be inductive rather than circular, since we can now form the following new games:

$$-1 := \{\ \mid 0\}, \qquad * := \{0 \mid 0\}, \qquad 1 := \{0 \mid\ \}$$

Now we can form the games where the left and right sets consist of 0, 1, -1 and $*$. For example:

$$\{0, 1 \mid\ \}, \qquad \{1 \mid -1\}, \qquad \{* \mid *\}, \qquad \{-1 \mid *\}, \qquad \{\ \mid -1, *\}, \qquad \ldots$$

Here we use a standard notational shorthand by leaving out the curly braces around the sets of left and right options.

This process of forming new games from previously created games can now continue indefinitely. In particular, observe that our definition does not demand the left and right sets to be finite. After having formed all games that can be created from 0 in finitely many steps, we can, for example, form the game that has all of those games in its left set. This means we are actually dealing with *transfinite induction*, where even after infinitely many steps, there is always another step to make. We further discuss games with infinitely many options in section 3.4 on ordinal numbers.

One consequence of the inductive nature of games is that we can also prove basically all properties of games by induction. In order to prove that all games have a certain property, we assume that all the left options g^{L} and right options g^{R} of an arbitrary game G have said property, and then show that this implies that G also has the property. Another way to interpret this is that we show that G has a certain property if all *simpler* games (i.e., games created before G) have this property. Note that we will never have to consider a base case for induction since trivially all games in the empty set will have the desired property, and thus by the inductive step 0 will have the desired property, and so on.

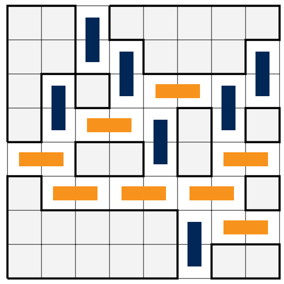


Figure 2.4: After a couple of moves, this game of Domineering turned into a sum of smaller games.

2.3 Arithmetic of Games

When playing Domineering on a large playfield, it often occurs that after a couple moves, the dominoes separate the playfield into multiple disjoint parts (see fig. 2.4). Now for each move, the players have to choose exactly one part in which they want to play, and they loose if they have no move in any of the parts. This is how we define the sum of games.

Definition 2.3.1 (Addition of Games)
For two games G and H, we define their sum as

$$G + H := \{g^{\mathrm{L}} + H, G + h^{\mathrm{L}} \mid g^{\mathrm{R}} + H, G + h^{\mathrm{R}}\}.$$

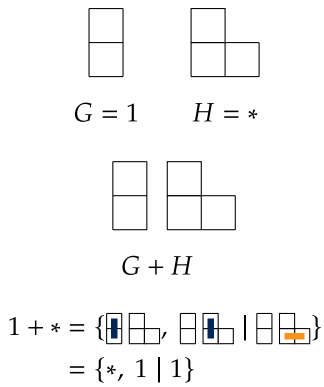


Figure 2.5: The sum of two simple domineering positions.

In definitions like this, g^{L} is to be understood to range over all left options of G, whereas g^{R} ranges over all right options of G, and similarly for h^{L} and h^{R}. This definition is again inductive: the sum of two games is defined in terms of sums where one game is simpler. Definitions like these are called *genetic*.

With the intuition that the sum of two Domineering or Hackenbush positions is just the two positions put next to each other, the following properties of addition should not come as a surprise.

Theorem 2.3.2 (Properties of Addition)
For games G, H, K we have

(a) $G + 0 = G$ (*neutral element*),
(b) $G + H = H + G$ (*commutativity*),
(c) $(G + H) + K = G + (H + K)$ (*associativity*).

Proof. We prove all of these properties by induction. The inductive hypothesis is going to be that the property we are trying to show holds if we replace G, H, or K by one of its left or right options. The spots where the inductive hypothesis is used are marked by I. H.

Neutral element: Since 0 has neither a left or a right option, those terms just do not appear in the sum. With that, we get:

$$G + 0 = \{g^{\mathrm{L}} + 0 \mid g^{\mathrm{R}} + 0\} \overset{\text{I.H.}}{=} \{g^{\mathrm{L}} \mid g^{\mathrm{R}}\} = G.$$

Commutativity:

$$\begin{aligned} G + H &= \{g^{\mathrm{L}} + H, G + h^{\mathrm{L}} \mid g^{\mathrm{R}} + H, G + h^{\mathrm{R}}\} \\ &\overset{\text{I.H.}}{=} \{h^{\mathrm{L}} + G, H + g^{\mathrm{L}} \mid h^{\mathrm{R}} + G, H + g^{\mathrm{R}}\} = H + G. \end{aligned}$$

Associativity:

$$\begin{aligned} (G + H) + K &= \{g^{\mathrm{L}} + H, G + h^{\mathrm{L}} \mid g^{\mathrm{R}} + H, G + h^{\mathrm{R}}\} + K \\ &= \{(g^{\mathrm{L}} + H) + K, (G + h^{\mathrm{L}}) + K, (G + H) + k^{\mathrm{L}} \mid \\ &\qquad \mid (g^{\mathrm{R}} + H) + K, (G + h^{\mathrm{R}}) + K, (G + H) + k^{\mathrm{R}}\} \\ &\overset{\text{I.H.}}{=} \{g^{\mathrm{L}} + (H + K), G + (h^{\mathrm{L}} + K), G + (H + k^{\mathrm{L}}) \mid \\ &\qquad \mid g^{\mathrm{R}} + (H + K), G + (h^{\mathrm{R}} + K), G + (H + k^{\mathrm{R}})\} \\ &= G + \{h^{\mathrm{L}} + K, H + k^{\mathrm{L}} \mid h^{\mathrm{R}} + K, H + k^{\mathrm{R}}\} \\ &= G + (H + K). \end{aligned}$$

□

Games can also be negated. The idea behind this is that we swap Left's and Right's roles in the game, i.e., recursively swapping Left's and Right's options.

Definition 2.3.3 (Negation of Games)
For a game G, we define

$$-G := \{-g^{\mathrm{R}} \mid -g^{\mathrm{L}}\}.$$

In our two example games, negation is simple to conceptualize. In Domineering, negation just means rotating the playfield by 90 degrees, whereas in Hackenbush, negation just means swapping the colors blue and orange.

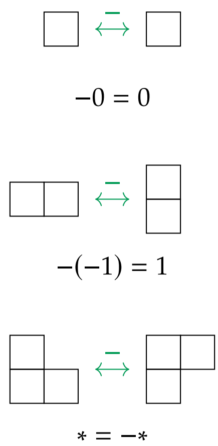


Figure 2.6: Negation of the four simplest games.

Theorem 2.3.4 (Properties of Negation)
For games G and H we have

(a) $-(-G) = G$,
(b) $-(G + H) = -G - H$.

Proof. With induction, we have

$$-(-G) = -\{-g^{\mathrm{R}} \mid -g^{\mathrm{L}}\} = \{-(-g^{\mathrm{L}}) \mid -(-g^{\mathrm{R}})\} \overset{\text{I.H.}}{=} \{g^{\mathrm{L}} \mid g^{\mathrm{R}}\} = G,$$

as well as

$$\begin{aligned}-(G+H) &= -\{g^{\mathrm{L}}+H, G+h^{\mathrm{L}} \mid g^{\mathrm{R}}+H, G+h^{\mathrm{R}}\} = \\ &= \{-(g^{\mathrm{R}}+H), -(G+h^{\mathrm{R}}) \mid -(g^{\mathrm{L}}+H), -(G+h^{\mathrm{L}})\} \\ &\overset{\text{I.H.}}{=} \{(-g^{\mathrm{R}})+(-H), (-G)+(-h^{\mathrm{R}}) \mid (-g^{\mathrm{L}})+(-H), (-G)+(-h^{\mathrm{L}})\} \\ &= (-G)+(-H).\end{aligned}$$

□

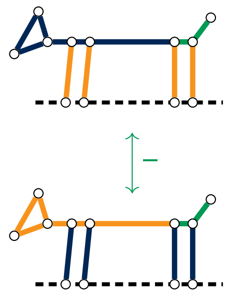


Figure 2.7: Hackenbush-dog, and its evil cousin.

Notice that we do not have $G - G = 0$ yet. For example:

$$1 + (-1) = \{-1 \mid 1\} \neq \{\ \mid\ \} = 0.$$

However, subtraction will behave like the inverse of addition under the right equivalence relation, which will be introduced in the next section.

2.4 Ordering Games

One thing we have not looked at yet is who wins in a given game. This will lead us to a partial order on the games, which intuitively orders the games by how much of an advantage one of the players has.

Definition 2.4.1 (Outcome Classes)
We say a player has a *winning strategy* in a game, if they can always win the game, no matter how their opponent plays. We call a game G

- *positive*, if Left has a winning strategy, no matter who begins.
- *negative*, if Right has a winning strategy, no matter who begins.
- *zero*, if the player who moves second has a winning strategy.
- *fuzzy*, if the player who moves first has a winning strategy.

The four simplest games we mentioned before, 0, 1, −1 and $*$, are zero, positive, negative and fuzzy, respectively.

The following lemma gives a way to determine which outcome class a game belongs to if one already knows the outcome classes of its left and right options. If one finds the above definition by winning strategies not rigorous enough, this could be used as an alternative, inductive definition of the outcome classes.

Lemma 2.4.2 (Genetic Characterization of Outcome)
A game G is

(a) positive, iff there exists a positive or zero left option, and all of its right options are positive or fuzzy.
(b) negative, iff there exists a negative or zero right option, and all of its left options are negative or fuzzy.
(c) zero, iff all left options are negative or fuzzy, and all right options are positive or fuzzy.
(d) fuzzy, iff there exists a positive or zero left option, and there exists a negative or zero left option.

Each game belongs to exactly one of these outcome classes.

Proof. The first four properties are proven simply by looking at all possible cases of who begins and which move this player could make. To prove that an arbitrary game G is either positive, negative, zero or fuzzy, suppose that such is true for all g^{L} and g^{R}, and notice that the cases (a) to (d) cover all possible situations and exclude each other. Thus the statement is true by induction. □

Theorem 2.4.3 (Outcome Classes and Addition)

(a) The sum of two positive games is again positive.
(b) The sum of two negative games is again negative.
(c) Addition of a zero game does not change the outcome class.

Proof.[2] (a) If we have two positive games, that means Left has a winning strategy for both of the games individually. For each move of Right in one of the games, Left can move in the same game according to their winning strategy for that game, and will thus never run out of moves. So Left has a winning strategy in the sum of the two games.
(b) This is proven analogously to the positive case.
(c) Let G be any game, and let Z be a zero game. Now whoever has a winning strategy in G can reply to their opponents move in G by picking a move according to that strategy. If their opponent however moves in Z, they can also move in Z according to the winning strategy for the second player to move. This means the player who has a winning strategy in G will never run out of moves in $G + Z$, and thus also have a winning strategy in this sum. □

2: If one prefers, this can also be proven symbolically rather than combinatorically using induction and lemma 2.4.2.

Theorem 2.4.4 (Outcome Classes and Negation)

(a) The negation of a positive game is negative.
(b) The negation of a negative game is positive.
(c) The negation of a zero game is zero.
(d) The negation of a fuzzy game is fuzzy.
(e) For any game G, the difference $G - G$ is zero.

Proof. The statements (a) to (d) are clear, since negation just swaps the roles of both players (or alternatively by induction using lemma 2.4.2).

As for (e): In the game $G - G$, the second player can always pick the negative of the move that the first player just picked. With this strategy, the second player will never run out of moves, which means $G - G$ is zero. □

With these properties, we see that the outcome classes can be used to order the games.

Definition 2.4.5 (Equivalence and Order)
For games G, H we define

- $G > H$ iff $G - H$ is positive.
- $G < H$ iff $G - H$ is negative.
- $G \equiv H$ iff $G - H$ is zero.[3]
- $G \parallel H$ iff $G - H$ is fuzzy.

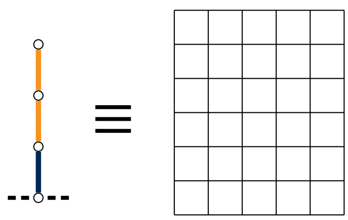


Figure 2.8: A surprising equivalence between Domineering and Hackenbush (cf. [3, p. 13]).

We also define the following combinations:

- $G \geq H$ iff $G - H$ is positive or zero.
- $G \leq H$ iff $G - H$ is negative or zero.
- $G \mathrel{\triangleright\!\!\!|} H$ iff $G - H$ is positive or fuzzy.
- $G \mathrel{|\!\!\!\triangleleft} H$ iff $G - H$ is negative or fuzzy.

For $G \equiv H$ we say that the games G and H are *equivalent*, for $G \parallel H$ we say that G and H are *incomparable*, or *confused* (with each other).

3: Note that we define this in the opposite way than Conway in [1]. In most sources on combinatorial game theory, $=$ is a defined relation and has the same meaning as our equivalence. Then Conway uses the symbol $\equiv$ to denote *identical* games, i.e., games with the same left and right sets. We chose the opposite notation since it is closer to how the symbol $\equiv$ is used in other areas of mathematics. Also we will soon only be talking about equivalence classes of games anyway, where we will be able to write $=$ instead of $\equiv$ again.

We also want to characterize order in a genetic way. This will be used frequently in the following chapters, since it will be easier to analyze games symbolically rather than combinatorically once we get into more abstract territory. Again, this characterization could also be used to inductively define the order on games in the first place.

Lemma 2.4.6 (Genetic Characterization of Order)
For games G and H, we have $G \leq H$ iff there is no g^L with $H \leq g^L$ and no h^R with $h^R \leq G$.[4]

4: This characterization gives us an intuition that is good to keep in mind: an inequality between games holds, *unless there is a reason to the contrary.*

Proof. We have $G \leq H$ iff $G - H \leq 0$, so iff

$$G - H = \{G - h^R, g^L - H \mid G - h^L, g^R - H\}$$

is zero or negative. By lemma 2.4.2, this is equivalent to one of the following two statements being true:

1. $(\forall h^R : G - h^R \mathrel{|\!\!\!\triangleleft} 0) \wedge (\forall g^L : g^L - H \mathrel{|\!\!\!\triangleleft} 0)$, and
   $(\forall h^L : G - h^L \mathrel{\triangleright\!\!\!|} 0) \wedge (\forall g^R : g^R - H \mathrel{\triangleright\!\!\!|} 0)$.
2. $(\exists h^R : G - h^R \geq 0) \vee (\exists g^L : g^L - H \geq 0)$, and
   $(\forall h^L : G - h^L \mathrel{\triangleright\!\!\!|} 0) \wedge (\forall g^R : g^R - H \mathrel{\triangleright\!\!\!|} 0)$.

The respective first lines in both bullet points are logical opposites, so one of them is certainly true. The respective second lines are the same. So in total, we get that $G \leq H$ is equivalent to

$$(\forall h^L : G - h^L \mathrel{\triangleright\!\!\!|} 0) \wedge (\forall g^R : g^R - H \mathrel{\triangleright\!\!\!|} 0),$$

which is in turn equivalent to there being no g^L with $H \leq g^L$ and no h^R with $h^R \leq G$. $\square$

By setting $G = H$, we immediately get the following result:

Corollary 2.4.7 (A game lies "between" its left and right options.)
For any game G, we have $g^L \mathrel{|\!\!\!\triangleleft} G \mathrel{|\!\!\!\triangleleft} g^R$.

Theorem 2.4.8 (Properties of Equivalence and Order)
For games G, H, K, we have

(a) $\equiv$ is an equivalence relation.
(b) $<$ and $>$ are transitive.
(c) $G > H$ iff $H < G$.
(d) If $G > H$ and $K \equiv G$, then $K > H$, and analogously for $<$ and $\parallel$.
(e) If for every g^L and every g^R, there are h^L and h^R, such that $g^L \equiv h^L$ and $g^R \equiv h^R$, and vice-versa, then $G \equiv H$.

(f) $G \equiv H$ iff $G + K \equiv H + K$, and analogously for the other relations.
(g) $G > H$ iff $-G < -H$.
(h) $G \equiv H$ iff $-G \equiv -H$ and $G \parallel H$ iff $-G \parallel -H$.

Proof. To prove all of these claims, we just need to collect results that we have already proven. The reflexivity of $\equiv$ follows from 2.4.4 (e), whereas the symmetry follows from 2.4.4 (c) and 2.3.4. For the transitivity, we assume that $G-H$ and $H-K$ are zero, which gives us that $(G-H)+(H-K)$ is zero using 2.4.3 (c), which by 2.3.2 equals $(G-K)+(H-H)$. Since $H-H$ is zero, $G-K$ is also zero. The transitivity of $<$ and $>$ follows similarly.

The equivalence of $G > H$ and $H < G$ is obtained from 2.4.4 (a) and (b), as well as 2.3.4.

If $K \equiv G$, that means that $K - G$ is zero, so $G - H$ is in the same outcome class as $(G-H)+(K-G) = (K-H)+(G-G)$, which again is in the same outcome class as $K - H$ since $G - G$ is zero. This proves (d).

For (e), consider the game $G - H$, and assume WLOG that Left plays first and picks the move g^{L} in G, resulting in the game $g^{\mathrm{L}} - H$. Then Right has a winning move by picking $-h^{\mathrm{L}}$ in $-H$ such that $h^{\mathrm{L}} \equiv g^{\mathrm{L}}$, resulting in the zero position $g^{\mathrm{L}} - h^{\mathrm{L}}$, which Right can now win. All cases considered, this means that the second player to move always has a winning strategy in $G - H$, and thus $G \equiv H$.

For the equivalence of $G \equiv H$ and $G + K \equiv H + K$, we notice that $G - H$ has the same outcome class as $G - H + (K - K) = (G + K) - (H + K)$ since $K - K$ is zero. The argument for the other relations works exactly the same.

The statements $G > H \Leftrightarrow -G < -H$, $G \equiv H \Leftrightarrow -G \equiv -H$ and $G \parallel H \Leftrightarrow -G \parallel -H$ follow directly from 2.4.4. □

With this, we can define the structure that is central to this thesis.

Definition 2.4.9 (Game Value)
For a game G, we define its *value* as the equivalence class of G under $\equiv$.
If $G = \{L \mid R\}$, we denote the equivalence class of G by $\langle L \mid R \rangle$.
We denote the collection of all game values by **Pg** (for *partisan games*).[5]

We define the arithmetic and order of game values via their underlying games. Theorem 2.4.8 (f), (h) and (d) tell us that addition, negation and order of elements of **Pg** are well-defined. Theorem 2.3.2, together with Theorem 2.4.4 (e) tell us that **Pg** forms an abelian Group.[6] Theorem 2.4.8 also tells us that $\leq$ and $\geq$ are partial order relations (where antisymmetry follows from the fact that the difference of two games cannot be positive and negative at the same time), and specifically from (f) we get addition is compatible with order. Thus, **Pg** forms a *partially ordered abelian Group* (POA Group for short).

5: There is a small problem with this: since the definition of games is so general, the equivalence class of any game is going to be a proper class. Therefore, we technically can not form the class of all of those equivalence classes, since proper classes cannot be members of other classes. Otherwise, one would get similar contradictions as the ones that arise when treating the collection of all sets as a set.
This problem is, however, easily sidestepped by identifying each equivalence class with its unique simplest representative, which is a standard technique in set theory known as *Scott's trick* [4, section 8.6]. Here *simplest* means the game with minimal birthday (def. 3.4.5), where one of course needs to use the standard set theoretic ordinals instead to avoid circular reasoning. For the sake of simplicity, we leave it at that and ignore this technicality in further discussions.

6: Technically **Pg** is not a *group*, since groups are usually defined via sets, whereas **Pg** is a proper class. We denote this distinction by saying that **Pg** is a *Group* (with a capitalized first letter), and not a group. We will do the same for other algebraic structures.

A *partially ordered abelian group* is an abelian group X together with a partial order relation $\leq$ on X, such that $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in X$.

Surreal Numbers | 3

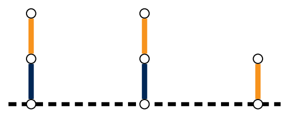
Some games behave a lot like numbers. We already gave this away a bit by naming games 0, 1 and −1, but there are many more. For example:

$$2 = \langle 0, 1 \mid \ \rangle, \quad 3 = \langle 0, 1, 2 \mid \ \rangle, \quad -3 = \langle \ \mid -2, -1, 0 \rangle, \quad \tfrac{1}{2} = \langle 0 \mid 1 \rangle.$$

Those names are all justified by the arithmetic of games. Let us look at the case of $\frac{1}{2}$:

$$\tfrac{1}{2} + \tfrac{1}{2} - 1 = \langle \tfrac{1}{2} \mid 1 + \tfrac{1}{2} \rangle - 1 = \langle \underbrace{\tfrac{1}{2} - 1}_{\langle -1 \mid 0, \frac{1}{2} \rangle} \mid \tfrac{1}{2}, 1 + \tfrac{1}{2} \rangle$$

Now we can use lemma 2.4.2 to first get that $\frac{1}{2}$ is positive (since it has a zero left option and only positive right options), then that $\frac{1}{2} - 1$ is negative (since it has only negative left options and a zero right option), and finally that $\frac{1}{2} + \frac{1}{2} - 1$ is zero (since it has only negative left and only positive right options).

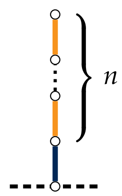


$$\langle 0 \mid 1 \rangle \ + \ \langle 0 \mid 1 \rangle \ + \ (-1) \ = \ 0$$

Figure 3.1: The second player to move can always win this game of Hackenbush, which implies $\langle 0 \mid 1 \rangle = \frac{1}{2}$.

What makes $\langle 0 \mid 1 \rangle$ behave like a number? When thinking of (real) numbers, we usually think of a total rather than a partial order, i.e., we want every number to be comparable with each other (which in particular rules out fuzzy games). For example, in corollary this means 2.4.7, instead of $g^{\mathrm{L}} \lhd G \lhd g^{\mathrm{R}}$ we would rather have $g^{\mathrm{L}} < G < g^{\mathrm{R}}$, which is true for $\langle 0 \mid 1 \rangle$.

This can also be understood nicely using *blue-red Hackenbush*: If we want to eliminate fuzzy games, we can try to just leave out all green lines in Hackenbush. This changes the game a lot, since now every time a player moves, they will be deleting one line in their own color, resulting in a playfield where they now have less of an advantage over their opponent (since even if a lot of their opponent's lines fall down, the line below them is always "worth more"). So left will always have to move to a position g^{L} which is less than G, whereas right will always move to a position g^{R} which is greater than G, giving us $g^{\mathrm{L}} < G < g^{\mathrm{R}}$.



$$\left\langle 0 \,\middle|\, \frac{1}{2^{n-1}}, \ldots, \frac{1}{2}, 1 \right\rangle = \frac{1}{2^n}$$

Figure 3.2: By adding and subtracting games of this form, all dyadic rationals (that is, rational numbers of the form $\frac{m}{2^n}$) can be realized within Hackenbush.

It turns out that it is actually enough to demand $g^{\mathrm{L}} < g^{\mathrm{R}}$. From this $g^{\mathrm{L}} < G < g^{\mathrm{R}}$ will follow inductively.

> **Definition 3.0.1 (Surreal Number)**
> A game $G = \{L \mid R\}$ is called a *form of a number*, if all $g^{\mathrm{L}} \in L$ and all $g^{\mathrm{R}} \in R$ are forms of numbers and satisfy $g^{\mathrm{L}} < g^{\mathrm{R}}$.
> A game value $x \in \mathbf{Pg}$ is called a *surreal number* (or for short just *number*), if x is the equivalence class of a form of a number. We denote the collection of all surreal numbers by **No**.

The games values 0, 1 and −1 are numbers, while $*$ is not. The above $\frac{1}{2}$ is also a number, since $0 < 1$. We commit some abuse of notation here: When writing the usual symbols for numbers (like 0, 1, $\frac{1}{2}$, etc.), we will not distinguish whether we mean numbers (equivalence classes) or games. We will also write numbers as options of numbers, even though strictly speaking the notation $\langle a, b, \ldots \mid x, y, \ldots \rangle$ is only defined when a,

b, etc. are games. This is justified by theorem 2.4.8 (e). When writing x^{L} and x^{R} for possible left and right options of a number x, we will, unless mentioned otherwise, implicitly assume x^{L} and x^{R} to be numbers as well, even though they could be other game values, as in $0 = \langle * \mid * \rangle$.

Theorem 3.0.2 (Properties of Numbers)

(a) For $x \in \mathbf{No}$, we have $x^{\mathrm{L}} < x < x^{\mathrm{R}}$ for all $x^{\mathrm{L}}, x^{\mathrm{R}}$.
(b) The surreal numbers are totally ordered.
(c) For $x, y \in \mathbf{No}$, we have $x \leq y$ iff $x^{\mathrm{L}} < y$ and $x < y^{\mathrm{R}}$ for all x^{L} and y^{R}.
(d) The surreal numbers are closed under addition and negation.

Proof. (a) We will show $x^{\mathrm{L}} < x$, the other inequality can be proven analogously. Assume for induction that x^{L} is greater than all its left options, i.e., $x^{\mathrm{LL}} < x^{\mathrm{L}}$. The value $x - x^{\mathrm{L}}$ has $x^{\mathrm{L}} - x^{\mathrm{L}} = 0$ as a left option, so to prove that $x - x^{\mathrm{L}}$ is positive, it suffices to show that all of its right options are positive or fuzzy (see lemma 2.4.2). Those are either of the form $x^{\mathrm{R}} - x^{\mathrm{L}}$ or of the form $x - x^{\mathrm{LL}}$. The former is positive since x is a number. For the latter we use $x \mathrel{|\!\!\triangleright} x^{\mathrm{L}}$ to get $x - x^{\mathrm{LL}} \mathrel{|\!\!\triangleright} x^{\mathrm{L}} - x^{\mathrm{LL}} > 0$, which can only be true if $x - x^{\mathrm{LL}}$ is positive or fuzzy.
(b) Assume $x \mathrel{\triangleleft\!\!|} y$ for numbers x, y. Then there is a x^{R} with $x^{\mathrm{R}} \leq y$, or there is a y^{L} with $x \leq y^{\mathrm{L}}$ (lemma 2.4.6). Since $x < x^{\mathrm{R}}$ and $y^{\mathrm{L}} < y$ by (a), we get $x < y$ in either case.
(c) This follows immediately from the previous point and lemma 2.4.6.
(d) For numbers x and y, we see that in

$$x + y = \langle x^{\mathrm{L}} + y, x + y^{\mathrm{L}} \mid x^{\mathrm{R}} + y, x + y^{\mathrm{R}} \rangle,$$

all left options are smaller than all the right options by (a). Similarly, $-x = \langle -x^{\mathrm{R}} \mid -x^{\mathrm{L}} \rangle$ is a number, since $x^{\mathrm{L}} < x^{\mathrm{R}}$ gives us $-x^{\mathrm{R}} < -x^{\mathrm{L}}$.

□

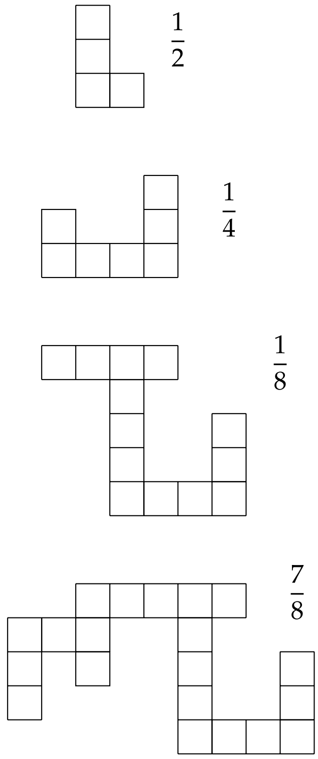


Figure 3.3: Albeit in a much more complicated way, the dyadic rationals can also be represented within Domineering. It can even be done if one requires the playing field to be connected (cf. [5, p. 10–13]).

3.1 Multiplication

In contrast to general game values, the surreal numbers can also be equipped with a sensible multiplication operation. We will first define the product of games and then show that it is well-defined for numbers.[1]

1: The following example shows that there cannot be a multiplication for general game values that has all the expected properties:

$0 = \frac{1}{2} \cdot 0 = \frac{1}{2} \cdot (* + *) = \frac{1}{2} \cdot (1 + 1) \cdot * = *.$

Definition 3.1.1 (Multiplication)
For games G and H, we define their *product* as

$$G \cdot H := \{g^{\mathrm{L}} \cdot H + G \cdot h^{\mathrm{L}} - g^{\mathrm{L}} \cdot h^{\mathrm{L}}, g^{\mathrm{R}} \cdot H + G \cdot h^{\mathrm{R}} - g^{\mathrm{R}} \cdot h^{\mathrm{R}} \mid \mid g^{\mathrm{L}} \cdot H + G \cdot h^{\mathrm{R}} - g^{\mathrm{L}} \cdot h^{\mathrm{R}}, g^{\mathrm{R}} \cdot H + G \cdot h^{\mathrm{L}} - g^{\mathrm{R}} \cdot h^{\mathrm{L}}\}.$$

The idea behind this definition is that for numbers x and y, we have $x^{\mathrm{L}} < x < x^{\mathrm{R}}$ and $y^{\mathrm{L}} < y < y^{\mathrm{R}}$. So if we want to have a multiplication with the usual properties, the following inequalities ought to hold:

$$(x - x^{\mathrm{L}})(y - y^{\mathrm{L}}) > 0 \qquad (x - x^{\mathrm{R}})(y - y^{\mathrm{R}}) > 0$$
$$(x - x^{\mathrm{L}})(y - y^{\mathrm{R}}) < 0 \qquad (x - x^{\mathrm{R}})(y - y^{\mathrm{L}}) < 0$$

These are equivalent to

$$xy > x^{\mathrm{L}}y + xy^{\mathrm{L}} - x^{\mathrm{L}}y^{\mathrm{L}} \qquad xy > x^{\mathrm{R}}y + xy^{\mathrm{R}} - x^{\mathrm{R}}y^{\mathrm{R}}$$
$$xy < x^{\mathrm{L}}y + xy^{\mathrm{R}} - x^{\mathrm{L}}y^{\mathrm{R}} \qquad xy < x^{\mathrm{R}}y + xy^{\mathrm{L}} - x^{\mathrm{R}}y^{\mathrm{L}},$$

which gives us the left and right options for xy.

Theorem 3.1.2 (Properties of Multiplication)
For all games G, H, K, we have:

(a) $G \cdot 0 = 0$
(b) $G \cdot 1 = 1$
(c) $G \cdot H = H \cdot G$
(d) $(G \cdot H) \cdot K \equiv G \cdot (H \cdot K)$
(e) $(G + H) \cdot K \equiv G \cdot K + H \cdot K$

Proof. The first property is true because 0 has no left or right options. The next two properties are immediate by induction.

The last two are also straight-forward proofs by induction, but they only give us equivalence, since the fact that $G - G \equiv 0$ for any game G is used. □

Theorem 3.1.3 (Multiplication of Numbers)
Let x, y, x_1, y_1, x_2, y_2 be forms of numbers. Then:

(a) The product xy is also a form of a number.
(b) If $x_1 \equiv x_2$, then $x_1 y \equiv x_2 y$.
(c) If $x_1 \le x_2$ and $y_1 \le y_2$, then $x_1 y_2 + x_2 y_1 \le x_1 y_1 + x_2 y_2$, where the last inequality is strict if the first two are.
(d) If x and y are positive, then so is xy.

Proof. We prove these claims simultaneously by induction, i.e., we assume for induction that all claims hold when the variables are replaced by any combination of the games x, y, x_1, y_1, x_2, y_2, as long as at least one of these games is replaced by one of its options. We denote the statement of (c) by $P(x_1, x_2 : y_1, y_2)$.

(a) By induction (and since the forms of numbers are closed under addition and negation), all of the options of xy are forms of numbers. So it suffices that all left options of xy are less than all right options of xy. For this we prove

$$x^{\mathrm{L}_1}y + xy^{\mathrm{L}} - x^{\mathrm{L}_1}y^{\mathrm{L}} < x^{\mathrm{L}_2}y + xy^{\mathrm{R}} - x^{\mathrm{L}_2}y^{\mathrm{R}},$$

where x^{L_1} and x^{L_2} are left options of x. The other cases are proven analogously. If $x^{\mathrm{L}_1} \le x^{\mathrm{L}_2}$, we can deduce

$$x^{\mathrm{L}_1}y + xy^{\mathrm{L}} - x^{\mathrm{L}_1}y^{\mathrm{L}} \le x^{\mathrm{L}_2}y + xy^{\mathrm{L}} - x^{\mathrm{L}_2}y^{\mathrm{L}} < x^{\mathrm{L}_2}y + xy^{\mathrm{R}} - x^{\mathrm{L}_2}y^{\mathrm{R}}$$

by using $P(x^{L_1}, x^{L_2} : y^L, y)$ and $P(x^{L_2}, x : y^L, y^R)$. If on the other hand $x^{L_2} \leq x^{L_1}$, then we get

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_1}y + xy^R - x^{L_1}y^R \leq x^{L_2}y + xy^R - x^{L_2}y^R$$

by using $P(x^{L_1}, x : y^L, y^R)$ and $P(x^{L_2}, x^{L_1} : y, y^R)$. So in both cases, the desired inequality is true.

(b) For this we show that x_1y lies between the left/right options of x_2y and vice-versa. Then we get $x_1y \equiv x_2y$ by Theorem 3.0.2 (c). We will only prove

$$x_1^Ly + x_1y^L - x_1^Ly^L < x_2y,$$

the other inequalities are proven analogously. Inductively we know $x_1y^L \equiv x_2y^L$, and

$$x_1^Ly + x_2y^L < x_1^Ly^L + x_2y$$

from $P(x_1^L, x_2 : y^L, y)$. Combining these two facts yields the desired inequality.

(c) If $x_1 \equiv x_2$ or $y_1 \equiv y_2$, the claim follows from (b). Now we consider $x_1 < x_2$ and $y_1 < y_2$. From the former, we get that either there is an x_1^R with $x_1 < x_1^R \leq x_2$, or there is an x_2^L with $x_1 \leq x_2^L < x_2$. In the first case, the desired inequality follows by adding $P(x_1, x_1^R : y_1, y_2)$ and $P(x_1^R, x_2 : y_1, y_2)$ after cancelling like terms. In the second case, we do the same with $P(x_1, x_2^L : y_1, y_2)$ and $P(x_2^L, x_2 : y_1, y_2)$.

(d) This is simply $P(0, x : 0, y)$.

□

This last theorem shows that multiplication is well-defined on **No**, thus making **No** a totally ordered Ring.

A *totally ordered ring* is a commutative ring R together with a total order relation $\leq$ on R, such that for all $a, b, c \in R$:

- If $a \leq b$, then $a + c \leq b + c$.
- If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

3.2 Division

Definition 3.2.1 (Inverse)

For a positive form of a number x, we define its *inverse* as the game y equal to

$$\left\{0, \frac{1 + (x^R - x)y^L}{x^R}, \frac{1 + (x^L - x)y^R}{x^L} \middle| \frac{1 + (x^L - x)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R}\right\},$$

where x^L ranges over all the positive left options of x, and $\frac{1}{x^L}, \frac{1}{x^R}$ are the inverses of x^L and x^R, respectively.

For a negative form of a number x, we define its inverse as the negative of the inverse of $-x$.

This definition is different from the inductive definitions we have encountered before. Not only does the definition of the inverse y of x make use of the "previously constructed" inverses of the options of x, it also makes use of the options of y. This is to be understood as feeding the already constructed options of y back into the definition, and taking the left and right sets of y as all the options y^L and y^R which have been created by this process in finitely many steps.[2]

2: For a more precise definition, we could set $Y_0^L := \{0\}$ and $Y_0^R := \emptyset$, and then recursively define Y_{n+1}^L as the set containing all

$$\frac{1 + (x^R - x)y^L}{x^R} \quad \text{and} \quad \frac{1 + (x^L - x)y^R}{x^L}$$

for $y^L \in Y_n^L$ and $y^R \in Y_n^R$, and similarly define Y_{n+1}^R. Then the inverse of x will be

$$y := \{\textstyle\bigcup_{n\in\mathbb{N}} Y_n^L \mid \bigcup_{n\in\mathbb{N}} Y_n^R\}.$$

However, for what follows, we will work with y as given in 3.2.1 as essentially a shorthand notation for this more rigorous definition.

As an example, let's look at $\frac{1}{3}$. We take the form $x = \{2 \mid \ \}$ of the surreal number 3. Since there is no x^{R}, the definition of y reduces to

$$\left\{0, \frac{1+(x^{\mathrm{L}}-x)y^{\mathrm{R}}}{x^{\mathrm{L}}} \,\middle|\, \frac{1+(x^{\mathrm{L}}-x)y^{\mathrm{L}}}{x^{\mathrm{L}}}\right\} = \left\{0, \frac{1-y^{\mathrm{R}}}{2} \,\middle|\, \frac{1-y^{\mathrm{L}}}{2}\right\}$$

since $x^{\mathrm{L}} - x = 2 - 3 = -1$. From the starting value $y^{\mathrm{L}} = 0$, we get the new right option $\frac{1}{2}(1-0) = \frac{1}{2}$, which in turn gives us the left option $\frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$. This then generates the right option $\frac{1}{2}(1-\frac{1}{4}) = \frac{3}{8}$, and so on, yielding

$$\tfrac{1}{3} = \{0, \tfrac{1}{4}, \tfrac{5}{16}, \ldots \mid \tfrac{1}{2}, \tfrac{3}{8}, \tfrac{11}{32}, \ldots\}.$$

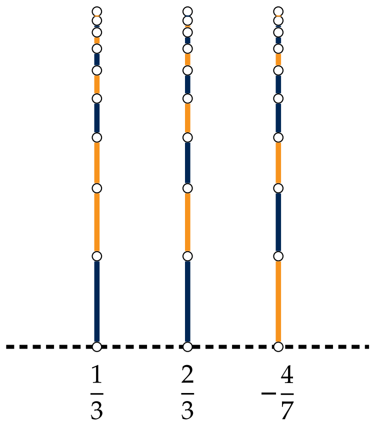


Figure 3.4: Rational numbers, whose denominator is not a power of 2, are not the value of any game with only finitely many options. They can however be represented by games with infinitely many options.

Now we show that this notion of inverse is well-defined for surreal numbers, and actually gives $xy = 1$. We will only consider positive x for this; it is clear that analogous statements follow for negative x.

Theorem 3.2.2 (Inverses of Numbers)
For a positive form of a number x, and y as in definition 3.2.1:

(a) $xy^{\mathrm{L}} < 1 < xy^{\mathrm{R}}$ for all y^{L} and y^{R}.
(b) y is a form of a number
(c) $xy = 1$

Proof. For $y^{\mathrm{L}} = 0$, the inequality from (a) is clear. All other options y'' of y are generated by an option y' through

$$y'' = \frac{1+(x'-x)y'}{x'},$$

where x' is a positive option of x. Multiplying by $(-x)$ and adding 1 gives

$$1 - xy'' = \frac{x' - x - x(x'-x)y'}{x'} = \frac{x'-x}{x'}(1 - xy')$$

Since 0 is a left option of $\frac{1}{x'}$, we have $\frac{1}{x'} > 0$. So $\frac{x'-x}{x'}$ is positive if x' is a right option, and negative if x' is a left option. Assuming that y' satisfies the inequality (a), we get that $1 - xy''$ is negative if x' and y' are both left or both right options, which is the case when y'' is a right option. On the other hand, $1 - xy''$ is positive if x' and y' are different kinds of options, which is the case when y'' is a left option. So (a) is true by induction.

From this we immediately get (b), since $y^{\mathrm{L}} \geq y^{\mathrm{R}}$ would contradict $xy^{\mathrm{L}} < xy^{\mathrm{R}}$ (x is positive).

Since x and y are positive, so is $z := xy$. For (c), it thus suffices to show $z^{\mathrm{L}} < 1 < z^{\mathrm{R}}$ for all $z^{\mathrm{L}}, z^{\mathrm{R}}$ (see theorem 3.0.2 (c)).

An option of $z = xy$ has the form

$$z' = x'y + xy' - x'y',$$

where again x' and y' are options of x and y, respectively. In the case that x' is negative, it must be a left option (since x is positive), and the inequalities reduce to the case of a positive option of x:

Let x^{L_1} be negative and x^{L_2} be positive. If y' is a left option, then z' is also a left option and we get

$$z' = \underbrace{x^{L_1}(y - y')}_{<0} + xy' < \underbrace{x^{L_2}(y - y')}_{>0} + xy',$$

whereas if y' is a right option, then z' is also a right option, and

$$z' = \underbrace{x^{L_1}(y - y')}_{>0} + xy' > \underbrace{x^{L_2}(y - y')}_{<0} + xy'.$$

So it suffices to show the inequalities for positive x'. Assuming $x' \cdot \frac{1}{x'} = 1$, we can multiply y'' as above with x' to get $x'y'' = 1 + x'y' - xy'$, and thus

$$z' = 1 + x'(y - y'').$$

If x' and y' are both left or both right options, then z' is a left option of z and y'' is a right option of y, so $y - y'' > 0$, giving us $z^L < 1$. If x' and y' are different kinds of options, then z' is a right option and y'' is a left option, so $y - y'' < 0$, yielding $z^R > 1$. □

This shows that for every surreal number x (other than zero), there exists a surreal number y such that $xy = 1$. The surreal numbers thus form an *ordered Field*. Since the multiplicative inverse in a field is unique, this implies that our definition of inverse is well-defined in **No**.

An *ordered field* is a field K together with a total order relation $\leq$ on K, such that for all $a, b, c \in K$:

- If $a \leq b$, then $a + c \leq b + c$.
- If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

3.3 Real Numbers

One of the main goals of this thesis is to show that **No** contains every ordered field as a subfield. Here we look at a special case of this, and show that **No** contains all real numbers. This furthers our understanding of **No**, and allows one to make a lot of statements about the structure of **No** (see e.g., [1, chapter 3] on the Conway normal form). In particular, through this, the property that $\mathbb{R}$ is a real closed field can be lifted from $\mathbb{R}$ to **No**, which we will make use of later.

[1]: Conway (2001), *On Numbers and Games*

One way to find the reals within the surreals is to notice that our construction of **No** is very similar to the construction of $\mathbb{R}$ using *Dedekind cuts* (cf. [6, p. 27]). Since **No** is an ordered Field, it must contain an isomorphic copy of the rational numbers, which means that we can replicate Dedekind cuts within **No**. For simplicity, we will not distinguish $\mathbb{Q}$ from its analogue inside **No**.

[6]: Rudin (1976), *Principles of Mathematical Analysis*

Lemma 3.3.1 (Monotone Function is Determined by Dense Set)
Let X, Y be totally ordered sets and $D \subseteq X$ a dense subset (in the sense that for any $a, b \in X$ with $a < b$, there is a $d \in D$ with $a < d < b$). Assume that $f, g \colon X \to Y$ are monotone functions such that $f(d) = g(d)$ for all $d \in D$, and $f(D) = g(D)$ is a dense subset in Y. Then $f = g$.

Proof. WLOG we consider f and g to be increasing. Assume, for the sake of contradiction, that $f \neq g$, so WLOG $f(x) < g(x)$ for an $x \in X$. Then, since $f(D)$ is dense in Y, there is a $d \in D$ with $f(x) < f(d) < g(x)$. Then we have $d > x$, since the opposite would imply $f(d) \leq f(x)$. But this on the other hand implies $f(d) = g(d) \geq g(x)$, a contradiction. □

Theorem 3.3.2 (The surreals contain the reals.)
Define for $x \in \mathbb{R}$ as well as for $x \in \mathbf{No}$:

$$L(x) := \{q : q \in \mathbb{Q}, q < x\} \qquad R(x) := \{q : q \in \mathbb{Q}, q > x\}.$$

Let $\iota \colon \mathbb{R} \to \mathbf{No} \colon x \mapsto \langle L(x) \mid R(x) \rangle$. Then:

(a) ι is strictly increasing.
(b) $z \in \iota(\mathbb{R})$, iff there is an $n \in \mathbb{N}$ with $-n < z < n$ and

$$z = \left\langle z-1, z-\frac{1}{2}, z-\frac{1}{3}, \ldots \middle| z+1, z+\frac{1}{2}, z+\frac{1}{3}, \ldots \right\rangle.$$

(c) $\iota(\mathbb{R})$ is closed under addition and multiplication.
(d) ι is a field homomorphism.

Proof. For real numbers $x < y$, pick a rational number q with $x < q < y$. Then $q \in R(x)$ and $q \in L(y)$, so $\iota(x) < q < \iota(y)$. This proves that ι is increasing.

Now we show (b). For $x \in \mathbb{R}$, there is of course an $n \in \mathbb{N}$ with $-n < x < n$. Thus $-n \in L(x)$ and $n \in R(x)$, so $-n < \iota(x) < n$. We define

$$z = \left\langle \iota(x)-1, \iota(x)-\frac{1}{2}, \iota(x)-\frac{1}{3}, \ldots \middle| \iota(x)+1, \iota(x)+\frac{1}{2}, \iota(x)+\frac{1}{3}, \ldots \right\rangle$$

and show $z = \iota(x)$ by proving $z \leq \iota(x)$ and $z \geq \iota(x)$ using 3.0.2 (c). Obviously the left options of z are less than $\iota(x)$. Now a right option of $\iota(x)$ is q for a rational $q > x$. We can pick an $m \in \mathbb{N}$ such that also $q - \frac{1}{m} > x$. Therefore $q > \iota(x) + \frac{1}{m}$, which is a right option of z, showing that the right option q of $\iota(x)$ is greater than z. This proves $\iota(x) \geq z$, the other inequality is proven analogously.

For the other direction, let z be a surreal number of the form given in (b). Since $-n < z < n$, both $L(z)$ and $R(z)$ are non-empty, and we have $l < r$ for all $l \in L(z)$, $r \in R(z)$. For $l \in L$, $l < z$ means that one of the following cases occurs:

- There is a z^{L} with $l \leq z^{\mathrm{L}}$, i.e., $l \leq z - \frac{1}{m}$ for some natural number m. So $l + \frac{1}{2m}$ is still less than z and thus an element of $L(z)$.
- There is an l^{R} with $l^{\mathrm{R}} \leq z$. From the definitions of the product and the inverse, it is easy to see that every rational number has a form where all of its options are also rational numbers. Since inequalities between game values are all independent of the chosen form, we can take l^{R} to be a rational less than or equal to z. Then $l < l^{\mathrm{R}}$ implies that $\frac{1}{2}(l + l^{\mathrm{R}})$ is a rational which is strictly less than z, and thus an element of $L(x)$.

In either case, we have found a rational number greater than l which is still in $L(x)$. This shows that $L(z)$ has no largest element, and similarly $R(z)$ has no smallest element, which means there is a unique real number x such that $l < x < r$ for all $l \in L(z)$, $r \in R(z)$. With this we again show

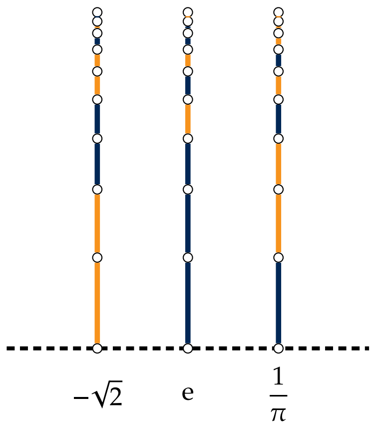


Figure 3.5: All real numbers can be represented as an infinitely tall Hackenbush stalk. This is done via the so-called *sign expansion* of a surreal number, which is related to the binary expansion of real numbers (see [1, p. 30–31, 89–91]).

$z \leq \iota(x)$ and $z \geq \iota(x)$ using 3.0.2 (c). The left options of $\iota(x)$ are exactly the members of $L(z)$, so less than z. A right option of z is of the form $z + \frac{1}{m}$ for a natural number m. We can pick a $q \in R(z)$ that is less than $z + \frac{1}{m}$, giving us $z + \frac{1}{m} > q > x$. This shows $z \geq \iota(x)$. The other inequality is analogous, concluding the proof of (b).

With this form, it becomes simple to show that $\iota(\mathbb{R})$ is closed under addition and multiplication. For $a, b \in \iota(\mathbb{R})$, a left option a^{L} looks like $a - \frac{1}{n}$, which means the left options of $a + b$ are of the form $a + b - \frac{1}{n}$. The same holds for the right options, showing that

$$a + b = \left\langle a + b - 1, a + b - \frac{1}{2}, \ldots \middle| a + b + 1, a + b + \frac{1}{2}, \ldots \right\rangle \in \iota(\mathbb{R}).$$

For multiplication, take $a^{\mathrm{L}} = a - \frac{1}{n}$ and $b^{\mathrm{L}} = b - \frac{1}{m}$. Then the corresponding left option of ab is

$$a^{\mathrm{L}}b + ab^{\mathrm{L}} - a^{\mathrm{L}}b^{\mathrm{L}} = ab - (a - a^{\mathrm{L}})(a - b^{\mathrm{L}}) = ab - \frac{1}{mn}.$$

The other options of ab have analogous expressions, which shows that also $ab \in \iota(\mathbb{R})$.

To prove $\iota(x + y) = \iota(x) + \iota(y)$ for all real numbers x and y, notice that this can be verified directly when x and y are rationals. Now for an arbitrary rational q, define the function $f_q \colon \mathbb{R} \to \mathbf{No} \colon x \mapsto \iota(x) + \iota(q)$. Since $\iota(\mathbb{R})$ is closed under addition, the image of f_q is contained in $\iota(\mathbb{R})$. Now the function $g_q \colon \mathbb{R} \to \mathbf{No} \colon x \mapsto \iota(x + q)$ has image $\iota(\mathbb{R})$ and agrees with f_q on $\mathbb{Q}$. Since f and g are monotone and $\mathbb{Q}$ is dense in $\mathbb{R}$, the image $f_q(\mathbb{Q}) = g_q(\mathbb{Q})$ is also dense in $\iota(\mathbb{R})$. Therefore we have $f_q = g_q$ by lemma 3.3.1.

Now, for all $x \in \mathbb{R}$, we can use the same argument on the functions $f_x \colon \mathbb{R} \to \mathbf{No} \colon y \mapsto \iota(x) + \iota(y)$ and $g_x \colon \mathbb{R} \to \mathbf{No} \colon y \mapsto \iota(x + y)$ to get $f_x = g_x$, since we have already shown that they are the same for rational y. This proves $\iota(x + y) = \iota(x) + \iota(y)$. The same reasoning works for multiplication, which concludes the proof. □

We have therefore shown that **No** contains a subfield which is isomorphic to $\mathbb{R}$. As is common, we will from now on no longer distinguish $\mathbb{R}$ from its isomorphic copy in **No**, and simply regard $\mathbb{R}$ as a subfield of **No**.

3.4 Ordinal Numbers

Beside the reals, the surreal numbers also include another important class of numbers: the *ordinal numbers*. These were first introduced by Georg Cantor and are usually regarded as a part of set theory. Here we will however introduce the ordinals as they are represented in **No**,[3] so no prerequisites are needed.

3: One important thing to note when discussing ordinal numbers as elements of **No** is, that the addition and multiplication of games is not the same as the usual (non-commutative) addition and multiplication of ordinals in the context of well-ordered sets. Here, + and · will always be the addition/multiplication of games as we defined it earlier (which are sometimes referred to as the *natural* sum/product of ordinals).

Definition 3.4.1 (Ordinal Number)
A game value $x \in \mathbf{Pg}$ is called an *ordinal number*, if it can be written as

$$x = \langle L \mid \ \rangle$$

for some set of games L. We denote the collection of all game values by $\mathbf{On}$.

In some sense, the ordinals are to $\mathbf{No}$ what the naturals are to $\mathbb{R}$. In particular, the ordinals contain the naturals:

$$0 = \langle \ \mid \ \rangle, \quad 1 = \langle 0 \mid \ \rangle, \quad 2 = \langle 0, 1 \mid \ \rangle, \quad 3 = \langle 0, 1, 2 \mid \ \rangle, \quad \ldots$$

But aside from these finite numbers, the ordinals contain many more infinitely large numbers, the smallest of which being

$$\omega := \langle 0, 1, 2, 3, \cdots \mid \ \rangle,$$

and then also

$$\omega + 1 = \langle 0, 1, 2, \ldots, \omega \mid \ \rangle, \quad \omega + 2 = \langle 0, 1, 2 \ldots, \omega, \omega + 1 \mid \ \rangle, \ldots$$

as well as

$$2\omega = \langle \omega, \omega + 1, \omega + 2, \ldots \mid \ \rangle \quad \text{and} \quad \omega^2 = \langle \omega, 2\omega, 3\omega, \ldots \mid \ \rangle,$$

and many more.

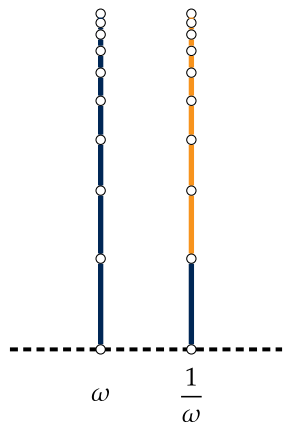


Figure 3.6: Games with infinitely many options allow for infinite as well as infinitesimal surreal numbers.

Theorem 3.4.2 (Properties of Ordinals)

(a) For $x \in \mathbf{Pg}$, the collection $\mathbf{On}_{<x} := \{\gamma \in \mathbf{On} \mid \gamma < x\}$ is a set.
(b) $\alpha = \langle \mathbf{On}_{<\alpha} \mid \ \rangle$ for all ordinals α.[4]
(c) All ordinals are surreal numbers.
(d) $\mathbf{On}$ is closed under addition and multiplication.
(e) The ordinals are well-ordered by $<$ (i.e., every non-empty class $C \subseteq \mathbf{On}$ has a smallest element).
(f) If $S \subset \mathbf{Pg}$ is a nonempty *set* of game values, there is an ordinal that is greater than all elements of S.

4: This provides the analogy to how ordinals are usually defined in set theory. There an ordinal ends up being the set of all smaller ordinals. For example:

$$0 = \emptyset$$
$$1 = \{0\}$$
$$2 = \{0, 1\}$$
$$3 = \{0, 1, 2\}$$
$$\omega = \{0, 1, 2, 3, \ldots\} = \mathbb{N}$$

So informally speaking, one obtain the set theoretic ordinal from corresponding surreal number by writing $\{-\}$ instead of $\langle - \mid \ \rangle$.

Proof. (a) Write $x = \langle L \mid R \rangle$. Assume for induction that $\mathbf{On}_{<x^{\mathrm{L}}}$ is a set for all $x^{\mathrm{L}} \in L$. The inequality $\gamma < x$ implies $\gamma \leq x^{\mathrm{L}}$ for some $x^{\mathrm{L}} \in L$ by lemma 2.4.6. So we have

$$\mathbf{On}_{<x} \subseteq L \cup \bigcup_{x^{\mathrm{L}} \in L} \mathbf{On}_{<x^{\mathrm{L}}}.$$

The union of a set of sets is again a set, and so $\mathbf{On}_{<x}$ is a set as a subclass of a set.

(b) Since $\mathbf{On}_{<\alpha}$ is a set, $\beta := \langle \mathbf{On}_{<\alpha} \mid \ \rangle$ is a game value. So we have $\beta^{\mathrm{L}} \lhd\!\!\mid \alpha$ for all β^{L} by 2.4.7. Since there is no β^{R}, we get $\beta \leq \alpha$ by 2.4.6. If β was less than α, we would get $\beta \in \mathbf{On}_{<\alpha}$. But $\beta^{\mathrm{L}} \lhd\!\!\mid \beta$ for all $\beta^{\mathrm{L}} \in \mathbf{On}_{<\alpha}$, which leads to $\beta \lhd\!\!\mid \beta$ a contradiction. So we must have $\beta = \alpha$.

(c) Assume for induction that all ordinals less than an ordinal α are numbers. Then (b) tells us that α has the form $\langle \mathbf{On}_{<\alpha} \mid \ \rangle$. All the left options of this form are numbers, and since the set of right options is empty, the inequality condition for a game value to be a number is trivially fulfilled. Therefore α is a number.

(d) It is clear from the definitions that, if there are no α^{R} and no β^{R}, then the right sets of $\alpha + \beta$ and $\alpha \cdot \beta$ are also empty.

(e) Let $L := \bigcap_{\gamma \in C} \mathbf{On}_{<\gamma}$. As an intersection of sets, L itself is a set, which means $\delta := \langle L \mid \ \rangle$ is a number. Since all $\delta^{\mathrm{L}} < \gamma$ for all $\gamma \in C$, and there are no δ^{R}, we again have $\delta \leq \gamma$ for all $\gamma \in C$ by 2.4.6. If δ was less than every $\gamma \in C$, we would have the contradiction $\delta = \langle L \mid \ \rangle \in L$. So δ must be the minimum of C, which shows that said minimum exists.

(f) Assume for induction that there is an ordinal α which is greater than all s^{L} and s^{R}, for all $s \in S$. Since there is no α^{R}, we once more get $s \leq \alpha$ by 2.4.6 for all $s \in S$. Therefore $\alpha + 1$ is an ordinal with the desired properties.

$\square$

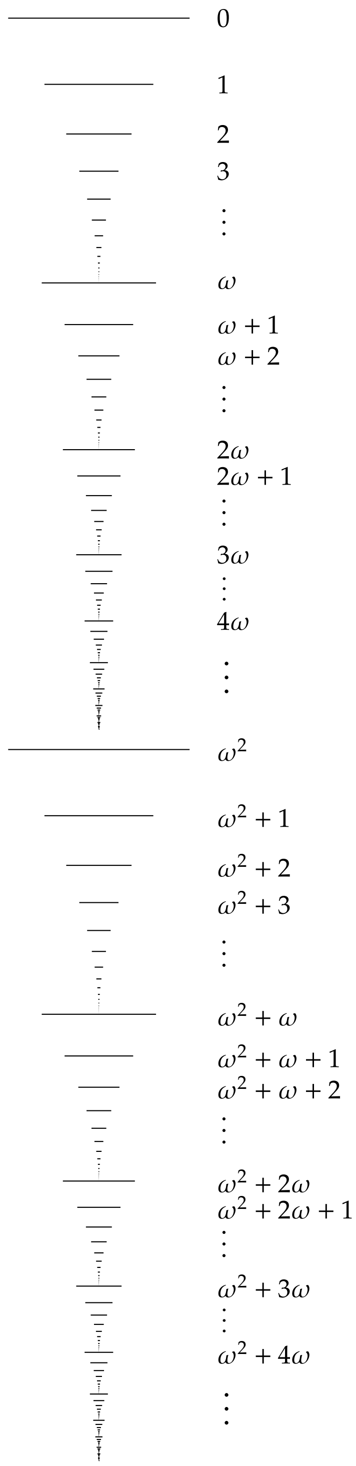


Figure 3.7: The first few ordinals.

Theorem 3.4.3 (Proper Classes)
The collections **On**, **No** and **Pg** are proper classes.

Proof. If **On** was a set, then $\langle \mathbf{On} \mid \ \rangle$ would be an ordinal that is greater than all ordinals — contradiction. So **On** must be a proper class. Since $\mathbf{On} \subset \mathbf{No} \subset \mathbf{Pg}$, the latter two collections must be proper classes as well. $\square$

Ordinals basically come in two kinds: those that can be reached from smaller ordinals simply by counting up, and those that can be seen as the "limits" of this counting.

Definition 3.4.4 (Successor and Limit Ordinal)
An ordinal α is called a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$. An ordinal that is not a successor ordinal and not 0 is called a *limit ordinal*.

As we have already seen, facts about ordinals can be proven by induction on games. Since any ordinal can be written as $\alpha = \langle \mathbf{On}_{<\alpha} \mid \ \rangle$, this specifically means the following: If all ordinals less than α having a certain property implies that α also has this property, then the property holds for all ordinals. However, it is often more useful to tackle the *successor stages* and *limit stages* individually. This means we can prove a statement holds for all ordinals by proving the following:

- The statement holds for 0.
- If the statement holds for α, it also holds for $\alpha + 1$ (*successor stage*).
- If the statement holds for all ordinals less than a limit ordinal λ, then it also holds for λ.

This procedure is particularly useful for constructing transfinite sequences of objects $(X_\alpha)_{\alpha \in \mathbf{On}}$, where most of the time, limit stages will be dealt with by taking the union of all previously constructed objects. Such constructions will appear in chapter 6.

Using ordinals, we can also formalize what we mean by the *simplest* game with a certain property. It is the game that is "born first".

Definition 3.4.5 (Birthday)
Define sets

$$M_\alpha = \left\{ \{L \mid R\} : L, R \subseteq \bigcup_{\beta<\alpha} M_\beta \right\}$$

for all $\alpha \in \mathbf{On}$. Then the *birthday* of a game G is defined as the smallest $\alpha \in \mathbf{On}$ such that $G \in M_\alpha$. The birthday of a game value $x \in \mathbf{Pg}$ is defined as the smallest $\alpha \in \mathbf{On}$ such that there is a game in the equivalence class x which has birthday α.

Such a smallest ordinal always exists by 3.4.2 (b). It is easy to see that the birthday of an ordinal α is just α itself. The concept of birthday is analogous to the concept of *rank* from set theory, where instead of the sets M_α as above, one looks at the cumulative hierarchy of sets (see [4, p. 192] or [7, p. 257]).

[4]: Forster (2003), *Logic, Induction and Sets*

[7]: Hrbacek et al. (1999), *Introduction to Set Theory*

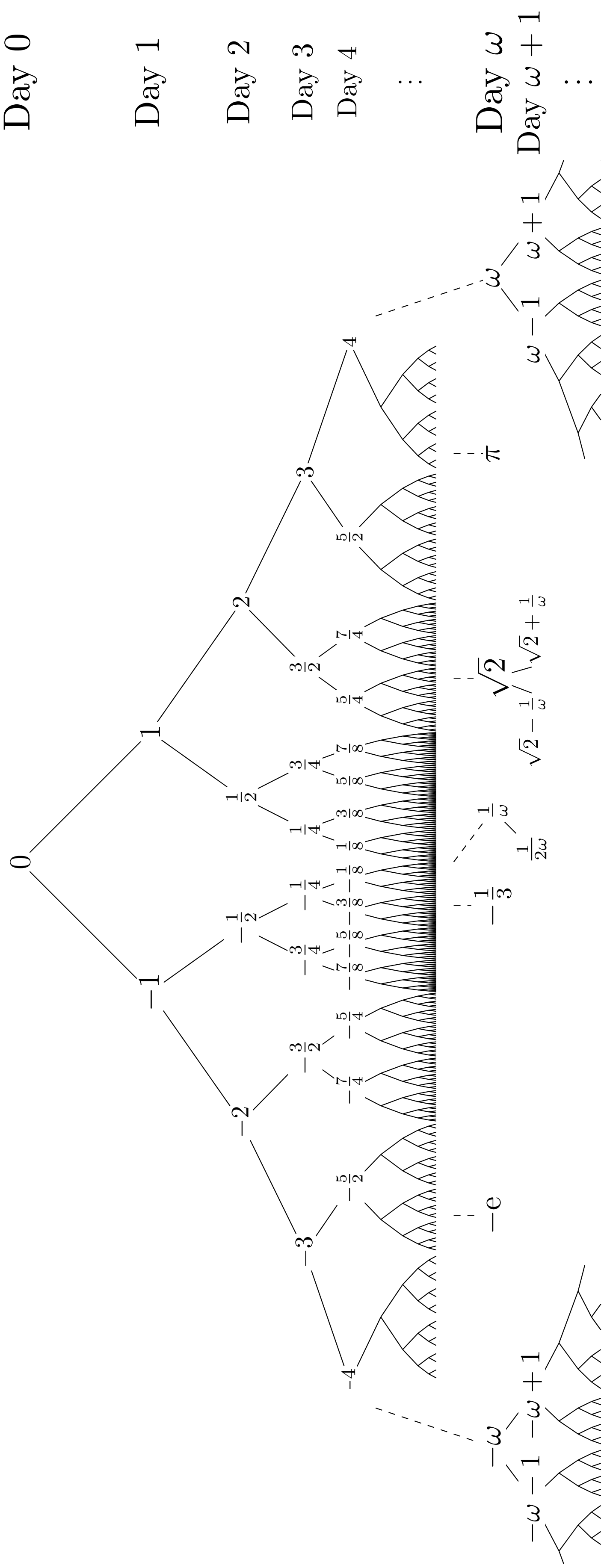


Figure 3.8: The tree of surreal numbers.

4 Universal Embedding Properties of the Surreal Numbers

The construction of the partisan games is basically as general as set theory, upon which almost all of modern mathematics is built. A consequence of this is that **Pg** and **No** have multiple universality properties: several common mathematical structures can be found within the games or the surreal numbers.

Fix a certain algebraic structure (for example "groups" or "ordered fields"). Basically, what we care about is a class **U** that has this structure, as well as the property that every *set* with this structure can be *embedded* into **U**. An embedding here is an injection that preserves the structure. In the example of groups, embeddings are injective group homomorphisms. In the example of ordered fields, they are strictly increasing field homomorphisms.

However, it would be even nicer if we could embed two sets in a way that the two embeddings are *compatible* with each other. We say that a class **U** is *universally embedding* for an algebraic structure if the following holds: If $X \subseteq Y$[1] are *sets* with this structure, and there is an embedding $\varphi\colon X \to \mathbf{U}$, then φ can be extended to an embedding $\psi\colon Y \to \mathbf{U}$ with $\psi|_X = \varphi$, i.e., $\varphi(x) = \psi(x)$ for all $x \in X$.[2]

1: When writing $X \subseteq Y$, we implicitly assume X and Y to have compatible structure. E.g., when X and Y are ordered fields, this means that X is a subfield of Y, and that the order relations on X is just the restriction of the order on Y to X (i.e., that the positive elements of X are also positive in Y).

2: These notions can be made more rigorous by using the language of model theory. Since this is not very important for understanding the universal embedding theorems in this thesis, we omit these precise definitions here and refer interested readers to [8].

The universal embedding property of a certain structure implies that every such structure can be embedded into **U**, as long as there is a *prime object* for the structure, i.e., an object that can be embedded into any other object with this structure. In the example of groups, this would be the trivial group, the group that only has one element. In the example of ordered fields, the prime object would be the rational numbers. If **U** is universally embedding and P is a prime object, we can take $X = P$ to see that every object Y can be embedded into **U**. In the cases we consider, there is always going to be such a prime object, so for us, universal embedding properties will always be stronger than being able to embed every set-sized object.

$$\begin{tikzcd}
Y \arrow[dr, hook, "\exists\psi"] & \\
 & \mathbf{U} \\
X \arrow[uu, hook] \arrow[ur, hook, "\phi"] &
\end{tikzcd}$$

Figure 4.1: A commutative diagram illustrating universal embedding properties. Assuming that the embeddings depicted by blue arrows exist, **U** being universally embedding means that there is an embedding ψ, depicted by an orange arrow, such that the diagram commutes.

Speaking of "set-sized": It is essential here that X and Y are sets, whereas **U** will usually be a proper class. If we allowed X and Y to be proper classes as well, we could take $X = \mathbf{U}$ and enlarge this to a bigger object Y, which cannot be embedded into **U**. If one wants to avoid proper classes, some other dichotomy of "small" and "large" objects than sets vs. proper classes can be used instead, e.g., countable vs. uncountable. We discuss this more in section 6.1 (p. 45).

We will first cover universal embedding properties of **No**, since these are much easier to prove than their analogues for **Pg**. The most important such property is that **No** forms a universally embedding ordered Field. However, to get a feel for how to prove such properties, we will first look at the simpler case of totally ordered sets, ignoring the field structure for now.

4.1 Totally Ordered Sets

To prove a universal embedding theorem, we will (unless it follows immediately from a previously proven theorem) argue in the following way:

1. Using Zorn's lemma, we will reduce the problem to showing that we can extend the embedding $\varphi\colon X \to \mathbf{U}$ to one additional element $s \in Y \setminus X$.
2. We then construct a game value/surreal number s' which behaves in the same way as s (under the lens of the currently considered structure). This allows us to extend φ by setting $\varphi(s) = s'$.

The Zorn's lemma argument is always the same. We will describe it in full detail in the proof of the following theorem, and then just refer to this in future uses. The second part is where the real work happens, and what differs between the different embedding properties. In the case of totally ordered sets, it is relatively straight-forward.

Theorem 4.1.1 (No has a universally embedding total order.) Let $X \subseteq Y$ be totally ordered sets. If there is an embedding $\varphi\colon X \to \mathbf{No}$, then there exists an embedding $\psi\colon Y \to \mathbf{No}$ such that $\psi|_X = \varphi$.

Proof. In this context, an embedding is a strictly increasing function. Let $\mathbf{\Phi}$ be the collection of all partial extensions of φ:

$$\mathbf{\Phi} := \{\psi\colon Z \to \mathbf{No} : X \subseteq Z \subseteq Y, \psi|_X = \varphi, \psi \text{ embedding}\}.$$

Note that $\mathbf{\Phi}$ is non-empty, since $\varphi \in \mathbf{\Phi}$. There is a partial order on $\mathbf{\Phi}$, given by $\psi \preceq \psi'$ iff ψ' extends ψ, which means for $\psi\colon Z \to \mathbf{No}$ and $\psi'\colon Z' \to \mathbf{No}$ that

$$\psi \preceq \psi' \quad :\iff \quad Z \subseteq Z' \quad \text{and} \quad \psi'|_Z = \psi.$$

$$\begin{tikzcd}
Y & \\
Z' \arrow[u, hook] \arrow[dr, "\psi'"] & \\
& \mathbf{No} \\
Z \arrow[uu, hook] \arrow[ur, "\psi"] & \\
X \arrow[u, hook] \arrow[uur, "\varphi"'] &
\end{tikzcd}$$

Figure 4.2: We have $\psi \preceq \psi'$ iff this diagram commutes.

Now let $C \subseteq \mathbf{\Phi}$ be a chain, and write $C = \{\psi_i\colon Z_i \to \mathbf{No} : i \in I\}$. Set $\overline{Z} := \bigcup_{i\in I} Z_i$. Since C is a chain, for two functions $\psi, \psi' \in C$ we have either $\psi \preceq \psi'$ or $\psi' \preceq \psi$, meaning that in either case, the two functions agree on all elements on which they are both defined. This means we can define

$$\begin{aligned} \overline{\psi}\colon \overline{Z} &\to \mathbf{No} \\ z &\mapsto \psi_i(z) \quad \text{for } i \in I \text{ such that } z \in Z_i. \end{aligned}$$

With this, we have $\psi \preceq \overline{\psi}$ for all $\psi \in C$. So we have shown that every chain $C \subseteq \mathbf{\Phi}$ has an upper bound $\overline{\psi} \in \mathbf{\Phi}$, which means that by Zorn's lemma,[3] $\mathbf{\Phi}$ has a maximal element $\psi\colon Z \to \mathbf{No}$.

3: There is a slight problem here since $\mathbf{\Phi}$ is a proper class. In general, using Zorn's lemma on a proper class requires the *axiom of global choice*. However, in this case, there is a technical trick that guarantees the existence of a maximal element while only assuming the regular axiom of choice. The reason this works is that there is an upper bound for the cardinality of a chain in $\mathbf{\Phi}$. We describe this in section 6.3, p. 47.

Now, for the sake of contradiction, we assume that $Z \neq Y$. Choose a $s \in Y \setminus Z$, and set

$$L := \{z \in Z : z < s\} \qquad R := \{z \in Z : z > s\},$$

as well as

$$s' := \langle \psi(L) \mid \psi(R) \rangle.$$

Since ψ is strictly increasing, s' is a surreal number, and so we have $\psi(z) < s'$ for all $z < s$, as well as $\psi(z) > s'$ for all $z > s$. Therefore we can extend ψ to $Z \cup \{s\}$ by setting $\psi(s) = s'$ and still have a strictly increasing function. This contradicts the maximality of ψ, which means that our assumption was wrong and $Z = Y$. Therefore $\psi\colon Y \longrightarrow \mathbf{No}$ is the desired extension of the embedding φ. □

This theorem can be seen as an extension of Cantor's isomorphism theorem, which implies that any *countable* linear order can be embedded into the rational numbers. For a proof of this, see [7, p. 83–84].

[7]: Hrbacek et al. (1999), *Introduction to Set Theory*

As a special case of theorem 4.1.1, we get that every well-ordering can be embedded into **No**, though for that, the class of ordinals **On** would have sufficed, see [7, p. 111]. However, **On** is *not* universally embedding with respect to well-orderings. To see this, take $Y = \mathbb{N}$ (including 0) and $X = \mathbb{N} \setminus \{0\}$. Then $\varphi\colon X \longrightarrow \mathbf{On}\colon n \mapsto n-1$ is an embedding of X which can not be extended to an embedding of Y.

4.2 Fields and Rings of Characteristic Zero

The essential ingredient for the universal embedding theorems for fields is that **No** forms a real closed field.

Definition 4.2.1 (Real Closed Field)
Let $\mathcal{R}$ be an ordered field. Then $\mathcal{R}$ is called *real closed*, if

- every positive element of $\mathcal{R}$ has a square root in $\mathcal{R}$, and
- every polynomial of odd degree with coefficients in $\mathcal{R}$ has at least one root in $\mathcal{R}$.

Theorem 4.2.2 (No is real closed.)
The surreal numbers **No** form a real closed Field.

We omit the proof here, since it would require more theory about the surreal numbers, like normal forms and surreal power series. The result is due to Conway, who described it in [1], along with the necessary prerequisites. Later, Gonshor gave a slightly different proof, which is closer to the classic argument for showing real closure using Hensel's lemma [9]. Conway's proof is elaborated upon by Siegel in [3], who used some of Gonshor's lemmata.

[1]: Conway (2001), *On Numbers and Games*

[9]: Gonshor (1986), *An Introduction to the Theory of Surreal Numbers*

[3]: Siegel (2013), *Combinatorial Game Theory*

A classic result of Galois theory is its generalization of the fundamental theorem of algebra, stating that if $\mathcal{R}$ is real closed, then $\mathcal{R}[\mathrm{i}]$ is algebraically closed, where $\mathrm{i}^2 = -1$. Thus the *surcomplex numbers* $\mathbf{No}[\mathrm{i}]$ are algebraically closed. We will first give an embedding theorem for $\mathbf{No}[\mathrm{i}]$, since it is easier to prove.

Theorem 4.2.3 (No[i] is a universally embedding Field of char. zero.)
Let $X \subseteq Y$ be fields of characteristic 0. If there is an embedding $\varphi\colon X \longrightarrow \mathbf{No}[\mathrm{i}]$, then there exists an embedding $\psi\colon Y \longrightarrow \mathbf{No}[\mathrm{i}]$ such that $\psi|_X = \varphi$.

Proof. As in previous embedding theorems, by Zorn's lemma, it suffices to extend φ to a single element $s \in Y$ (cf. page 27). That means we want to construct a field homomorphism

$$\psi\colon X(s) \to \mathbf{No}[\mathrm{i}].$$

In the case that s is algebraic over X, the field $X(s)$ is contained in the algebraic closure $\overline{X}$ of X. Let $X' := \varphi(X)$ be the isomorphic copy of X inside $\mathbf{No}[\mathrm{i}]$, and let $\overline{X'}$ be its algebraic closure. Because $\mathbf{No}[\mathrm{i}]$ is algebraically closed, $\overline{X'} \subseteq \mathbf{No}[\mathrm{i}]$. Since the algebraic closure is unique, there is an isomorphism between $\overline{X}$ and $\overline{X'}$. Restricting this isomorphism to $X(s)$, we obtain an embedding from $X(s)$ into $\mathbf{No}[\mathrm{i}]$.

Now we consider the case that s is transcendental over X. Since we can, as we just saw, extend the embedding φ to the algebraic closure of X, we can WLOG assume that X is algebraically closed. Then algebraically, $X(s)$ is just the field of rational functions in one variable with coefficients in X. The same is true within $\mathbf{No}[\mathrm{i}]$: If s' is transcendental over X', then $X'(s')$ is the field of rational functions in one variable with coefficients in X', which is isomorphic to $X(s)$, since X and X' are isomorphic. Since X' is algebraically closed, any $s' \in \mathbf{No}[\mathrm{i}]$ that is not in X' will be transcendental over X', and such an s' definitely exists, since X' is a set and $\mathbf{No}[\mathrm{i}]$ is a proper class. Mapping s to s' induces a field homomorphism $\psi\colon X(s) \to \mathbf{No}[\mathrm{i}]$ given by

$$\frac{p(s)}{q(s)} \mapsto \frac{\tilde{p}(s')}{\tilde{q}(s')} \quad \text{for any polynomials } p, q \in X[t],$$

where for a polynomial $p = a_0 + \cdots + a_n t^n$ with $a_0, \ldots, a_n \in X$, we write $\tilde{p} = \varphi(a_0) + \cdots + \varphi(a_n)t^n$ for the polynomial corresponding to p in $X'[t]$. It is simple to check that such a ψ is indeed a field homomorphism, and that $\psi|_X = \varphi$. □

This generalizes immediately to integral domains.

Corollary 4.2.4 (Embedding Theorem for Integral Domains of Char. 0) Let $X \subseteq Y$ be integral domains of characteristic 0. If there is an embedding $\varphi\colon X \to \mathbf{No}[\mathrm{i}]$, then there exists an embedding $\psi\colon Y \to \mathbf{No}[\mathrm{i}]$ such that $\psi|_X = \varphi$.

An *integral domain* is a commutative ring R with the property that the product of any nonzero elements of R is again nonzero.

Proof. This follows immediately from the previous theorem by embedding the integral domain into its *quotient field* (the smallest field containing the integral domain). □

4.3 Ordered Fields and Rings

For the universal embedding theorem for ordered fields, we will also need the following classic properties of real closed fields.

Theorem 4.3.1 (Artin-Schreier)

(a) Every ordered field has a real closure: If K is an ordered field, then there is an algebraic extension R/K such that R is real closed and the order on R extends the order of K. This R is unique up to (order-preserving) isomorphism and is called the *real closure* of K.
(b) An ordered field R is real closed iff all polynomial functions with coefficients in R satisfy the intermediate value property.

For a proof and additional details, see [10, p. 451–457].

[10]: Lang (2002), *Algebra*

With this, we can now prove that **No** is a universally embedding ordered Field. This theorem is also due to Conway. In [1], Conway sketches his proof, in which he uses the well ordering principle (which is equivalent to Zorn's lemma). We phrase it using Zorn's lemma here.

Theorem 4.3.2 (No is a universally embedding ordered Field.)
Let $X \subseteq Y$ be ordered fields. If there is an embedding $\varphi\colon X \to \mathbf{No}$, then there exists an embedding $\psi\colon Y \to \mathbf{No}$ such that $\psi|_X = \varphi$.

Proof. The proof for embedding the field structure of Y is essentially the same as in the previous embedding theorem (4.2.3). However, additional work will be required to preserve the ordering.

Again, by Zorn's lemma, it suffices to extend φ to a single element $s \in Y$ (cf. page 27). That means we want to construct a *strictly increasing* field homomorphism

$$\psi\colon X(s) \to \mathbf{No}.$$

We first look at the case that s is algebraic over X. Let $\mathcal{R}$ be the real closure of $X(s)$. Since the field extensions $X \subseteq X(s)$ and $X(s) \subseteq \mathcal{R}$ are algebraic, the extension $X \subseteq \mathcal{R}$ is also algebraic, which means that $\mathcal{R}$ is also the real closure of X. Let $X' := \varphi(X)$ be the isomorphic copy of X inside **No**, and let $\mathcal{R}'$ be its real closure. Then $\mathcal{R}' \subseteq \mathbf{No}$, because **No** is also real closed. Since the real closure is unique, there is an order-preserving isomorphism between $\mathcal{R}$ and $\mathcal{R}'$. Restricting this isomorphism to $X(s)$, we obtain an embedding from $X(s)$ into **No**.

Now we consider the case that s is transcendental over X. Since we can, as we just saw, extend the embedding φ to the real closure of X, we can WLOG assume that X is real closed. Then algebraically, $X(s)$ is just the field of rational functions in one variable with coefficients in X. The same is true for $X'(s')$ if s' is transcendental over X'. Since X' is real closed and **No** is ordered, any $s' \in \mathbf{No}$ that is not in X' will be transcendental over X', yielding that $X(s)$ is (as a field) isomorphic to $X'(s')$, since X and X' are isomorphic. So, as far as the field structure is concerned, we can choose any s' which is transcendental over X' for $\varphi(s)$ and thus receive a field homomorphism $\psi\colon X(s) \to \mathbf{No}$ given by

$$\frac{p(s)}{q(s)} \mapsto \frac{\tilde{p}(s')}{\tilde{q}(s')} \quad \text{for any polynomials } p, q \in X[t],$$

where for a polynomial $p = a_0 + \cdots + a_n t^n$ with $a_0, \ldots, a_n \in X$, we write $\tilde{p} = \varphi(a_0) + \cdots + \varphi(a_n)t^n$ for the polynomial corresponding to p in $X'[t]$.

It is simple to check that such a ψ is indeed a field homomorphism, and that $\psi|_X = \varphi$.

So what is left to do is to pick s' such that ψ is strictly increasing. Elements $x, y \in X(s)$ are of the form $x = \frac{p_1(s)}{q_1(s)}$ and $y = \frac{p_2(s)}{q_2(s)}$ for polynomials p_1, q_1, p_2, $q_2 \in X[t]$. This means we have to find an s' such that

$$\frac{p_1(s)}{q_1(s)} > \frac{p_2(s)}{q_2(s)} \quad\Longrightarrow\quad \frac{\tilde{p}_1(s')}{\tilde{q}_1(s')} > \frac{\tilde{p}_2(s')}{\tilde{q}_2(s')}$$

for all p_1, q_1, p_2, $q_2 \in X[t]$. Bringing everything to one side in both inequalities (and using that $p \mapsto \tilde{p}$ is a homomorphism between the two polynomial rings), this is equivalent to

$$\frac{p(s)}{q(s)} > 0 \quad\Longrightarrow\quad \frac{\tilde{p}(s')}{\tilde{q}(s')} > 0$$

for all $p, q \in X[t]$, which means that it suffices to find an s' which satisfies the same rational function inequalities as s. Actually, it is enough to have s' satisfy the same polynomial inequalities as s, i.e., $p(s) > 0 \Rightarrow \tilde{p}(s') > 0$ for all $p \in X[t]$, since then we can clear the denominators and still keep the direction of the two inequalities the same.

To produce such a surreal number, we again take

$$L := \{x \in X : x < s\} \qquad R := \{x \in X : x > s\}$$
$$s' = \langle \varphi(L) \mid \varphi(R) \rangle.$$

Each $x' \in X'$ is in either L or R, implying that s' greater than or less than x', respectively. So s' is not in and therefore transcendental over X'. To show that s' satisfies the same polynomial inequalities, let $p \in X[t]$ with $p(s) > 0$. Pick an $l \in L$ that is greater than or equal to the largest root of p in L, and pick an $r \in R$ that is less than or equal to the smallest root of p in R. Therefore $p(x) > 0$ for all $x \in X$ with $l < x < r$.

Now, since X and X' are isomorphic as ordered fields, it will also hold that $\tilde{p}(x') > 0$ for all $x' \in X'$ with $\varphi(l) < x' < \varphi(r)$. By construction, the surreal number s' also lies between $\varphi(l)$ and $\varphi(r)$. Assume $\tilde{p}(s') \leq 0$. Then $\tilde{p}(s') < 0$ since s' is transcendental. That means $\tilde{p}$, as a function from **No** to **No**, changes sign between $\varphi(l)$ and $\varphi(r)$. Therefore, since **No** is real closed, $\tilde{p}$ has a root in that interval of **No**. However, since X' is real closed, this root must be in X', which contradicts the choice of l and r. So we have $\tilde{p}(s') > 0$ for any $p \in X[t]$ with $p(s) > 0$, which implies that ψ is strictly increasing, as desired. ☐

Again, this can easily be generalized to the respective kind of rings.

Corollary 4.3.3 (Universal Embedding Theorem for Ordered Rings) Let $X \subseteq Y$ be totally ordered rings. If there is an embedding $\varphi\colon X \to \mathbf{No}$, then there exists an embedding $\psi\colon Y \to \mathbf{No}$ such that $\psi|_X = \varphi$.

A *totally ordered ring* is a commutative ring R together with a total order relation $\leq$ on R, such that for all $a, b, c \in R$:

- If $a \leq b$, then $a + c \leq b + c$.
- If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

Proof. This follows immediately from the previous theorem by embedding the ordered ring into its *quotient field* (equipped with the corresponding order). ☐

4.4 Vector Spaces and Free Modules

Theorem 4.4.1 (Universal embedding theorems for vector spaces.)

(a) Let $K \subseteq \mathbf{No}[\mathrm{i}]$ be a field, and let $X \subseteq Y$ be K-vector spaces. If there is an embedding $\varphi\colon X \to \mathbf{No}[\mathrm{i}]$, then there exists an embedding $\psi\colon Y \to \mathbf{No}[\mathrm{i}]$ such that $\psi|_X = \varphi$.

(b) Let $K \subseteq \mathbf{No}$ be a field, and let $X \subseteq Y$ be totally ordered K-vector spaces. If there is an embedding $\varphi\colon X \to \mathbf{No}$, then there exists an embedding $\psi\colon Y \to \mathbf{No}$ such that $\psi|_X = \varphi$.

A *totally ordered K-vector space* over an ordered field K is a vector space V together with a total order relation $\leq$ on V, such that for all $a, b, c \in V$ and all $\lambda \in K$:

- If $a \leq b$, then $a + c \leq b + c$.
- If $0 \leq a$ and $0 \leq \lambda$, then $0 \leq \lambda a$.

Proof. Let $\mathbf{U}$ be $\mathbf{No}$ or $\mathbf{No}[\mathrm{i}]$. Again, by Zorn's lemma, it suffices to extend φ to a single element $s \in Y$ (cf. page 27). Let V be the vector space spanned by all vectors in X as well as s. If $s \in X$, the statement is trivial. Otherwise s is linearly independent of all vectors in X, so for each $s' \in \mathbf{U}$ there is a unique linear map $\psi\colon V \to \mathbf{U}$ extending φ with $\psi(s) = s'$, given by

$$\psi(x + ks) = \varphi(x) + ks' \quad \text{for } x \in X, k \in K.$$

In order to make ψ injective, we just need $s' \notin \varphi(X)$. Since X is a set and $\mathbf{U}$ is a proper class, such a s' definitely exists. This is enough to proof (a).

For (b), we need to pick s' such that ψ is strictly increasing. We take

$$L := \{x \in X : x < s\} \qquad R := \{x \in X : x > s\}$$
$$s' = \langle \varphi(L) \mid \varphi(R) \rangle.$$

To show that this choice makes ψ strictly increasing, let $u, v \in V$ with $u < v$. Those are of the form $u = x_1 + k_1 s$, $v = x_2 + k_2 s$ with $x_1, x_2 \in X$ and $k_1, k_2 \in K$. In the case that $k_1 < k_2$, the condition $u < v$ can be rewritten as

$$\frac{x_1 - x_2}{k_2 - k_1} < s,$$

which means that the left hand side is in L, and thus

$$\varphi\left(\frac{x_1 - x_2}{k_2 - k_1}\right) < s',$$

which can be rearranged to $\psi(u) < \psi(v)$. If instead $k_1 > k_2$, the same holds with the directions of both inequalities changed, using instead that the left hand side is in R. If $k_1 = k_2$, the terms involving s' cancel and $\psi(u) < \psi(v)$ reduces to $\varphi(u) < \varphi(v)$, which is true by assumption. This shows that ψ is the desired embedding. □

Together with the universal embedding theorems for fields, this shows that every vector space over a field of characteristic 0 can be embedded in $\mathbf{No}[\mathrm{i}]$, and every totally ordered vector space over an ordered field can be embedded in $\mathbf{No}$.

There is again a generalization from fields to rings, but this time, it is a bit more subtle.

Corollary 4.4.2 (Universal Embedding Theorems for Free Modules)

(a) Let $R \subseteq \mathbf{No}[\mathrm{i}]$ be a ring, and let $X \subseteq Y$ be free R-modules. If there is an embedding $\varphi\colon X \to \mathbf{No}[\mathrm{i}]$, then there exists an embedding $\psi\colon Y \to \mathbf{No}[\mathrm{i}]$ such that $\psi|_X = \varphi$.
(b) Let $R \subseteq \mathbf{No}$ be a ring, and let $X \subseteq Y$ be totally ordered free R-modules. If there is an embedding $\varphi\colon X \to \mathbf{No}$, then there exists an embedding $\psi\colon Y \to \mathbf{No}$ such that $\psi|_X = \varphi$.

A *totally ordered R-module* over a totally ordered ring R is a module M together with a total order relation $\leq$ on M, such that for all $a, b, c \in M$ and all $\lambda \in R$:

- If $a \leq b$, then $a + c \leq b + c$.
- If $0 \leq a$ and $0 \leq \lambda$, then $0 \leq \lambda a$.

An R-module is called *free* if it has a basis.

Proof. The key point is that X and Y are *free*. For X, this means that it has a basis $B \subseteq X$, i.e., every $x \in X$ can be written as a linear combination

$$\sum_{b \in B} r_b \cdot b,$$

where $r_b \in R$ for all $b \in B$, and $r_b = 0$ for all but finitely many $b \in B$. Let K be the quotient field of R. Then we can embed X into the K-vector space spanned by B. The same is possible for Y, which means that we can deduce the universal embedding theorems for modules from the universal embedding theorem for vector spaces. □

If X and Y are not taken to be free, the theorem is false: The ring $\mathbb{Z}$ is contained in $\mathbf{No}[\mathrm{i}]$, and thus $X := \mathbb{Z}$ can be embedded into $\mathbf{No}[\mathrm{i}]$. However, $\mathbb{Z}$ can be embedded into $Y := \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$, which can not be embedded into $\mathbf{No}[\mathrm{i}]$.

4.5 Further Universality Properties

For further reading, we want to mention that **No** also satisfies universality properties for several other algebraic structures.

- Many of the universal embedding theorems we covered in this chapter can also be extended to respect the tree structure (also called the “simplicity hierarchy”, cf. fig. 3.8) of **No**, which comes from its inductive construction. [11]
- Martin Kruskal and Harry Gonshor discorvered that there is also a suitable definition of the exponential function on **No**, which has all the properties one would expect [9, chapter 10]. This makes **No** an ordered exponential Field.
- Recently, the surreal numbers have been also been used in the study of so-called transseries. Here **No** again satisfies several universality properties, for example as a transserial Hahn Field. [12] [13]

[11]: Ehrlich (2001), *Number Systems with Simplicity Hierarchies: A Generalization of Conway's Theory of Surreal Numbers*

[9]: Gonshor (1986), *An Introduction to the Theory of Surreal Numbers*

[12]: Mantova et al. (2017), *Surreal Numbers With Derivation, Hardy Fields and Transseries: A Survey*

[13]: Ehrlich et al. (2021), *Surreal Ordered Exponential Fields*

Universal Embedding Properties of the Partisan Games 5

After proving that the surreal numbers form a universal embedding ordered Field, Conway conjectured that a similar theorem would be true for the games as a partially ordered abelian Group. David Meows proved in 2002 that **Pg** forms a universal embedding abelian Group, where the embedding however does not generally preserve the partial order [14]. In the same year, Jacob Lurie proved the full conjecture for partially ordered abelian groups (which we will more briefly call POA groups) [15].[1] This chapter will follow Lurie's paper, with some minor elaboration.

[14]: Moews (2002), *The Abstract Structure of the Group of Games*

[15]: Lurie (2002), *On a Conjecture of Conway*

1: Note that every abelian group can be seen as a POA group if one takes equality as the partial order relation. So while ordered fields are special kinds of fields, POA groups are actually more general than abelian groups.

We want to prove statements regarding the partial order inductively, since lemma 2.4.6 is essentially the only tool at our disposal. However, to make use of induction in a set $S \subset \mathbf{Pg}$, we must make sure that S is itself structured inductively. This motivates the following definition.

Definition 5.0.1 (Hereditary)
A set of games $\widetilde{M}$ is called *hereditary* if, for every $G \in \widetilde{M}$, all of its options g^{L} and g^{R} are also in $\widetilde{M}$.
A set of game values $M \subseteq \mathbf{Pg}$ is called hereditary, if it is the set of equivalence classes of a hereditary set of games.

Note that any set of games can be enlarged to a hereditary set by adding all the left and right options of the games in the set, then again adding all of those games' options, and so on. Therefore, we can also enlarge every set of game values M to a hereditary set, by taking a member g of each equivalence class $x \in M$ and collecting all those games g in a set $\widetilde{M}$. If we then enlarge $\widetilde{M}$ to a hereditary set of games, and take the set of all equivalence classes of elements of $\widetilde{M}$, this will be a hereditary set of game values which contains M.

5.1 Partially Ordered Sets

First, we will prove the universal embedding theorem for partially ordered sets (*posets* for short). On the one hand, this will already demonstrate how embedding a partial order is more complex than embedding a total order. On the other hand, this will later be used in the proof of the embedding theorem for POA groups. We will phrase everything in this section in terms of games rather than game values since the group structure of **Pg** is not relevant when only considering the partial order. It is clear that the same property will then also hold for **Pg**.

Our basic strategy for proving universal embedding theorems is still the same as in the previous section (cf. p. 27): We pick an element s of a poset and extend the embedding by mapping s to a game s'. To achieve this extension, s' will have to satisfy the same *partial* inequalities as s, which here (as opposed to the total orders we tackled before) also means that s' does *not* satisfy an inequality if s does not. Because of this, instead of strictly increasing maps, we will speak of *order-preserving* maps.

Definition 5.1.1 (Order-Preserving Map)
Let A and B be partially ordered sets. Then we call a function $f : A \longrightarrow B$ *order-preserving*, when for all $x, y \in A$, $x \leq y$ holds iff $f(x) \leq f(y)$ holds.

To construct a game that satisfies a collection of specified inequalities, we proof the following lemma.

Lemma 5.1.2 (Constructing a game satisfying the right inequalities.)
Let M be a hereditary set of games and let $L, R \subseteq M$ with the following properties:

- L is closed downwards: If $y \leq x$ for $x \in L$ and $y \in M$, then $y \in L$.
- R is closed upwards: If $y \geq x$ for $x \in R$ and $y \in M$, then $y \in R$.
- For all $l \in L, r \in R$, we have $l \leq r$.

Set $G := \{M \setminus R \mid M \setminus L\}$. Then for all $x \in M$, the inequality $x \leq G$ holds iff $x \in L$, and the inequality $G \leq x$ holds iff $x \in R$.

Proof. We use induction on x, which works because M is hereditary. If $x \leq G$, then x cannot be a right option of G. So x must be in L. If however $x \nleq G$, i.e., $x \rhd G$, one of the following two cases must occur (lemma 2.4.6):

- We have $x^{\mathrm{L}} \geq G$ for some x^{L}. Then by the inductive hypothesis, $x^{\mathrm{L}} \in R$ and thus $x \notin L$ since $x^{\mathrm{L}} \lhd x$ would contradict $x \leq y$ for $x \in L, y \in R$.
- We have $x \geq g^{\mathrm{R}}$ for some g^{R}. If x was in L, then g^{R} would be too, since L is downwards closed. This cannot be, because $g^{\mathrm{R}} \in M \setminus L$. So we must have $x \notin L$.

The claim $G \leq x \Leftrightarrow x \in R$ follows analogously. □

Theorem 5.1.3 (Universal Embedding Theorem for Posets)
Let $X \subseteq Y$ be posets. If there is an embedding φ from X to the games, then there exists an embedding ψ from Y to the games such that $\psi|_X = \varphi$.

Proof. As in previous embedding theorems, by Zorn's lemma, it suffices to extend φ to a single element $s \in Y$ (cf. page 27). That means we want to construct an order-preserving map from $X \cup \{s\}$ to the games.

Enlarge $\varphi(X)$ to a hereditary set M, and set

$$L := \{l \in M : \exists x \in X : (l \leq \varphi(x) \wedge x \leq s)\}$$
$$R := \{r \in M : \exists y \in X : (r \geq \varphi(y) \wedge y \geq s)\}.$$

This L is closed downwards, since for a smaller l, the same x can be used (since φ is order-preserving). The same reasoning shows R is closed upwards. Also for $l \in L$, $r \in R$, pick corresponding $x, y \in X$, with $l \leq \varphi(x)$ and $x \leq s$, as well as $r \geq \varphi(y)$ and $y \geq s$. Then $x \leq y$ and therefore $\varphi(x) \leq \varphi(y)$, implying $l \leq r$.

This means all conditions of lemma 5.1.2 are fulfilled, guaranteeing the existence of a game G with $l \leq G$ iff $l \in L$, and $G \leq r$ iff $r \in R$, for all l,

$r \in M$. Looking only at the l which are in $\varphi(X) \subseteq M$, we can write such an l as $\varphi(t)$ for $t \in X$. Then $l \in L$ iff there is an $x \in X$ with $\varphi(t) \leq \varphi(x)$ and $x \leq s$. We show that this is the case iff $t \leq s$:

- On the one hand, if $\varphi(t) \leq \varphi(x)$, we get $t \leq x$ since φ is order-preserving. Then $x \leq s$ implies $t \leq s$.
- On the other hand, if $t \in X$ with $t \leq s$, we can just take $x = t$ and trivially obtain $\varphi(t) \leq \varphi(x)$ and $x \leq s$.

So in total, for all $t \in X$ we get $\varphi(t) \leq G$ iff $t \leq s$. Using the same argument for R, we for all $t \in X$ obtain $G \leq \varphi(t)$ iff $s \leq t$. Therefore we can extend φ to $X \cup \{s\}$ by setting $\varphi(s) = G$ and still have an order-preserving map, which completes the proof. □

5.2 Partially Ordered Abelian Groups

5.2.1 Construction of Auxiliary Games

Compared to the previously considered totally ordered fields, partially ordered abelian groups are a lot more general and typically less well-behaved. This section focuses on constructing games that mimic the strange inequalities that can hold in a POA group. These constructions will be used later in the proof.

> **Lemma 5.2.1 (Auxiliary games I)**
> Let $A \subseteq \mathbf{Pg}$ be a set of game values such that $a \mathrel{\rhd\!\!\!\!|} 0$ for all $a \in A$. Then for any ordinal α, there exists a game value $x \geq 0$ such that $x \mathrel{|\!\!\!\!\lhd} a$ for all $a \in A$, and at the same time $nx \geq \alpha$ for any natural number $n \geq 2$.

Proof. First, note that if the statement holds for one $\alpha \in \mathbf{On}$, it will also hold for all smaller ordinals. Therefore it suffices to prove the statement for $\alpha > -a$ for all $a \in A$, since such an α exists by 3.4.2 (f). Now set

$$x := \langle 2\alpha \mid a \rangle,$$

where a ranges over A. This satisfies the desired inequality $x \mathrel{|\!\!\!\!\lhd} a$ by 2.4.7. It also satisfies $x \geq 0$ by lemma 2.4.6, since 0 has no left option, and since each x^{L} is some $a \in A$, which satisfies $a \mathrel{\rhd\!\!\!\!|} 0$ by assumption.

Therefore $nx \geq 2x$ for all $n \geq 2$. So it suffices to prove $2x \geq \alpha$. Suppose otherwise. Then by 2.4.6, one of the following two cases must occur:

- We have $2x \leq \alpha^{\mathrm{L}}$ for some α^{L}. By 3.4.2 (b), we may take α^{L} to be some ordinal $\beta < \alpha$. Then clearly $\beta < 2\alpha$, from which we get

  $$2x \leq \beta < 2\alpha \leq 2\alpha + x$$

  since $x \geq 0$, which implies $x < 2\alpha$, contradicting that $x \mathrel{\rhd\!\!\!\!|} 2\alpha$ by construction.
- We have $(2x)^{\mathrm{R}} \leq \alpha$, where $(2x)^{\mathrm{R}} = (x + x)^{\mathrm{R}} = a + x$ for some $a \in A$. Now $a + x \leq \alpha$ implies that $\alpha \mathrel{|\!\!\!\!\lhd} (a + x)^{\mathrm{L}}$, where such a left option could be $a + x^{\mathrm{L}} = a + 2\alpha$. So $\alpha \mathrel{|\!\!\!\!\lhd} a + 2\alpha$ and therefore $\alpha \mathrel{|\!\!\!\!\lhd} -a$, contrary to our initial assumption.

□

We now use this to construct more general auxiliary games.

Lemma 5.2.2 (Auxiliary games II)
Let A, $(B_n)_{n\geq 1}$ and $(C_n)_{n\geq 1}$ be sets of game values, such that $a \mathrel{|\!\!\!>} b_1$ for all $a \in A$, $b_1 \in B_1$. Then there exists a game value x satisfying the following properties:

(a) $a \mathrel{|\!\!\!>} x$ for all $a \in A$.
(b) $nx \geq b_n$ for all $b_n \in B_n$ and all naturals $n \geq 1$.
(c) $nx \mathrel{|\!\!\!>} c_n$ for all $c_n \in C_n$ and all naturals $n \geq 1$.

Proof. For starters, we want to find a game value x that satisfies the conditions for $n = 1$, so we want

$$a \mathrel{|\!\!\!>} x, \qquad x \geq b_1, \qquad x \mathrel{|\!\!\!>} c_1$$

for all $a \in A$, $b_1 \in B_1$, $c_1 \in C_1$. If additionally an inequality of the form $a \leq b_1$ were to hold, then with $b_1 \leq x$ this would yield $a \leq x$, which contradicts $a \mathrel{|\!\!\!>} x$. This is why we need the assumption $a \mathrel{|\!\!\!>} b_1$. With that assumption in place however, the desired inequalities are all consistent with each other, meaning they induce a partial order on the set

$$A \cup B_1 \cup C_1 \cup \{\tilde{x}\},$$

where $\tilde{x}$ is just an arbitrary symbol not in $A \cup B_1 \cup C_1$. By theorem 5.1.3, we can embed this poset into **Pg**, mapping $\tilde{x}$ to a game value x with the desired properties.

Now to construct an x satisfying the conditions for all $n \geq 1$, let x_1 be the previously constructed game value which satisfies the conditions for $n = 1$. We now perform the following substitution:

Replace x by $x - x_1$
Replace A by $\{a - x_1 : a \in A\}$
Replace B_n by $\{b - nx_1 : b \in B_n\}$ for each $n \geq 1$
Replace C_n by $\{c - nx_1 : c \in C_n\}$ for each $n \geq 1$.

It is clear that finding an x satisfying all conditions in this new setting will in turn yield an x satisfying all conditions in the original setting. By construction of x_1, after the substitution we know that

$$\begin{aligned} a &\mathrel{|\!\!\!>} 0 \quad \text{for all } a \in A \\ b_1 &\leq 0 \quad \text{for all } b_1 \in B_1 \\ c_1 &\mathrel{<\!\!\!|} 0 \quad \text{for all } c_1 \in C_1. \end{aligned}$$

Theorem 3.4.2 (f) implies that there exists an ordinal α greater than all b_n and c_n for $n \geq 1$, $b_n \in B_n$, $c_n \in C_n$. Then, by lemma 5.2.1, we can find $x \in \mathbf{Pg}$ with

$$x \geq 0, \qquad nx \geq \alpha, \qquad x \mathrel{<\!\!\!|} a$$

for all $a \in A$. The latter means the condition (a) is satisfied. Indeed, x also satisfies (b) and (c):

- Since $x \geq 0$ and $b_1 \leq 0$, we get $x \geq b_1$ for all $b_1 \in B_1$. Also, $x \leq c_1$ would give $0 \leq c_1$, contradicting $c_1 \lhd\!\!| \ 0$. So we have $x \lhd\!\!| \ c_1$ for all $c_1 \in C_1$.
- From $\alpha > b_n$, $\alpha > c_n$ and $nx' \geq \alpha$, we get $nx' > b_n$ and $nx' > c_n$ for all $n \geq 2$, $b_n \in B_n$, $c_n \in C_n$.

□

5.2.2 Framings and Justification

Where with posets we only needed to consider how one element $s \in Y \setminus X$ compares to the elements of X, in the case of POA groups, we also need to consider how the *multiples of s* compare to the elements of X. This is the topic of the present section.

Definition 5.2.3 (Framing)
Let X be a subgroup of **Pg**. A *framing* of X is a family $(X_i)_{i \in \mathbb{Z}}$ of subsets of X with the following properties:

(a) $X_i + X_j \subseteq X_{i+j}$ for all $i, j \in \mathbb{Z}$.
(b) $x \in X_0$ iff $x \in X$ and $x \geq 0$.

If $X \subseteq Y$ are subgroups of **Pg**, we say a framing of Y *extends* a framing of X if $X_i = Y_i \cap X$ for all $i \in \mathbb{Z}$. In this case we also say the two framings are *compatible*.

The intention behind this definition is, that X_i is going to be set of all $x \in X$ with $x \geq is$, where s is the game we need to construct. This intention suggests the following property.

Lemma 5.2.4 (Frames are closed upwards.)
Let X be a framed subgroup of **Pg**. Then for every $i \in \mathbb{Z}$, the set X_i is closed upwards: If $y \geq x$ for $x \in X_i$ and $y \in X$, then $y \in X_i$.

Proof. We have $y - x \in X_0$, and so since $X_i + X_0 \subseteq X_i$, we get that $y = x + (y - x) \in X_i$. □

Lemma 5.2.5 (Every framing can be extended.)
Let $X \subseteq Y$ be subgroups of **Pg**. Then for any framing $(X_i)_{i \in \mathbb{Z}}$ of X, there is a framing of Y that extends $(X_i)_{i \in \mathbb{Z}}$.

Proof. To construct such a framing of Y, simply take the "upwards closure" of X_i in Y:

$$Y_i := \{y \in Y : \exists x \in X_i : y \geq x\} \quad \text{for every } i \in \mathbb{Z}.$$

To verify the condition (a) in definition 5.2.3, let $i, j \in \mathbb{Z}$. Write $y \in Y_i + Y_j$ as $y = y_1 + y_2$ for $y_1 \in Y_i$, $y_2 \in Y_j$. Then by definition, there are $x_1 \in X_i$, $x_2 \in X_j$ such that $y_1 \geq x_1$, $y_2 \geq x_2$. Then $y_1 + y_2 \geq x_1 + x_2 \in X_{i+j}$, so $y_1 + y_2 \in Y_{i+j}$.

For condition (b), take $y \in Y$ with $y \geq 0$. Since X is a group, we have $0 \in X$, and therefore $0 \in X_0$, which gives us $y \in Y_0$. On the other hand, if $y \in Y_0$, then there is an $x \in X_0$ with $y \geq x$. This x must satisfy $x \geq 0$, yielding $y \geq 0$. □

Our main goal from now on is to show that if $(X_i)_{i\in\mathbb{Z}}$ is a framing of X, then there is an $s \in \mathbf{Pg}$ such that

$$X_i = \{x \in X : x \geq is\}$$

for all $i \in \mathbb{Z}$. Now an inequality like $g \geq is$ will hold *unless there is a reason to the contrary*, in the sense of lemma 2.4.6. We define (g, i) to be *justified* if there is such a reason, i.e., if "it is justified that g is not in X_i".

Definition 5.2.6 (Justification)
Let X be a framed subgroup of $\mathbf{Pg}$ and $n \geq 2$ be a natural number.

- If $g \in X \setminus X_n$, we say (g, n) is justified if there is an $x \in X_{-1}$ with $g + x \notin X_{n-1}$.
- If $g \in X \setminus X_{-n}$, we say $(g, -n)$ is justified if there is an $x \in X_1$ with $g + x \notin X_{-n+1}$.

If X is a framed subgroup of $\mathbf{Pg}$ such that for all $i \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $g \in X \setminus X_i$, the pair (g, i) is justified in X, we say that X is justified.

Lemma 5.2.7 (Justified hereditary framings have the desired form.)
Let X be a justified and hereditary framed subgroup of $\mathbf{Pg}$. Then there exists an $s \in \mathbf{Pg}$ such that

$$X_i = \{x \in X : x \geq is\}$$

for all $i \in \mathbb{Z}$.

Proof. Define

$$s := \langle X \setminus X_1 \mid X \setminus (-X_{-1})\rangle.$$

Now we prove $X_i = \{x \in X : x \geq is\}$ by induction on $|i|$. For $i = 0$, this is clear from the definition of a framing. From $X_1 + X_{-1} \subseteq X_0$ and $x \geq 0$ for all $x \in X_0$, we get that $l \leq r$ holds for all $r \in X_1$, $l \in -X_{-1}$. Also X_1 is closed upwards, while $-X_{-1}$ is closed downwards. So we can use lemma 5.1.2 on $-s$ to get that the assertion holds for $|i| = 1$.

Now let $n \geq 2$ and assume the assertion holds for all $i \in \mathbb{Z}$ with $|i| < n$. We demonstrate the proof for $n + 1$, the argument for $-n - 1$ works analogously.

First we look at $g \notin X_n$. Then (g, n) is justified, which implies the existence of an $x \in X_{-1}$ such that $g + x \notin X_{n-1}$. By the inductive hypothesis, $x \in X_{-1}$ means $x \geq -s$, whereas $g + x \notin X_{n-1}$ means $g + x \lhd\!| \ (n-1)s$. Now the inequality $g \geq ns$ is false, cause otherwise it would imply

$$g + x \geq ns + x \geq (n-1)s$$

contradicting the earlier inequality. So by contrapositive, we get that $g \geq ns$ implies $g \in X_n$ (and analogously for $-n$).

Now we take care of the other direction. Let $x \in X_n$. By lemma 2.4.6, there are two cases in which $ns \leq x$ does not hold:

- $x \leq (ns)^{\mathrm{L}}$, where the left option of ns is of the form $(n-1)s+s^{\mathrm{L}}$. This can be rearranged to $s^{\mathrm{L}} - x \geq (1-n)s$, which means $s^{\mathrm{L}} - x \in X_{1-n}$ by the inductive hypothesis. Now since $x \in X_n$ and $X_n + X_{1-n} \subseteq X_1$, we get $s^{\mathrm{L}} = x + (s^{\mathrm{L}} - x) \in X_1$, contradicting the construction of s.
- $x^{\mathrm{R}} \leq ns$, or in other words $-x^{\mathrm{R}} \geq -ns$. By what we have already shown, this implies $-x^{\mathrm{R}} \in X_{-n}$. Then $x \in X_n$ and $X_n + X_{-n} \subseteq X_0$ gives $x - x^{\mathrm{R}} \in X_0$, so $x - x^{\mathrm{R}} \geq 0$ and therefore $x \geq x^{\mathrm{R}}$, which contradicts 2.4.7.

So neither of those cases can occur, meaning that $ns \leq x$. This concludes the induction. Note that we used that X is hereditary in order to work with $x^{\mathrm{L}}, x^{\mathrm{R}} \in X$. □

It may seem like we are not really any closer to our goal, as not all framed subgroups $X \subset \mathbf{Pg}$ are justified or hereditary. But there is a way around that: if we show that $Y_i = \{y \in Y : y \geq is\}$ for some *extension* Y of X, then the same will easily follow for the framing of X. So in the next section, our goal will be to construct a justified extension Y for any framed subgroup $X \subset \mathbf{Pg}$.

5.2.3 Everything Can Be Justified

First, let us make use that we do not lose justified pairs by an extension.

Lemma 5.2.8 (Justification carries over to extension.)
Let $X \subseteq Y$ be subgroups of $\mathbf{Pg}$ with compatible framings $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$. If (g, i) is justified in X, then (g, i) is also justified in Y.

Proof. We give the proof for $i = n \geq 2$, the proof for $i = -n$ is analogous. Since (g, n) is justified, $g \notin X_n$, and there is an $x \in X_{-1}$ with $g + x \notin X_{n-1}$. Since the framings of X and Y are compatible, we also have $x \in Y_{-1}$. Also $g, g + x \in X$ together with $X_{n-1} = Y_{n-1} \cap X$ and $X_n = Y_n \cap X$ implies that also $g + x \notin Y_{n-1}$ and $x \notin Y_n$. This means (g, n) is justified in Y. □

Now we can start to construct our desired extension. Most of the work will go into constructing an extension that justifies one particular pair.

Lemma 5.2.9 (There is an extension that justifies (g, i).)
Let X be a framed subgroup of $\mathbf{Pg}$, and $g \in X \setminus X_i$. Then there exists a framed subgroup $Y \subset \mathbf{Pg}$ extending X, such that (g, i) is justified in Y.

Proof. Again we only give the proof for $i = n \geq 2$, the proof for $i = -n$ is analogous. We need an x as in definition 5.2.6 that makes (g, n) justified. So we take Y to be the group generated by X and some $x \in \mathbf{Pg}$, which we frame by

$$Y_k := \{y \in Y : \exists j \geq 0 \colon \exists z \in X_{k+j} \colon z + jx \leq y\}$$

for every $k \in \mathbb{Z}$. We will now make a "wishlist" of properties we want x to have so that this all works out. Some conditions will already be satisfied, no matter the choice of x. Others will require us to choose x in the right way.

First, let us see what properties x must have to ensure that this is indeed a framing:

- Let $y \in Y_k + Y_l$, so $y = y_1 + y_2$ for $y_1 \in Y_k$, $y_2 \in Y_l$. That means there are $j_1, j_2 \geq 0$ and $z_1 \in X_{k+j_1}$, $z_2 \in X_{l+j_2}$ such that $z_1 + j_1 x \leq y_1$ and $z_2 + j_2 x \leq y_2$. Adding those two inequalities, we get $(z_1+z_2)+(j_1+j_2)x \leq y_1+y_2 = y$, and since $X_{k+j_1}+X_{l+j_2} \subseteq X_{k+l+j_1+j_2}$, we have $z_1 + z_2 \in X_{k+l+j_1+j_2}$. This means that $y \in Y_{k+l}$. So this is true independent of x.
- Let $y \in Y$ with $y \geq 0$. Then since $0 \in X_0$, we can take $j = 0$ and $z = 0$ to see $y \in Y_0$. For the other direction, let $y \in Y_0$. Then there are $j \geq 0$ and $z \in X_j$ such that $z + jx \leq y$. So to show $y \geq 0$, we would want x to be a game value such that $jx \geq -z$ for all $j \geq 0$, $z \in X_j$.

We also need the framing $(Y_k)_{k\in\mathbb{Z}}$ to be compatible with $(X_k)_{k\in\mathbb{Z}}$.

- If $y \in X_k$, then using $j = 0$ and $z = y$ we get $z + jx \leq y$, so $y \in Y_k$. This means the inclusion $X_k \subseteq Y_k \cap X$ holds independently of x.
- For the inclusion $Y_k \cap X \subseteq X_k$, we need to show that for $y \in X$, the existence of $j \geq 0$ and $z \in X_{k+j}$ with $y \geq z + jx$ implies $y \in X_k$. For $j = 0$, this means $z \in X_k$ and $y \geq z$ implies $y \in X_k$, i.e., that X_k is closed upwards, which we know is true. For $j \geq 1$, we rewrite the condition as $jx \mathrel{\|\!\!\triangleright} y - z$ for all $z \in X_{k+j}$ and $y \in X \setminus X_k$ and add it to our wishlist.

Finally, we need that x justifies (g, n), so $x \in Y_{-1}$ and $g + x \notin Y_{n-1}$.

- We can take $j = 1$ and $z = 0 \in X_{j+k} = X_{1-1} = X_0$ to get that $x \in Y_{-1}$. So this part holds independently of x.
- For $g + x \notin Y_{n-1}$, we need that there are no $j \geq 0$, $z \in X_{n-1+j}$ such that $z + jx \leq g + x$. In other words, we need

$$z + jx \mathrel{\|\!\!\triangleright} g + x$$

  for all $j \geq 0$, $z \in X_{n-1+j}$. For $j = 1$, this is already true: If $z \leq g$ for some $z \in X_n$ was the case, then since X_n is closed upwards, also $g \in X_n$, contrary to our assumption. So $z \mathrel{\|\!\!\triangleright} g$ for all $z \in X_n$. For all other j, we rewrite this a bit and add it to our wishlist.

So our final wishlist for x is:

- $jx \geq -z$ for all $j \geq 0$, $z \in X_j$.
- $jx \mathrel{\|\!\!\triangleright} y - z$ for all $j \geq 1$, $z \in X_{k+j}$, $y \in X \setminus X_k$.
- $z - g \mathrel{\|\!\!\triangleright} x$ for all $z \in X_{n-1}$.
- $(j-1)x \mathrel{\|\!\!\triangleright} g - z$ for all $j \geq 2$, $z \in S_{n+j-1}$.

To construct such an $x \in \mathbf{Pg}$, we use lemma 5.2.2. We take

$$\begin{aligned}
A &= \{z - g : z \in X_{n-1}\}, \\
B_j &= \{-z : z \in X_j\}, \\
C_j &= \{g - z : z \in X_{n+j}\} \cup \{y - z : k \in \mathbb{Z}, z \in X_{k+j}, y \in X \setminus X_k\},
\end{aligned}$$

for all $j \geq 1$. The only thing we need to make sure of is that there is no contradiction caused by $a \leq b_1$ for any $a \in A$, $b_1 \in B_1$. This would mean $z_a - g \leq z_b$ for $z_a \in X_{n-1}$, $z_b \in X_1$, so $z_a + z_b \leq g$. But since $z_a + z_b \in X_{n-1} + X_1 \subseteq X_n$, that X_n is closed upwards implies $g \in X_n$, which is contrary to our initial assumption. □

Now that we know that any one (g, i) can be justified, it is straightforward to show that we can also find an extension in which all (g, i) are justified.

> **Lemma 5.2.10 (There is an extension such that any (g, i) is justified.)**
> Let X be a framed subgroup of **Pg**. Then there exists a framed subgroup $Y \subset \mathbf{Pg}$ extending X, such that for all $i \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $g \in X \setminus X_i$, the pair (g, i) is justified in Y.

Proof. Much like we did with universal embedding theorems, we reduce this to creating an extension where one additional (g, i) is justified, using Zorn's lemma (see p. 27). Let $\mathbf{\Phi}$ be the collection of all extensions of the framed group X. For $Z \in \mathbf{\Phi}$, define

$$J(Z) := \big\{(g, i) : i \in \mathbb{Z} \setminus \{-1, 0, 1\}, g \in X \setminus X_i, (g, i) \text{ justified in } Z\big\}.$$

We can partially order $\mathbf{\Phi}$ by defining

$$Z < Z' \qquad :\Longleftrightarrow \qquad Z' \text{ extends } Z \quad \text{and} \quad J(Z) \subset J(Z'),$$

and $Z \leq Z'$ iff $Z = Z'$ or $Z < Z'$. Here, demanding $J(Z)$ to be a proper subset of $J(Z')$ is necessary because otherwise we could keep going bigger after already having justified all pairs.

Now let $C \subseteq \mathbf{\Phi}$ be a chain. Then $\overline{Z} := \bigcup_{Z \in C} Z$ with the framing $\overline{Z}_i := \bigcup_{Z \in C} Z_i$ extends all framed subgroups in the chain. Also by lemma 5.2.8, every (g, i) that is justified in a $Z \in C$ will be justified in $\overline{Z}$, i.e., $J(Z) \subseteq J(\overline{Z})$ for all $Z \in C$.

What we have just shown is that any chain $C \subseteq \mathbf{\Phi}$ has an upper bound $\overline{Z} \in \mathbf{\Phi}$. Thus, by Zorn's lemma,[2] $\mathbf{\Phi}$ has a maximal element Y. For the sake of contradiction, assume that there is a pair (g, i) that is not justified in Y. Then we can use lemma 5.2.9 to extend Y to a framed subgroup Y' of **Pg** where (g, i) is justified, contradicting the maximality of Y. This means every pair (g, i) is justified in Y, as desired. □

2: As with universal embedding theorems, the fact that $\mathbf{\Phi}$ is a proper class causes a technicality, which can again be avoided since every chain in $\mathbf{\Phi}$ is bounded in size by the cardinality of the set

$$\{(g, i) : i \in \mathbb{Z} \setminus \{-1, 0, 1\}, g \in X \setminus X_i\}.$$

We discuss this further in section 6.3, p. 47.

We now know that there is an extension Y of X where all pairs (g, i) with $g \in X$ are justified. But we do not yet know whether we can do this in such a way that (g, i) is also justified for all g that are in Y, i.e., that Y is justified itself. This is what we tackle in the following lemma. We will, at the same time, make sure that Y is hereditary.

> **Lemma 5.2.11 (There is a justified and hereditary extension.)**
> Let X be a framed subgroup of **Pg**. Then there exists a framed subgroup $Y \subset \mathbf{Pg}$ extending X that is justified and hereditary.

Proof. We define a sequence $(X^n)_{n \in \mathbb{N}}$ of framed subgroups of **Pg** inductively as follows. First set $X^0 := X$. Now assume that X^n has already been defined. Enlarge X^n to an hereditary set, and let $\overline{X^n}$ be the group generated by this set. Then $\overline{X^n}$ is also hereditary, since the definitions of addition (2.3.1) and negation (2.3.3) are genetic. By lemma 5.2.5, we can extend the framing of X^n to a framing of $\overline{X^n}$. Now let X^{n+1} be the extension of X^n such that for every $i \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $g \in \overline{X^n} \setminus \overline{X^n_i}$, the pair (g, i) is justified in X^{n+1} (such an extension exists by the previous lemma 5.2.10).

Finally, take $Y := \bigcup_{n\in\mathbb{N}} X^n$, framed by $Y_i := \bigcup_{n\in\mathbb{N}} X_i^n$ for all $i \in \mathbb{Z}$. This Y has the desired properties.

- Since $X^0 = X$, obviously Y extends X.
- The union of hereditary sets is again hereditary.
- Let $i \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $g \in Y \setminus Y_i$. Then there is an $n \in \mathbb{N}$ such that $g \in X^n$, and since $g \notin Y_i$ we have $g \notin X_i^n$. Therefore (g, i) is justified in X^{n+1}, and so also justified in Y by lemma 5.2.8.

□

5.2.4 Finalizing the Proof

Using this, we can finally show that every framing has its intended meaning, even if the framed subgroup is not justified or hereditary.

Corollary 5.2.12 (Classification of Framings)
Let X be subgroup of **Pg**. Then $(X_i)_{i\in\mathbb{Z}}$ is a framing of X iff there exists an $s \in$ **Pg** such that

$$X_i = \{x \in X : x \geq is\}$$

for all $i \in \mathbb{Z}$.

Proof. It is simple to check that $X_i = \{x \in X : x \geq is\}$ defines a framing of X. Now let X be a framed subgroup of **Pg**. By 5.2.11, there is a justified and hereditary framed subgroup Y of **Pg** which extends X. Lemma 5.2.7 now guarantees that

$$Y_i = \{y \in Y : y \geq is\}$$

for some $s \in$ **Pg**. Since the framing of Y extends the framing of X, we have $X_i = Y_i \cap X$, i.e.,

$$X_i = \{x \in X : x \geq is\}$$

for all $i \in \mathbb{Z}$. □

From this, we get the game value necessary to proof the universal embedding theorem for POA groups.

Theorem 5.2.13 (Pg is a universally embedding POA Group.)
Let $X \subseteq Y$ be partially ordered abelian groups. If there is an embedding $\varphi\colon X \to$ **Pg**, then there exists an embedding $\psi\colon Y \to$ **Pg** such that $\psi|_X = \varphi$.

Proof. As with previous embedding theorems, by Zorn's lemma, it suffices to extend φ to a single element $s \in Y$ (cf. page 27). Let Z be the group generated by X and s. We want to construct an order-preserving group homomorphism

$$\psi\colon Z \to \mathbf{Pg}$$

with $\psi|_X = \varphi$. Let $X' := \varphi(X)$ be the isomorphic copy of X inside **Pg**. Define

$$X'_i := \{\varphi(x) : x \in X, is \leq x\}$$

for all $i \in \mathbb{Z}$. Since φ is an order-preserving group homomorphism, it is easy to check that $(X'_i)_{i\in\mathbb{Z}}$ is a framing of X'. Therefore by lemma 5.2.12, there exists an $s' \in$ **Pg** such that

$$X'_i = \{x' : x' \in X', is' \leq x'\} \tag{$*$}$$

for all $i \in \mathbb{Z}$. Now, since every element of Z is of the form $x + ks$ for $k \in \mathbb{Z}$, $x \in X$, there is a unique group homomorphism $\psi\colon Z \to$ **Pg** extending φ with $\psi(s) = s'$ given by

$$\psi(x + ks) = \varphi(x) + ks' \quad \text{for } x \in X, k \in \mathbb{Z}.$$

In order to verify that ψ is order-preserving, let $z_1, z_2 \in Z$ with $z_1 \leq z_2$. Those are of the form $z_1 = x_1 + k_1 s$, $z_2 + k_2 s$ for $x_1, x_2 \in X$, $k_1, k_2 \in \mathbb{Z}$. Now $z_1 \leq z_2$ can be rewritten as

$$(k_1 - k_2)s \leq x_2 - x_1,$$

which means that $\varphi(x_2 - x_1) \in X'_{k_1-k_2}$. Using ($*$), we get

$$(k_1 - k_2)s' \leq \varphi(x_2 - x_1)$$

which, using that φ is a group homomorphism, rearranges to

$$\psi(z_1) = \varphi(x_1) + k_1 s' \leq \varphi(x_2) + k_2 s' = \psi(z_2).$$

The same argument can also be done in reverse, showing that $\psi(z_1) \leq \psi(z_2)$ implies $z_1 \leq z_2$ for all $z_1, z_2 \in Z$. This shows that ψ is an order-preserving group homomorphism, as desired. □

Table 5.1: Overview of the universal embedding properties proved in this thesis. Here K and R are a field/ring of the kind mentioned in the previous rows of the respective column.

Pg	**No**	**No**$[i]$
partial order	total order	—
partially ordered abelian group	ordered field	field of char. zero
—	totally ordered rings	integral domain of char. zero
—	totally ordered K-vector space	K-vector space
—	totally ordered free R-module	free R-module

Further Remarks 6

6.1 Large Versus Small

We have used the notion of a proper class several times over the course of this thesis. This can be formalized, so there is no logical problem. If, however, one wants to avoid proper classes anyway, dichotomies between “large” and “small” collections other than proper classes and sets may be used.

For example, one could change the definition of games (def. 2.2.1), adding the condition that the left and right sets have to be countable. The collection of all games will then actually be a set (although uncountable), and the same goes for the corresponding collections of game values and surreal numbers. These will have the same universal embedding properties for algebraic structures with countably many elements, since in all our proofs, we never construct a game or a number in such a way that the sets of options would become uncountable.

However, not all notions of large versus small work. Crucially, many of our universal embedding theorems do not hold if we change the definition of a game to only allow for *finitely* many options. These games are called *short games*.

Definition 6.1.1 (Short Game)
A game $\{L \mid R\}$ is called *short* if L and R are finite sets, and all members of L and R are short games.
We denote the collection of all values of short games by $\mathbb{G}$.

Most of combinatorial game theory only concerns $\mathbb{G}$, since most actual games only give their players a finite number of moves to choose from. The collection $\mathbb{G}$ is a countable set that forms a partially ordered abelian group (with the same proofs as we gave in chapter 2). The short games are universally embedding for finite posets — the same proof given in section 5.1 works, since at no point a finite amount of options is increased to an infinite amount of options.

However, $\mathbb{G}$ is *not* a universally embedding abelian group, even if one only considers finite abelian groups. One reason for this is that all elements of $\mathbb{G}$ have an order that is either infinite or a power of 2. The place where the proof for **Pg** fails for $\mathbb{G}$ is in the lemmata 5.2.7 and 5.2.11: the justified and hereditary extension Y of a finite framed subgroup $X \subset \mathbb{G}$ is not necessarily finite anymore, so the game value s defined in the proof of 5.2.7 is not necessarily in $\mathbb{G}$. The algebraic structure of $\mathbb{G}$ was fully analyzed by David Moews [14] and is also described in [3]. For the group structure of $\mathbb{G}$, Moews has shown that

$$\mathbb{G} \cong \mathbb{D}^{\mathbb{N}} \times (\mathbb{D}/\mathbb{Z})^{\mathbb{N}},$$

where $\mathbb{D}$ denotes the set of dyadic rationals.

[14]: Moews (2002), *The Abstract Structure of the Group of Games*

[3]: Siegel (2013), *Combinatorial Game Theory*

The surreal numbers one gets in $\mathbb{G}$ are also just the dyadic rationals $\mathbb{D}$, which do not even form a field. However, the universal embedding theorem for total orders still works for $\mathbb{D}$ when restricting to countable total orders. As we noted before, this is Cantor's isomorphism theorem, a proof of which can be found in [7, p. 83–84].

[7]: Hrbacek et al. (1999), *Introduction to Set Theory*

6.2 Uniqueness

For the end of this thesis, we want to mention that the universal embedding properties in this thesis uniquely characterize **Pg**, **No** and **No**[i] up to isomorphism. The proof of this uses the so-called *back-and-forth method*, a standard technique from model theory. We present the case of **Pg** as a universally embedding POA group, as featured in [15]. The argument works in the exact same way for all the other universal embedding properties we discussed.

[15]: Lurie (2002), *On a Conjecture of Conway*

Suppose that both **U** and **U**′ are universally embedding POA groups. For each ordinal α, take subgroups $U_\alpha \subset \mathbf{U}$ and $U'_\alpha \subset \mathbf{U}'$ such that U_α and U'_α are sets, and

$$\bigcup_{\alpha\in\mathbf{On}} U_\alpha = \mathbf{U}, \qquad \bigcup_{\alpha\in\mathbf{On}} U'_\alpha = \mathbf{U}'.^1$$

1: For example, one could take U_α to be the group generated by all elements of **U** of rank at most α (cf. p. 24), and similarly for U'_α.

These make sure that the following process actually collects all elements of **U** and **U**′. The goal is now to define subgroups $V_\alpha \subseteq \mathbf{U}$ and $V'_\alpha \subseteq \mathbf{U}'$ for each $\alpha \in \mathbf{On}$ with the following properties:

- There is an isomorphism $\varphi_\alpha \colon V_\alpha \to V'_\alpha$.
- The collection V_α (and therefore also V'_α) is a set.
- If $\beta < \alpha$, then $U_\beta \subseteq V_\alpha$ and $U'_\beta \subseteq V'_\beta$.
- If $\beta < \alpha$, then $V_\beta \subseteq V_\alpha$ and $\varphi_\alpha|_{V_\beta} = \varphi_\beta$ (and therefore also $V'_\beta \subseteq V'_\alpha$).

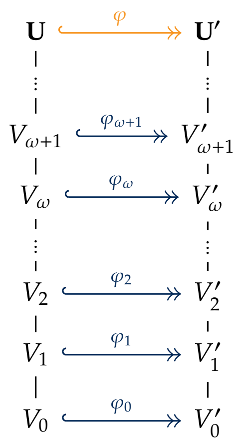


Figure 6.1: We inductively build an isomorphism of **U** and **U**′ by climbing the ladder of ordinals all the way up to the proper classes **U** and **U**′

We define these by induction on **On**. First set $V_0 := V'_0 := \{0\}$. Now let α be an ordinal, and assume that V_β and V'_β have already been constructed for all $\beta < \alpha$ and have the properties mentioned above. If α is a limit ordinal, we take

$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta, \qquad V'_\alpha := \bigcup_{\beta<\alpha} V'_\beta$$

and define

$$\begin{aligned} \varphi_\alpha \colon V_\alpha &\to V'_\alpha \\ x &\mapsto \varphi_\beta(x) \quad \text{for } \beta < \alpha \text{ such that } x \in V_\beta. \end{aligned}$$

It is simple to check that this is an isomorphism.

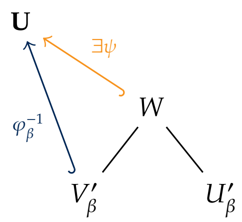


Figure 6.2: First, we use the universal embedding property of **U** to go from a subgroup W of **U**′ *back* to a subgroup of **U**′.

In the case that α is a successor ordinal, we write $\alpha = \beta+1$ for $\beta \in \mathbf{On}$. Let W be the subgroup of **U**′ generated by V'_β and U'_β. Because the latter two are sets, W is also a set. Now we use the universal embedding property of **U**′ to extend the isomorphism $\varphi_\beta^{-1} \colon V'_\beta \to V_\beta \subset \mathbf{U}$ to an embedding $\psi \colon W \to \mathbf{U}$.

Let V_α be the subgroup of $\mathbf{U}$ generated by U_β and $\psi(W)$. Again, $V_{\alpha+1}$ is a set, so by the universal embedding property of $\mathbf{U}'$, we can extend the isomorphism $\psi^{-1}\colon \psi(W) \to W \subset \mathbf{U}'$ to an embedding $\varphi_\alpha\colon V_\alpha \to \mathbf{U}'$. Setting $V'_\alpha := \varphi_\alpha(V_\alpha)$ completes the induction. All our desired properties are clearly satisfied.

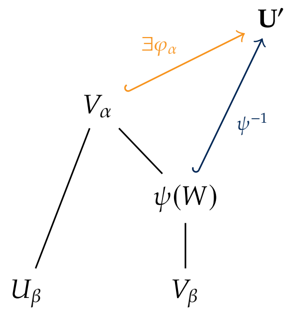


Figure 6.3: Then, we use the universal embedding property of $\mathbf{U}'$ to go from a subgroup V_α of $\mathbf{U}$ *forth* to a subgroup of $\mathbf{U}'$.

Since $U_\beta \subseteq V_\alpha \subset \mathbf{U}$ and $U'_\beta \subseteq V'_\alpha \subset \mathbf{U}'$ for all ordinals $\beta < \alpha$, we have

$$\bigcup_{\alpha\in\mathbf{On}} V_\alpha = \mathbf{U}, \qquad \bigcup_{\alpha\in\mathbf{On}} V'_\alpha = \mathbf{U}'.$$

Therefore we can define

$$\begin{aligned} \varphi\colon \mathbf{U} &\to \mathbf{U}' \\ x &\mapsto \varphi_\alpha(x) \quad \text{for } \alpha \in \mathbf{On} \text{ such that } x \in V_\alpha. \end{aligned}$$

It is again to check that this is an isomorphism, which means we have shown that any two universally embedding POA groups are isomorphic.

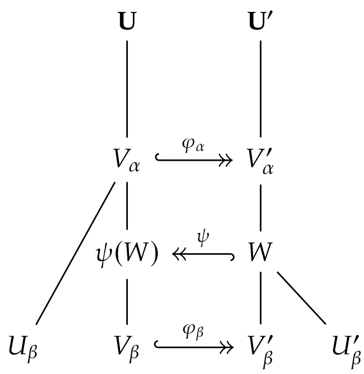


Figure 6.4: The full back-and-forth argument.

Note that in this proof, we needed to choose an extension $\varphi_{\beta+1}$ of the embedding φ_β for each $\beta \in \mathbf{On}$. Those extensions are not unique, and there is no canonical way of choosing them. So in order to make any one of those choices, the axiom of choice is needed. However, since we need to do this for every $\beta \in \mathbf{On}$, we need to make a proper class of choices. The regular axiom of choice is not strong enough to do this, it only guarantees the existence of a choice function on sets. What we have implicitly used here is a stronger principle, called the *axiom of global choice*. One of the consequences of this axiom is that every proper class has the "same cardinality". In other words, if global choice were to fail, there might not even be a bijection between $\mathbf{U}$ and $\mathbf{U}'$. This is one reason why Von Neumann–Bernays–Gödel set theory, which has global choice and direct way of dealing with proper classes, can be better suited for dealing with $\mathbf{No}$ and $\mathbf{Pg}$.

6.3 Avoiding Global Choice

While global choice is needed for $\mathbf{Pg}$ to be *the unique* universally embedding POA Group, it suffices to assume the regular axiom of choice to show that $\mathbf{Pg}$ is *a* universally embedding POA Group (and likewise for all our other embedding theorems).

Let $\mathbf{U}$ be $\mathbf{Pg}$, $\mathbf{No}$ or $\mathbf{No}[i]$. Our first step when proving a universal embedding theorem for $\mathbf{U}$ was always to use Zorn's lemma to reduce the problem to extending an embedding to one new element (see p. 27). Recall that for a given embedding $\varphi\colon X \to \mathbf{U}$, we defined the collection of all partial extensions

$$\boldsymbol{\Phi} := \{\psi\colon Z \to \mathbf{U} : X \subseteq Z \subseteq Y, \psi|_X = \varphi, \psi \text{ embedding}\},$$

which we partially ordered by $\psi \preceq \psi'$ iff ψ' extends ψ.

The issue now is that since $\mathbf{U}$ is a proper class, $\mathbf{\Phi}$ is too. If one accepts the axiom of global choice, there is no problem, since then Zorn's lemma can also be used on proper classes. However, one can also get by with only the regular axiom of choice, which makes sure that all universal embedding theorems also hold in the most commonly used Zermelo–Fraenkel set theory with choice. We describe how this is done in this section.[2] Note that this requires more set theoretic preliminaries than the rest of this thesis. All concepts used can, for example, be found in [7].

2: We thank Asaf Karagila for explaining this procedure on Mathematics Stack Exchange. [16]

[7]: Hrbacek et al. (1999), *Introduction to Set Theory*

Let $C \subseteq \mathbf{\Phi}$ be a chain. Now for every $\psi \in C$, the image of ψ is a subset of Y. Also if $\psi, \psi' \in C$ have the same image, then $\psi \le \psi'$ implies that $\psi = \psi'$, and so does $\psi' \le \psi$. Since C is a chain, one of these inequalities must hold, meaning that any two elements of C with the same image are equal. This means that the image of each member of C is a *different* subset of Y. The set of all subsets of Y is again a set, the power set of Y. So if κ be the cardinality of the power set of Y, our argument shows that every chain in $\mathbf{\Phi}$ has at most cardinality κ.

Let μ be the successor cardinal to κ. Then (since we are assuming choice) μ is a *regular* cardinal. We now filter the elements of $\mathbf{\Phi}$ by inductively defined subclasses:

- $\Phi_0 := \{\varphi\}$.
- At successor stages, define $\Phi_{\alpha+1}$ by adding to Φ_α all the elements of $\mathbf{\Phi}$ that are strictly greater than all elements of Φ_α, and have minimal rank in the von Neumann hierarchy.
- For limit ordinals α, take the union $U := \bigcup_{\beta<\alpha} \Phi_\beta$. Now define Φ_α by adding to U all upper bounds of minimal rank for chains in U which did not yet have an upper bound in U.

Now consider Φ_μ, and let $C \subseteq \Phi_\mu$ be a chain. Then C is also a chain in $\mathbf{\Phi}$, which means C has cardinality less than μ. Now the collection

$$S := \{\alpha \in \mathbf{On} : \exists \psi \in C \colon \psi \in \Phi_\alpha\}$$

is a set of ordinals less than μ, and the cardinality of S is less than the cardinality of C, so in particular less than μ. By the regularity of μ, we get that S must be bounded by an ordinal $\lambda < \mu$. So $C \subseteq \Phi_\lambda$, which means an upper bound of C was added by $\Phi_{\lambda+\omega}$, since $\lambda + \omega$ is the next limit ordinal after λ. Therefore every chain in Φ_μ has an upper bound in Φ_μ, so by Zorn's lemma, there is a maximal element $\psi \in \Phi_\mu$. Here the regular axiom of choice suffices because Φ_μ is a set.

Now $\psi \in \Phi_\mu$ means that either $\psi \in \bigcup_{\alpha<\mu} \Phi_\alpha$, or ψ is an upper bound to a chain in $\bigcup_{\alpha<\mu} \Phi_\alpha$. But, as we have just seen, every chain in Φ_μ already has an upper bound in $\Phi_{\lambda+\omega}$ for some $\lambda < \mu$. So we definitely have $\psi \in \Phi_\alpha$ for some $\alpha < \mu$. If ψ was not maximal in $\mathbf{\Phi}$, then some extension of it exists, which means some extension of minimal rank has been added to $\Phi_{\alpha+1}$, contradicting the maximality of ψ in Φ_μ. So ψ is the desired maximal element of $\mathbf{\Phi}$.

Bibliography

[1] John H. Conway. *On Numbers and Games*. 2nd ed. A K Peters, Ltd., 2001 (cited on pages 5, 12, 19, 20, 28, 30).

[2] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways for Your Mathematical Plays*. 2nd ed. Vol. 1. 4 vols. A K Peters, Ltd., 2001 (cited on page 5).

[3] Aaron N. Siegel. *Combinatorial Game Theory*. Graduate studies in mathematics 146. American Mathematical Society, 2013 (cited on pages 5, 11, 28, 45).

[4] Thomas Forster. *Logic, Induction and Sets*. London Mathematical Society Student Texts 56. Cambridge University Press, 2003 (cited on pages 13, 24).

[5] Jos W.H.M. Uiterwijk and Michael Barton. “New results for Domineering from combinatorial game theory endgame databases”. In: *Theoretical Computer Science* 592 (2015), pp. 72–86 (cited on page 15).

[6] Walter Rudin. *Principles of Mathematical Analysis*. 3rd ed. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976 (cited on page 19).

[7] Karel Hrbacek and Thomas Jech. *Introduction to Set Theory*. 3rd ed. Monographs and Textbooks in Pure and Applied Mathematics 220. Marcel Dekker, Inc., 1999 (cited on pages 24, 28, 46, 48).

[8] Alexander Prestel and Charles N. Delzell. *Mathematical Logic and Model Theory. A Brief Introduction*. Universitext. Springer-Verlag London, 2011 (cited on page 26).

[9] Harry Gonshor. *An Introduction to the Theory of Surreal Numbers*. London Mathematical Society lecture note series 110. Cambridge University Press, 1986 (cited on pages 28, 33).

[10] Serge Lang. *Algebra*. 3rd ed. Graduate Texts in Mathematics 211. Springer-Verlag New York, 2002 (cited on page 30).

[11] Philip Ehrlich. “Number Systems with Simplicity Hierarchies: A Generalization of Conway’s Theory of Surreal Numbers”. In: *The Journal of Symbolic Logic* 66.3 (2001), pp. 1231–1258 (cited on page 33).

[12] Vincenzo Mantova and Mickaël Matusinski. “Surreal Numbers With Derivation, Hardy Fields and Transseries: A Survey”. In: *Ordered Algebraic Structures and Related Topics*. Jan. 2017, pp. 265–290 (cited on page 33).

[13] Philip Ehrlich and Elliot Kaplan. “Surreal Ordered Exponential Fields”. In: *The Journal of Symbolic Logic* 86.3 (2021), pp. 1066–1115 (cited on page 33).

[14] David Moews. “The Abstract Structure of the Group of Games”. In: *More Games of No Chance*. Mathematical Sciences Research Institute publications 42. Cambridge University Press, 2002 (cited on pages 34, 45).

[15] Jacob Lurie. “On a Conjecture of Conway”. In: *Illinois Journal of Mathematics* 46.2 (2002), pp. 497–506 (cited on pages 34, 46).

[16] Asaf Karagila (https://math.stackexchange.com/users/622/asaf-karagila). *Zorn's lemma on proper class with chains bounded in cardinality*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/4757973. (version: 2023-08-24) (cited on page 48).