Bachelor thesis

On Van der Waerden's Conjecture

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July 31, 2023

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1 Introduction

The conjecture of Van der Waerden stood unproven for several years. In 1926 Van der Waerden stated that
if $A \in \mathbb{R}^{n\times n}$ is a doubly stochastic matrix then

$$per(A) \geq \frac{n!}{n^n}$$

and equality holds if and only if $a_{ij} = 1/n$ for all $i, j \in \{1, \dots, n\}$. This conjecture which is now a theorem, was first proven in 1980 by the russian mathematician G.P. Egorychev. The purpose of this bachelor's thesis is to make an almost self contained elementary proof which needs little to no prior knowledge. The first part of my thesis is very much guided by the work of Donald E. Knut [1], we will follow the same path to prove Van der Waerden's Theorem but with more eye for detail. In chapter 7 we will discuss an approach to define the permanent for block matrices.

2 Quadratic forms

A quadratic form $f(x_1, \dots, x_n)$ of n variables is an expression of the following form

$$f(x_1, \dots, x_n) = \sum_{i,j} f_{ij} x_i x_j. \tag{2.1}$$

Every quadratic form can be defined by a matrix $F \in \mathbb{R}^{n\times n}$ with entries f_{ij} which are the coefficients of the monomial $x_i x_j$. Due to the identity $x_i x_j = x_j x_i$ we can assume F to be in upper triangle or even symmetric form. We will always consider F to be a symmetric matrix. For example these 3 matrices all describe the same quadratic form

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & 10 \\ 0 & 5 & 14 \\ 0 & 0 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix}.$$

Quadratic forms are called equivalent if their corresponding matrices are congruent. Two matrices A and B are called congruent if there is an invertible matrix P such that $A = P^\intercal BP$. We use the notation $r(F)$ for the rank and $p(F)$ for the number of positive eigenvalues of F. We will now look at a few lemmas to further grasp the concept of quadratic forms so we can later make a connection to the permanent of a matrix.

Lemma 2.1. *Let $f(x_1, \dots, x_n) = \sum_{i,j} f_{i,j} x_i x_j$ be a quadratic form and let the vector $(a_1, \dots, a_n)$ of real numbers be such that $a_1 \neq 0$ and $f(a_1, \dots, a_n) = c \neq 0$. Then the nonsingular transformation defined by*

$$x_i = a_i \left(y_1 - \sum_{j\geq 2} y_j \sum_k f_{jk} a_k \frac{1}{c} \right) + y_i * (i \geq 2) \tag{2.2}$$

$$y_1 = \sum_{i,j} f_{ij} a_i x_j \frac{1}{c}, \qquad y_i = x_i - x_1 a_i \frac{1}{a_1} \qquad \text{for } i \geq 2 \tag{2.3}$$

makes $f(x_1, \dots, x_n) = c y_1^2 + g(y_2, \dots y_n)$*, where* g *is a quadratic form in* $n-1$ *variables. The notation* $(i \geq 2)$ *denotes* 1 *if* $i \geq 2$ *and* 0 *otherwise.*

Proof. It is easy to verify that (2.2) and (2.3) are inverses of each other by substituting y_i into (2.2) and x_i into (2.3). That concludes the nonsingularity of the transformation. To complete the proof we have to show that the coefficient of y_1^2 is c and the coefficient of $y_1 y_u$ is 0 for any $u \geq 2$. If we take a look at our quadratic form $f(x_1, \dots, x_n) = \sum_{i,j} f_{ij} x_i x_j$ and use the given definition of x_i we can write

$$\begin{aligned} f(x_1, \dots, x_n) = \sum_{i,j} f_{ij} \Bigg[a_i a_j \Bigg(y_1^2 - 2 y_1 \sum_{l \geq 2} y_l \sum_k f_{lk} a_k \frac{1}{c} + \Bigg(\sum_{l \geq 2} y_l \sum_k f_{lk} a_k \frac{1}{c} \Bigg)^2 \Bigg) \\ + a_i y_1 y_j (j \geq 2) - a_i \sum_{l \geq 2} y_l \sum_k f_{lk} a_k \frac{1}{c} y_j (j \geq 2) + a_j y_1 y_i (i \geq 2) \\ - a_j \sum_{l \geq 2} \sum_k f_{lk} a_k \frac{1}{c} y_i (i \geq 2) + y_i y_j (i \geq 2)(j \geq 2) \Bigg] \end{aligned}$$

By extracting the only term that includes y_1^2 we can see that the coefficient must be $\sum_{i,j} f_{ij} a_i a_j = c$. To find the coefficient of $y_1 y_u$ with $u \geq 2$ we only consider the terms that contribute to it and the following coefficient remains

$$\begin{aligned} &\sum_{i,j} f_{ij} \Bigg(-2 a_i a_j \sum_k f_{lk} a_k \frac{1}{c} + a_i (j = k) + a_j (i = k) \Bigg) \\ &= -2 \sum_k f_{lk} a_k + \sum_i f_{ik} a_i + \sum_j f_{kj} a_j = 0. \end{aligned}$$

For the last equality we used the symmetry of F. □

A crutial theorem for quadratic forms is Sylvester's law of inertia we can use this law to find a very simple form for every quadratic form. We will only need the following part of it.

Theorem 2.2. *Every symmetric matrix* $A \in \mathbb{R}^{n \times n}$ *is congruent to a matrix* $D = diag(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$*. The number of ones equals the number of positive eigenvalues and the number of* -1 *equals the number of negative eigenvalues of* A*. Two symmetric matrices are congruent if and only if they have the same number of positive and negative eigenvalues.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. According to the finite-dimensional spectral Theorem there exists an orthogonal matrix P such that $diag(\lambda_1, \dots, \lambda_n) = P^\intercal AP$. Where $\lambda_1 \geq \dots \geq \lambda_n$ are the eigenvalues of A, w.l.o.g. $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_n = 0$. By permuting the columns with permutation matrices we can get the eigenvalues in the correct order and all there is left to do is scaling them to 1 or -1 if they are nonzero. This can be done by multiplication from left and right by matrices $Q = Q^\intercal = diag(\frac{1}{\sqrt{|\lambda_1|}}, \dots, \frac{1}{\sqrt{|\lambda_k|}}, 1, \dots, 1)$. If we multiply all the used matrices together and call the result of that product T, we get $diag(1, \dots, 1, -1 \dots, -1, 0 \dots, 0) = T^\intercal AT$ where T is nonsingular. □

Lemma 2.3. *Every quadratic form is equivalent to a simple quadratic form*

$$g(y_1, \dots, y_n) = y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_r^2$$

and the numbers p and r are unique. In other words, if we have equivalent quadratic forms

$$y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_r^2 \quad \text{and} \quad z_1^2 + \dots + z_q^2 - z_{q+1}^2 - \dots - z_s^2$$

then $p = q$ and $r = s$

Proof. Follows directly by Theorem 2.2. □

Lemma 2.4. *Let $f_\theta(x_1, \dots, x_n)$ be the quadratic form $f_\theta(x_1, \dots, x_n) = (1-\theta)f_0(x_1, \dots, x_n) + \theta f_1(x_1, \dots, x_n)$ that changes from f_0 to f_1 as θ varies from 0 to 1. If $r(f_\theta) = n$ for $0 \leq \theta \leq 1$ then $p(f_0) = p(f_1)$*

Proof. f_θ is a quadratic form and the corresponding matrix is therefore symmetric and diagonalizable. $r(F_\theta) = n$ is equivalent to saying that F_θ has no zeroes as eigenvalues. If we use the symmetric Gauss algorithm we get a diagonalized matrix congruent to F_θ ($P^\intercal F_\theta P = D_\theta$). The symmetric Gauss algorithm can be looked at as a composition of continuous functions. Therefore we can in this case state, that the determination of eigenvalues is continuous. Due to the continuity the sign of the eigenvalues can't change if we vary θ from 0 to 1. □

3 Quadratic forms and permanents

The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$per(A) = \sum_\pi \prod_{i=1}^n a_{i\pi(i)},$$

taken over all permutations π of $\{1, \dots, n\}$. We will write a_i for the i-th row $(a_{i1}, \dots a_{in})$ of A, and $per(A) = per(a_1, \dots, a_n)$ if we enumerate its rows. From now on A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by removing row i and column j from A.

A basic fact about permanents that we will need is the following identity which looks very similar to the Laplace-expansion of the determinant.

$$per(A) = \sum_j a_{ij} per(A_{ij}) = \sum_i a_{ij} per(A_{ij}) \tag{3.1}$$

The next part of the theory is a proof of two lemmas, which are proved simultaneously by induction on n.

Lemma 3.1. *Let $a_1 \dots a_{n-1}$ be vectors of nonnegative numbers in which at least $n+1-i$ elements of a_i are positive, and suppose that $b = (b_1, \dots b_n)$ is any vector of real numbers such that*

$$per(a_1, \dots a_{n-1}, b) = 0. \tag{3.2}$$

Then

$$per(a_1, \dots, a_{n-2}, b, b) \leq 0 \tag{3.3}$$

Furthermore, $per(a_1, \dots, a_{n-2}, b, b) = 0$ if and only if $b_1 = \cdots = b_n = 0$.

Lemma 3.2. *Let $a_1, \dots, a_{n-2}$ be as in Lemma 3.1 and let f be the quadratic form*

$$f(x_1, \dots, x_n) = per(a_1, \dots, a_{n-2}, x, x) = \sum_{i,j} f_{i,j} x_i x_j \tag{3.4}$$

where x stands for the vector $(x_1, \dots, x_n)$. Then $r(F) = n$ and $p(F) = 1$.

Proof. We begin the induction by showing both results for $n = 2$. Let's start with Lemma 3.1 where $a_1 = (a_{11}, a_{12})$ with $a_{11}, a_{12} > 0$ and $b = (b_1, b_2)$ such that

$$per \begin{pmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{pmatrix} = a_{11}b_2 + a_{12}b_1 = 0.$$

Now this can only hold true if b_1 and b_2 have different *sign* or both are equal to zero. If they have different *sign* then $per(b, b) = 2b_1b_2 < 0$. Therefore Lemma 3.1 holds true for $n = 2$. We continue with Lemma 3.2.

$$f(x_1, x_2) = per \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} = 2x_1x_2 \quad \Rightarrow \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By diagonalizing the matrix F we can obtain that the eigenvalues of F are 1 and -1. Therefore Lemma 3.2 also holds true for $n = 2$.

From now on we assume $n \geq 3$ and both lemmas hold true for $n - 1$. In the quadratic form (3.4) f_{ij} is the permanent of the $(n-2) \times (n-2)$ matrix $(a_1, \dots, a_{n-2})$ obtained by removing columns i and j. Obviously there is no x_i^2 in (3.4) therefore $f_{ii} = 0$ for all i. Let's assume $r(F) < n$, it follows that the matrix F is singular so there is a nonzero vector $(c_1, \dots c_n)$ such that $\sum_j f_{ij} c_j = 0$ for all i.

This is equivalent to saying that

$$per(a_1, \dots, a_{n-2}, c, x) = \sum_{i,j} f_{ij} c_i x_j = 0$$

for all x, in particular $per(a_1, \dots, a_{n-2}, c, c) = 0$.
Furthermore we have $per_j(a_1, \dots, a_{n-2}, c) = 0$ for all j where per_j denotes the permanent obtained by removing column j. This can be seen, by looking at the Laplace-expansion of the matrix $A = (a_1, \dots, a_{n-2}, c, x)$:

$$0 = per(a_1, \dots, a_{n-2}, c, x) = \sum_k per(A_{nk}) x_k,$$

where A_{nk} is obtained by removing row n and column k of A. This equation holds true for all x. We now choose $x = (0, \dots 0, 1, 0, \dots, 0)$ where the 1 is at position j. It follows that $per_j(a_1, \dots, a_{n-2}, c) = per(A_{nj}) = 0$. By using the induction step for $n-1$ we get $per_j(a_1, \dots, a_{n-3}, c, c) \leq 0$. Now

$$0 = per(a_1, \dots, a_{n-2}, c, c) = \sum_j a_{(n-2)j} per_j(a_1, \dots, a_{n-3}, c, c) \leq 0$$

hence we have $per_j(a_1, \dots, a_{n-3}, c, c) = 0$ whenever $a_{(n-2)j} > 0$, this occurs at least 3 times and by using the induction step for $n-2$ it follows that $c_1 = \cdots = c_n = 0$, a contradiction. We have now proven the first part of Lemma 3.2, the fact that $r(F) = n$. For the other half it suffices to compute $p(F)$ in the special case that $a_1 = \cdots = a_{n-2} = (1, \dots, 1)$ since we can transform the rows one by one from this case into (3.4) by Lemma 2.4

$$\theta \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-2)1} & \dots & a_{(n-2)n} \\ x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix} + (1-\theta) \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \\ x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix}$$

and $r(F) = n$ for all intermediate quadratic forms. In our special case $f(x_1, \dots, x_n) = per(a_1, \dots, a_{n-2}, x, x)$ is defined by the Matrix

$$F = (n-2)! \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & \dots & 0 \end{pmatrix}$$

for simplification we can ignore the factor $(n-2)!$ and by multiplying F with the vector $(1, \dots, 1)$ we see that $(n-1)$ is an eigenvalue with multiplicity 1. Similarly by multiplying F with the vector $(1, \dots, 0, -1, 0, \dots, 0)$ where the -1 is at position j we get the eigenvalue -1 for every $1 < j \leq n$, hence $p(F) = 1$.

We now turn to the proof of Lemma 3.1. The hypothesis on $a_1, \dots, a_{n-1}$ implies $per(a_1, \dots, a_{n-1}, a_{n-1}) > 0$, hence we have $f(a_{(n-1)1}, \dots, a_{(n-1)n}) = c > 0$, in terms of the quadratic form (3.4). We assume without loss of generality that $a_{(n-1)1} > 0$. Therefore if we apply Lemma 2.1 we obtain

$$f(x_1, \dots, x_n) = cy_1^2 + g(y_2, \dots, y_n).$$

We know by Lemma 3.2 that $p(F) = 1$. Therefore we know that $g(y_2, \dots, y_n) \leq 0$ for all $(y_2, \dots, y_n)$ and $g(y_2, \dots, y_n) = 0$ if and only if $(y_2, \dots, y_n) = 0$. By Lemma 2.1 we have

$$y_1 = \sum_{1 \leq i,j \leq n} f_{ij} a_{(n-1)i} x_j / c = per(a_1, \dots, a_{n-1}, x)/c \tag{3.5}$$

The second equality can be achieved by definition of the quadratic form in Lemma 3.2. This together with our hypothesis (3.2), implies that $f(b_1, \dots, b_n) \leq 0$. At last we have to show that $f(b_1, \dots, b_n) = 0$ if and only if $b_1 = \dots = b_n = 0$. Obviously, if $b_1 = \dots = b_n = 0$ it follows that $f(b_1, \dots, b_n) = 0$. And if we have $f(b_1, \dots, b_n) = 0$ it follows by (3.5) that $y_1 = 0$ and our argument from before implies that $y_2 = \dots = y_n = 0$ □

Theorem 3.3. *Let $a_1, \dots a_{n-1}$ be nonnegative vectors such that a_i contains at least $n + 1 - i$ positive entries, and let a_n be any vector of real numbers. Then*

$$per(a_1, \dots, a_{n-1}, a_n)^2 \geq per(a_1, \dots, a_{n-1}, a_{n-1}) per(a_1, \dots, a_n, a_n), \tag{3.6}$$

and equality holds if and only if $a_n = \lambda a_{n-1}$ for some real number λ.

Proof. Let $per(a_1, \dots, a_{n-1}, a_n) = \lambda per(a_1, \dots, a_{n-1}, a_{n-1})$, then λ is well defined since $per(a_1, \dots, a_{n-1}, a_{n-1}) > 0$. If we set $b = a_n - \lambda a_{n-1}$, we get (3.2) since the permanent is a multilinear function in all of it's rows. Hence Lemma 3.1 tells us that

$$\begin{aligned}
0 &\geq per(a_1, \dots, a_{n-2}, b, b) \\
&= per(a_1, \dots, a_{n-2}, b, a_n) - \lambda per(a_1, \dots, a_{n-2}, b, a_{n-1}) \\
&= per(a_1, \dots, a_{n-2}, a_n, a_n) - 2\lambda per(a_1, \dots, a_n) + \lambda^2 per(a_1, \dots, a_{n-2}, a_{n-1}, a_{n-1}) \\
&= per(a_1, \dots, a_{n-2}, a_n, a_n) - \lambda^2 per(a_1, \dots, a_{n-2}, a_{n-1}, a_{n-1}) \\
&= per(a_1, \dots, a_{n-2}, a_n, a_n) - \frac{per(a_1, \dots, a_n)^2}{per(a_1, \dots, a_{n-2}, a_{n-1}, a_{n-1})}.
\end{aligned}$$

Equality holds if and only if $b = 0$. □

Corollary 3.4. *Let $a_1, \dots, a_{n-1}$ be nonnegative vectors and let a_n be arbitrary. Then the inequality* (3.6) *holds.*

Proof. This holds true because of the monotonicity of the limit and can be observed by looking at the vectors $a_i + (\epsilon, \dots, \epsilon)$. □

4 Doubly stochastic matrices

We will now take a closer look at doubly stochastic matrices, which brings us a step closer to proving Van der Waerden's Theorem. A doubly stochastic matrix A has the following properties.

- $a_{ij} \geq 0$
- $\sum_i a_{ij} = \sum_j a_{ij} = 1$

The sum over each row and column must always equal 1. Thus each row and column can be seen as a probability distribution and that is why these matrices are stochastic in a double sense. Doubly stochastic matrices have some nice properties which are easy to prove.

Proposition 4.1. *If A and B are doubly stochastic so is $AB = C$*

Proof. Obviously it holds that $c_{ij} \geq 0$ but we still have to prove that the row and column sums are equal to 1. Let's consider an arbitrary $i \in \{1, \ldots, n\}$

$$\sum_i c_{ij} = \sum_i \sum_k a_{ik} b_{kj} = \sum_k b_{kj} \sum_i a_{ik} = \sum_k b_{kj} 1 = 1$$

The proof works analogous for the row sums. □

Proposition 4.2. *If A and B are doubly stochastic so is $\theta A + (1-\theta)B$ for $\theta \in [0,1]$.*

Proof. Obviously the entries of the resulting matrix are still nonnegative. We have to show that the row and column sums are egual to 1 for all $\theta \in [0,1]$. Let's consider an arbitrary $i \in \{1, \ldots, n\}$ and $\theta_0 \in [0,1]$

$$\sum_i c_{ij} = \theta_0(a_{1j} + \cdots + a_{nj}) + (1-\theta_0)(b_{1j} + \cdots + b_{nj}) = \theta_0 + (1-\theta_0) = 1$$

The proof works analogous for the sum over j. □

Now that we are more familiar with the concept of a doubly stochastic matrix we want to deepen our understanding and we will do this by looking at a lemma by Garrett Birkhoff which he proved in 1946. The simplest kind of a doubly stochastic matrix is a permutation matrix. Garrett Birkhoff stated that permutation matrices can be seen as the corners of a polygon in which all doubly stochastic matrices are contained. In simple terms this means that all doubly stochastic matrices are convex combinations of permutation matrices.

Lemma 4.3. *The $n \times n$ matrix A is doubly stochastic if and only if there exist nonnegative numbers t_π and permutation matrices P_π such that*

$$A = \sum_\pi t_\pi P_\pi \quad \text{and} \quad \sum_\pi t_\pi = 1 \tag{4.1}$$

where the sums are over all $n!$ permutations π of $\{1, \ldots, n\}$.

Proof. Obviously every matrix of the form (4.1) is doubly stochastic, the more difficult part is to show that every doubly stochastic matrix can be represented in terms of permutation matrices. □

Lemma 4.4. *Consider n men and n women such that each man-woman pair is either ,,compatible" or ,,incompatible". If there is no way to match the men and women into n compatible marriages, then for some $k > 0$ there is a set of k men who are compatible with only $k-1$ women.*

Proof. Suppose that there is a way to obtain m compatible marriages but no way to obtain $m+1$ for some $m < n$, and let x be an unmarried man in one of these maximum matchings. Consider all chains of relationship of the form

$$i_1 \to j_1 \Rightarrow i_2 \to j_2 \Rightarrow \cdots \Rightarrow i_r \to j_r \tag{4.2}$$

where $x = i_1$, and the relation $i \to j$ means ,,man i is compatible with woman j" while $j \Rightarrow i$ means ,,woman j is married to man i". In every such chain the woman j_r must be married because otherwise it would be possible to create $m+1$ marriages by performing $r-1$ divorces and then marrying i_l with j_l for $1 \leq l \leq r$. Consider now the set S of all men i_l appearing in chains (4.2) and the set T of all women j_l that appear. Then each women in T is married to a man in S, and each man in S (except x) is married to a woman in T. Therefore S contains k elements while T contains only $k-1$. □

Proof. Returning now to the proof of Lemma 4.3, let A be doubly stochastic and let us imagine n men and women such that man i is compatible with woman j if and only if $a_{ij} > 0$. In these circumstances a set of n compatible marriages is possible, for if S were a set of k men that are compatible with only $k-1$ women in a set T we get a block matrix that looks like this

$$\left(\begin{array}{c|c} \star & 0 \\ \hline \star & \star \end{array}\right)$$

Where the $\star$ block at the top left is of dimension $k \times (k-1)$. This already cannot happen since the sum $\sum_{i \in S, j \in T} a_{ij} = k$ because it includes all nonzero elements of k rows. But the sum also can't be larger than $k-1$ because it involves only $k-1$ columns, a contradiction.

Thus there is a permutation π such that $a_{l\pi(l)} > 0$ for $1 \leq l \leq n$. Let $t_\pi = min(a_{1\pi(1)}, \ldots, a_{n\pi(n)})$. If $t_\pi = 1$, A is a permutation matrix, and has trivially the form (4.1). Otherwise we write A as

$$A = t_\pi P_\pi + (1 - t_\pi) B$$

Where B is constructed in the following way

$$b_{ij} = \frac{a_{ij}}{1 - t_\pi} \quad \text{if } j \neq \pi(i)$$
$$b_{i\pi(i)} = \frac{a_{i\pi(i)} - t_\pi}{1 - t_\pi}.$$

We will now show that $A = t_\pi P_\pi + (1 - t_\pi)B$ and that B is doubly stochastic.

$$(t_\pi P_\pi + (1-t_\pi)B)_{i,j} = 0 + (1-t_\pi)\frac{a_{ij}}{1-t_\pi} = a_{ij} \quad \text{if } j \neq \pi(i)$$
$$(t_\pi P_\pi + (1-t_\pi)B)_{i,\pi(i)} = t_\pi + (1-t_\pi)\frac{a_{i\pi(i)} - t_\pi}{1-t_\pi} = a_{i\pi(i)}$$

Trivially all entries of B are nonnegative, so it remains to show that column and row sums of B are equal to 1.

$$\sum_i b_{ij} = \sum_{i,j\neq\pi(i)} \frac{a_{ij}}{1-t_\pi} + \frac{a_{i\pi(i)} - t_\pi}{1-t_\pi} = \frac{1}{1-t_\pi}\left(\sum_i a_{ij} - t_\pi\right) = \frac{1}{1-t_\pi}(1-t_\pi) = 1$$

The proof remains the same for the row sums, which concludes that B is doubly stochastic and we can indeed decompose A in that way. Due to the construction, B contains at least one more zero than A. By induction on the number of nonzero entries, we can do the same process with B, and this yields the desired representation of A. □

We will now look at yet another property of matrices which we also need to proof our desired result. The matrix A is called decomposable if the set $\{1, \ldots, n\}$ can be partitioned into nonempty disjoint subsets S and T such that $a_{ij} = 0$ whenever $i \in S$ and $j \in T$. This is equivalent to saying that there is a permutation matrix P such that $PAP^\intercal$ is a block upper triangle form. Before we finish this chapter we will proof two lemmas which will help us on the coming pages.

Lemma 4.5. *If $x = (x_1, \ldots, x_n)$ is an eigenvector for the eigenvalue 1 of a doubly stochastic matrix A ($Ax = x$) having some components unequal, then A is decomposable.*

Proof. If $(x_1, \ldots, x_n)$ is a vector such that $Ax = x$, so is $(x_1 + c, \ldots, x_n + c)$, hence we can assume that all components of x are nonnegative and at least one is zero. Let $S = \{i|x_i = 0\}$ and $T = \{i|x_i > 0\}$, then $\sum_j a_{ij}x_j = x_i$ implies that $a_{ij} = 0$ whenever $i \in S$ and $j \in T$. □

This proof of Lemma 4.5 uses only the „single stochastic" property of A but doubly stochastic matrices satisfy an even stronger condition.

Lemma 4.6. *If A is a doubly stochastic decomposable matrix then $a_{ij} = 0$ unless $i, j \in T$ or $i, j \in S$, which is equivalent to saying that there is a permutation matrix P such that $PAP^\intercal$ is a block diagonal matrix.*

Proof. Due to the decomposability we already know that $a_{ij} = 0$ whenever $i \in S$ and $j \in T$. It is left to prove that $a_{ij} = 0$ for $i \in T$ and $j \in S$ if additionally A is doubly stochastic.

$$\sum_{i,j\in S} a_{ij} = \sum_{i\in S}\sum_{1\leq j\leq n} a_{ij} = \sum_{i\in S} 1 = \sum_{j\in S} 1 = \sum_{j\in S}\sum_{1\leq i\leq n} a_{ij} = \sum_{i,j\in S} a_{ij} + \sum_{i\in T,j\in S} a_{ij}$$

□

5 Minimal matrices

For the purpose of this discussion we shall say that an $n \times n$ matrix A is minimal if it is doubly stochastic and if it has the smallest permanent among all $n \times n$ doubly stochastic matrices. There is at least one minimal matrix since the permanent is a continuos function of the matrix elements and the set of doubly stochastic matrices is a closed and bounded subset of n^2-dimensional space. Lemma 3.1 implies that the permanent of A must be bigger than 0, because we can write for any doubly stochastic matrix $A = t_{\pi_1} P_{\pi_1} + B$ where B is the rest of the convex combination. Obviously $per(t_{\pi_1} P_{\pi_1}) = t_{\pi_1}^n$ and all entries of B are nonnegative which implies that $per(A) \geq t_{\pi_1}^n > 0$. Another basic fact about permanents is the following equation.

$$per(A + \epsilon B) = per(A) + \epsilon \sum_{i,j} b_{ij} per(A_{ij}) + \mathcal{O}(\epsilon^2) \tag{5.1}$$

This can be obtained by looking at the definition of the permanent:

$$per(A + \epsilon B) = per \begin{pmatrix} a_{11} + \epsilon b_{11} & \dots & a_{1n} + \epsilon b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + \epsilon b_{n1} & \dots & a_{nn} + \epsilon b_{nn} \end{pmatrix} = \sum_{\pi} \prod_{i=1}^{n} (a_{i\pi(i)} + \epsilon b_{i\pi(i)})$$

Obviously the terms without ϵ is exactly $per(A)$. If we look for the terms with ϵ^1 we find that those are the ones where we multiply with $\epsilon b_{i\pi(i)}$ only once and for the rest of the product always choose $a_{i\pi(i)}$. If we do that we get $\epsilon \sum_{i,j} b_{ij} per(A_{ij})$ and we use the big-$\mathcal{O}$ notation for the terms with higher order.

If A is doubly stochastic we call B a valid modification for A if the row sums and column sums of B are zero and if $b_{ij} \geq 0$ whenever $a_{ij} = 0$. It follows immediately by definition of B that the column and row sums of $A + \epsilon B$ are equal to 1. We also know that $b_{ij} < 0$ is only possible if $a_{ij} > 0$ so if we choose ϵ small enough then $a_{ij} + \epsilon b_{ij} > 0$. Hence $A + \epsilon B$ is doubly stochastic.

Lemma 5.1. *If A is a minimal matrix and if B is a valid modification for A, then*

$$\sum_{i,j} b_{ij} per(A_{ij}) \geq 0$$

Proof. If we look at $per(A + \epsilon B)$ as a polynomial function in ϵ then the slope at $\epsilon = 0$ is given by $\sum_{i,j} b_{ij} per(A_{ij})$. Because of $per(A) \leq per(D)$ for all doubly stochastic matrices D we know that $per(A) \leq per(A + \epsilon B)$ for all $\epsilon > 0$. Therefore it holds that $\sum_{i,j} b_{ij} per(A_{ij}) \geq 0$. □

Lemma 5.2. *A minimal matrix is indecomposable.*

Proof. Suppose that A is a minimal matrix and decomposable. Therefore it follows by Lemma 4.6 that $a_{ij} > 0$ if and only if $i, j \in S$ or $i, j \in T$, where S and T are sets of decomposability for A.

We know that $per(A) > 0$, so there must be a permutation π with $a_{i\pi(i)} > 0$ for all i. Let's consider $s \in S$, $t \in T$ and let B be a matrix that is entirely zero except that $b_{s\pi(s)} = b_{t\pi(t)} = -1$ and $b_{s\pi(t)} = b_{t\pi(s)} = 1$. We know that $\pi(s) \in S$ and $\pi(t) \in T$ because $a_{s\pi(s)}, a_{t\pi(t)} > 0$ and since $b_{i\pi(j)} \geq 0$ for all $i \neq j$ the matrix B is a valid modification for A. Therefore

$$per(A_{s\pi(t)}) + per(A_{t\pi(s)}) - per(A_{s\pi(s)}) - per(A_{t\pi(t)}) \geq 0$$

by Lemma 5.1. But this cannot happen, because $per(A_{s\pi(s)}), per(A_{t\pi(t)}) > 0$ since A without row $s, (t)$ and column $\pi(s), (\pi(t))$ still has a positive permanent if we use the same permutation π from before. Furthermore $per(A_{s\pi(t)})$ and $per(A_{t\pi(s)})$ are zero. $per(A_{t\pi(s)})$ is zero because $A_{t\pi(s)}$ has k rows corresponding to S in which all nonzero entries occur in only $k-1$ columns corresponding to $S - \{\pi(s)\}$. □

Lemma 5.3. *If A is a minimal matrix, then $per(A_{ij}) > 0$ for all i, j.*

Proof. If $per(A_{ij}) = 0$, it has some set S of $k > 0$ rows in which all nonzero entries occur in $k-1$ columns, this follows directly from the proof of Lemma 4.3. Let $T = \{1, \dots, n\} - S$ and note that T is not empty because we know from the proof of Lemma 5.2 that i and j cannot be in the same set otherwise $per(A_{ij}) > 0$. Let's assume w.l.o.g that $i \in T$ and $j \in S$. We can now permute the columns of A such that all of the nonzero entries for the k rows of S appear in the k columns of S. A is obviously still minimal and in upper triangle form, hence decomposable, which contradicts Lemma 5.2. □

Lemma 5.4. *If A is a minimal matrix and $a_{ij} > 0$,then $per(A_{ij}) = per(A)$.*

Proof. To prove this lemma we are first going to show that there are constants λ_i and μ_j such that

$$per(A_{ij}) = \lambda_i + \mu_j \quad \text{if } a_{ij} > 0. \tag{5.2}$$

We know that $per(A) > 0$ therefore we can assume, by permuting the columns of A, that $a_{ii} > 0$ for all i. We will now have to prove a small lemma from graph theory for the following part.
Let us write $i \to j$ if $a_{ij} > 0$, thus $i \to i$ for all i. If A is decomposable there must be a „path"

$$1 = j_0 \to j_1 \to \cdots \to j_l = j \tag{5.3}$$

from 1 to all j.

Proof. Let's assume there is an index x with no path to index y. Then we consider the set $T = \{j \in \{1, \dots, n\} |$ there is a path from j to $y\}$. Obviously $x \notin T$ and $y \in T$. Now we consider $S = \{1, \dots, n\} \backslash T = \{i \in \{1, \dots, n\} |$ there is no path from i to $y\}$, so S and T are both nonempty strict subsets of $\{1, \dots, n\}$. In total we have $T \cap S = \emptyset$ and $S \cup T = \{1, \dots, n\}$ with $a_{ij} = 0$ for $i \in S$ and $j \in T$.

This is because if we had $a_{ij} \neq 0$ for $i \in S$ and $j \in T$ then there must be a path from i to j by our definition of a path above. Due to the path from j to y it follows immediately that there is a path from i to y. Therefore S and T are sets which satisfy the definition of decomposability and we get a contradiction. □

We say that j is at distance l if the shortest such path is of length l. To make these paths unique we call $p(j)$ the smallest index at distance $l-1$ such that $p(j) \to j$ for every $j > 1$ and from now on insist that $j_{k-1} = p(j_k)$ for all $1 \leq k \leq l$. Now we can interpret every path as a branch of an oriented tree emanating from point 1. We call i an ancestor of j (and we write $i \prec j$) if $i = p(j)$ or $i = p(p(j))$ or ..., the notation $i \preceq j$ means that $i = j$ or $i \prec j$.
Now we can start our proof of (5.2). We will define λ_i and μ_j inductively and begin by saying $\lambda_1 = 0$ and $\mu_1 = per(A_{11})$. Then for the indices $j > 1$ with $1 = p(j)$ we can define μ_j so that $per(A_{1j}) = 0 + \mu_j$ holds. After that we can define λ_j for the same indices such that $per(A_{jj}) = \mu_j + \lambda_j$ holds. Then we can repeat this procedure and use λ_j to define μ_k for $k = p(j)$ and we go on with that until we have assigned values for all λ_i and μ_i.
The construction above assigns values for $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ in the situation that $a_{ij} > 0$, $i = p(j)$ or $i = j$. We must now proof, that this construction also holds for the pairs $(\hat{i}, \hat{j})$ such that $a_{\hat{i}\hat{j}} > 0$, $\hat{i} \neq \hat{j}$ and $\hat{i} \neq p(\hat{j})$. Consider the matrix B whose entries are all zero except $b_{\hat{i}\hat{j}} = 1$ and

$$\begin{aligned} b_{jj} &= (j \prec \hat{j}) - (j \preceq \hat{i}), \text{ for } 1 \leq j \leq n \\ b_{p(j)j} &= (j \preceq \hat{i}) - (j \preceq \hat{j}), \text{ for } 1 \leq j \leq n \end{aligned} \tag{5.4}$$

where the notation means that $(j \prec \hat{j}) = 1$ if $j \prec \hat{j}$ and 0 otherwise. Now we show that B is a valid modification for A (row and column sums must be zero). The following part needs some explanation so we will first only look at the column sums. Keep in mind that the j in the following equation is arbitrary but fixed.

$$\sum_i b_{ij} = (j = \hat{j}) + ((j \prec \hat{j}) - (j \preceq \hat{i})) + ((j \preceq \hat{i}) - (j \preceq \hat{j})) = 0$$

The iterating i will at some point reach $\hat{i}$ and if $j = \hat{j}$ we need to add 1 because $b_{\hat{i}\hat{j}} = 1$ so that's where the fist summand comes from. Now the second and third summand is just for the cases $i = j$ and $i = p(j)$ which will both only happen once in the sum because for every j there is exactly one $i = p(j)$. Now we look at the row sums which are more difficult to explain.

$$\sum_j b_{ij} = (i = \hat{i}) + ((i \prec \hat{j}) - (i \preceq \hat{i})) + ((i \prec \hat{i}) - (i \prec \hat{j})) = 0$$

Now the first two summands can be explained in the same way as before, the third one is a bit trickier. Firstly it doesn't matter if we write $j \preceq \hat{i}$ or $i \prec \hat{i}$ because in this case the i is fixed but arbitrary and $i = p(j)$. Essentially we have to show that the sum over all b_{ij} where $i = p(j)$ is equal to $(i \prec \hat{i}) - (i \prec \hat{j})$. For this we will perform a distinction of all possible cases.

1. $i \prec \hat{i}$ and $i \prec \hat{j}$

   1.1 $\hat{i}$ and $\hat{j}$ are in the same path
   W.l.o.g $j_1 \prec \hat{i}$ and $j_1 \prec \hat{j}$, so it follows that

$$\sum_{j,i=p(j)} b_{ij} = ((j_1 \preceq \hat{i}) - (j_1 \preceq \hat{j})) + \cdots + ((j_n \preceq \hat{i}) - (j_n \preceq \hat{j}))$$
$$= (1-1) + (0-0) + \cdots + (0-0) = 0$$

   1.2 $\hat{i}$ and $\hat{j}$ are not in the same path
   W.l.o.g $j_1 \prec \hat{i}$ and $j_2 \prec \hat{j}$, so it follows that

$$\sum_{j,i=p(j)} b_{ij} = ((j_1 \preceq \hat{i}) - (j_1 \preceq \hat{j})) + \cdots + ((j_n \preceq \hat{i}) - (j_n \preceq \hat{j}))$$
$$= (1-0) + (0-1) + \cdots + (0-0) = 0$$

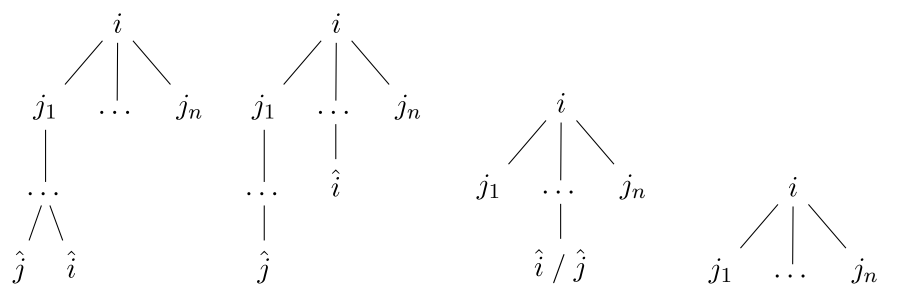
2. $i \prec \hat{i}$ and $i \not\prec \hat{j}$
   W.l.o.g $j_1 \prec \hat{i}$ so it follows that

$$\sum_{j,i=p(j)} b_{ij} = ((j_1 \preceq \hat{i}) - (j_1 \preceq \hat{j})) + \cdots + ((j_n \preceq \hat{i}) - (j_n \preceq \hat{j}))$$
$$= (1-0) + \cdots + (0-0) = 1$$

3. $i \not\prec \hat{i}$ and $i \prec \hat{j}$
   Analog to case 2.

4. $i \not\prec \hat{i}$ and $i \not\prec \hat{j}$

$$\sum_{j,i=p(j)} b_{ij} = ((j_1 \preceq \hat{i}) - (j_1 \preceq \hat{j})) + \cdots + ((j_n \preceq \hat{i}) - (j_n \preceq \hat{j}))$$
$$= (0-0) + \cdots + (0-0) = 0$$



case 1.1 case 1.2 case 2 / 3 case 4

We can now see, that all these cases have the same result as $((i \prec \hat{i}) - (i \prec \hat{j}))$ thus we have shown that the column sums are also equal to zero.
Thus B is a valid modification for A and so is $-B$ because of $a_{ii}, a_{ip(i)} > 0$. Therefore Lemma 5.1 suggests that $\sum_{i,j} b_{ij} per(A_{ij}) = 0$, hence it finally follows that

$$\begin{aligned}\sum_{i,j} b_{ij} per(A_{ij}) &= \sum_{i,j} (\lambda_i - \lambda_i + \mu_j - \mu_j) b_{ij} \\ &= \sum_{i,j} (per(A_{ij}) - \lambda_i - \mu_j) b_{ij} \\ &= per(A_{\hat{i}\hat{j}}) - \lambda_{\hat{i}} - \mu_{\hat{j}} = 0.\end{aligned}$$

For the second to last equality we used that $per(A_{ij}) = \lambda_i + \mu_j$ holds for all pairs (i, j) such that $b_{ij} \neq 0$ except possibly for the given pair $(\hat{i}, \hat{j})$. Thus (5.2) holds in general.
Now we can complete the proof. We know that $a_{ij} per(A_{ij}) = a_{ij}(\lambda_i + \mu_j)$ for all i, j, hence by (3.1) we get

$$per(A) = \lambda_i + \sum_j a_{ij}\mu_j = \mu_j + \sum_i a_{ij}\lambda_i \tag{5.5}$$

for all i, j. In matrix notation this equals, $\lambda + A\mu = \mu + A^\intercal\lambda = per(A)e$, where e is a column vector of all one's. Since $Ae = A^\intercal e = e$, we have $A^\intercal\lambda + A^\intercal A\mu = per(A)e$ and $A\mu + AA^\intercal\lambda = per(A)e$, hence

$$\mu = A^\intercal A\mu \quad \text{and} \quad \lambda = AA^\intercal\lambda.$$

If we can proof that $A^\intercal A$ and $AA^\intercal$ are indecomposable then our proof is completed by using Lemma 4.5. So for the final step we proof the following implication which we can use for our purpose by contraposition.
If $A^\intercal A$ or $AA^\intercal$ is decomposable then A is also decomposable.

Proof. We will do the proof for $A^\intercal A$, the proof for $AA^\intercal$ works analogous. It obviously holds that $(A^\intercal A)_{i,j} = \langle a_i, a_j \rangle$ where a_i is the i-th row of A. Due to the decomposability of $A^\intercal A$ there must be sets S and T such that $(A^\intercal A)_{i,j} = \langle a_i, a_j \rangle = 0$ if $i \in S$ and $j \in T$. Because of the nonnegativity of a_{ij} this implies that if $a_{ik} > 0$ then $a_{jk} = 0$. At the beginning of the proof we assumed w.l.o.g that $a_{ii} > 0$. Therefore if $i \in S$ and $j \in T$ it follows that $a_{ij} = 0$, which is the definition of decomposability and concludes the proof. □

Now we know that $A^\intercal A$ and $AA^\intercal$ are also indecomposable. As μ is an eigenvector for $A^\intercal A$ and λ is an eigenvector for $AA^\intercal$ with eigenvalue 1 we know by contraposition of Lemma 4.5 that $\lambda_1 = \cdots = \lambda_n = 0$ and $\mu_1 = \cdots = \mu_n = per(A_{11})$. Remembering (5.2) we get that $per(A_{ij}) = \mu_j$ if $a_{ij} > 0$ and if we put it all together we get by (5.5)

$$per(A) = \mu_j + \sum_i a_{ij}\lambda_i = \mu_j = per(A_{ij}) \quad \text{if } a_{ij} > 0$$

□

Lemma 5.5. *If A is a minimal matrix, then $per(A_{ij}) \geq per(A)$ for all i and j.*

Proof. Because of Lemma 5.4 we only have to consider (i, j) such that $a_{ij} = 0$, w.l.o.g assume that $i = j = 1$ and $a_{11} = 0$. By Lemma 5.3 we know that $per(A_{11}) > 0$, hence we can achieve that $a_{jj} > 0$ for $2 \leq j \leq n$ by permuting rows. Now let $B = I - A$ a valid modification for A, obviously it holds that $b_{ij} \geq 0$ if $a_{ij} = 0$ and the row and column sums are zero. By Lemma 5.1 and equation (3.1) we have

$$\begin{aligned} 0 \leq \sum_{i,j} b_{ij} per(A_{ij}) &= \sum_{j} per(A_{jj}) - \sum_{i,j} a_{ij} per(A_{ij}) \\ &= per(A_{11}) + (n-1)per(A) - n per(A) = per(A_{11}) - per(A) \end{aligned}$$

Where the first equality follows by the definition of B and by Lemma 5.4 $per(A_{jj}) = per(A)$ for $j > 1$. □

6 Egorychev's Theorem

We can now combine our results to complete the proof of Van der Waerden's Theorem.

Lemma 6.1. *If A is a minimal matrix, then $per(A_{ij}) = per(A)$ for all i and j.*

Proof. Due to Lemma 5.5 we only have to proof that $per(A_{ij}) > per(A)$ cannot happen. Permuting rows and columns doesn't change the value of the permanent, therefore we can assume w.l.o.g that $i = j = n$. We further assume that $per(A_{nn}) > per(A)$ and $a_{(n-1)n} > 0$. Then $n > 1$ and Corollary 3.4 implies that

$$per(A)^2 = per(a_1, \ldots, a_n)^2 \geq per(a_1, \ldots, a_{n-2}, a_{n-1}, a_{n-1}) per(a_1, \ldots, a_{n-2}, a_n, a_n).$$

But $per(a_1, \ldots, a_{n-2}, a_{n-1}, a_{n-1}) = \sum_j a_{(n-1)j} per(A_{nj}) > \sum_j a_{(n-1)j} per(A) = per(A)$, and $per(a_1, \ldots, a_{n-2}, a_n, a_n) = \sum_j a_{nj} per(A_{(n-1)j}) \geq \sum_j a_{nj} per(A) = per(A)$. Which leads to the contradiction $per(A)^2 > per(A)^2$. □

Lemma 6.2. *If A is a minimal matrix of order n, with $a_{ij} > 0$ for all i and j except possibly when $i = n$, then $a_{ij} = 1/n$ for all i and j.*

Proof. We already know that

$$per(a_1, \ldots, a_{n-2}, a_n, a_n) = \sum_j a_{nj} per(A_{(n-1)j}) = \sum_j a_{nj} per(A) = per(A)$$

by Lemma 5.1, (3.1) and the fact that $\sum_j a_{nj} = 1$.
Similarly we get $per(a_1, \ldots, a_{n-2}, a_{n-1}, a_{n-1}) = per(A)$. Therefore equality holds in (3.6) and Theorem 3.3 implies that $a_n = \lambda a_{n-1}$ for some λ. Obviously $\lambda = 1$ because A is doubly stochastic, hence $a_n = a_{n-1}$. Similarly, all rows of A are equal. Therefore all columns of A consist of identical elements. It follows directly that all elements of A must be equal to $1/n$. □

Theorem 6.3. *If A is a minimal matrix of order n, then $a_{ij} = 1/n$ for all i and j, hence*

$$per(A) = n!/n^n$$

Proof. Let B be the matrix obtained from A by replacing some row a_i by some other row a_k, then $per(B) = \sum_j a_{kj} per(A_{ij}) = \sum_j a_{kj} per(A) = per(A)$. Similarly we can construct a matrix C by replacing the row a_k by a_i. Obviously it holds that $per(A) = per(B) = per(C)$. Now B and C don't have to be doubly stochastic but

$$D = \frac{1}{2}(B+C) = \begin{pmatrix} -a_1- \\ \vdots \\ -\frac{1}{2}(a_i + a_k)- \\ \vdots \\ -\frac{1}{2}(a_i + a_k)- \\ \vdots \\ -a_n- \end{pmatrix}$$

surely is. The row sums are trivially equal to 1 and all entries are nonnegative. To see that D is doubly stochastic we make sure that the column sums are equal to 1. So we choose an arbitrary index j and calculate

$$\sum_l d_{lj} = a_{1j} + \cdots + \frac{1}{2}(a_{ij} + a_{kj}) + \cdots + \frac{1}{2}(a_{ij} + a_{kj}) + \cdots + a_{nj} = 1$$

Now we will confirm that $per(D) = per(A)$ by using the multilinearity of the permanent.

$$\begin{aligned} per(D) &= \frac{1}{4}\left(per\begin{pmatrix} -a_1- \\ \vdots \\ -a_k- \\ \vdots \\ -a_k- \\ \vdots \\ -a_n- \end{pmatrix} + per\begin{pmatrix} -a_1- \\ \vdots \\ -a_i- \\ \vdots \\ -a_k- \\ \vdots \\ -a_n- \end{pmatrix} + per\begin{pmatrix} -a_1- \\ \vdots \\ -a_k- \\ \vdots \\ -a_i- \\ \vdots \\ -a_n- \end{pmatrix} + per\begin{pmatrix} -a_1- \\ \vdots \\ -a_i- \\ \vdots \\ -a_i- \\ \vdots \\ -a_n- \end{pmatrix} \right) \\ &= \frac{1}{4}(per(B) + 2per(A) + per(C)) \\ &= per(A) \end{aligned}$$

So D is a minimal matrix.

By a finite number of averaging steps like the ones that formed D we are able to achieve a matrix E with the same bottom row as A, but with $e_{ij} = 0$ only if $i = n$ or $a_{1j} = a_{2j} = \cdots = a_{(n-1)j} = 0$. The second case cannot happen since A would then be decomposable with the sets $S = \{n\}$ and $T = \{1, \ldots, n-1\}$.
Hence E is a minimal matrix satisfying the condition from Lemma 6.2 it follows that all it's rows are $e_i = (1/n, \ldots, 1/n)$ and so $a_n = (1/n, \ldots, 1/n)$. With the same procedure we can achieve the same result for all other rows of A. Therefore A is the matrix with all entries $a_{ij} = 1/n$ and the permanent then calculates to $\frac{n!}{n^n}$ □

7 Block matrices

In this chapter we will discuss an approach to define the permanent for block matrices.

Definiton 7.1. *Let A be a $n \times n$ block matrix with entries $A_{ij} \in \mathbb{R}^{m \times m}$. A_{ij} is positive semidefinit (psd) and $\sum_i A_{ij} = \sum_j A_{ij} = I$ for all i, j. Then we define the permanent of A as follows*

$$per(A) = \frac{1}{n!} \sum_{\pi} \sum_{\tau} \prod_{i=1}^{n} A_{\tau(i)\pi(\tau(i))} \tag{7.1}$$

where both τ and π iterate over all permutations of $\{1, \ldots, n\}$

The idea behind the definition is to take the arithmetic mean of all possible matrix products to counter the non-commutativity of matrices. The symmetry of the permanent is also secured this way, because every combination of every product is part of the sum and therefore transposing doesn't change the result. If we want this definition to be reasonable we require $per(A) \in \mathbb{R}^{m \times m}$ to be a psd matrix to match the nonnegativity of the permanent. Let's take a look at an instructive example

Example 7.2.

$$per \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{2} \left(A_{11}A_{22} + A_{22}A_{11} + A_{21}A_{12} + A_{21}A_{12} \right)$$

If the block matrix A is of dimension 2×2 then it is quite easy to prove that the above definition for the permanent is psd.

Theorem 7.3. *If $A \in \mathbb{R}^{2 \times 2}$ is a block matrix with entries $A_{ij} \in \mathbb{R}^{m \times m}$ which are psd with $\sum_i A_{ij} = \sum_j A_{ij} = I$, then $per(A)$ by Definition 7.1 is psd.*

Proof. If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with corresponding permanent as in Example 7.2 we get this system of linear equations:

$$A_{11} + A_{12} = I \qquad A_{11} + A_{21} = I$$
$$A_{12} + A_{22} = I \qquad A_{21} + A_{22} = I$$

which implies that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{11} \end{pmatrix}$ and the permanent simplifies to $per(A) = A_{11}^2 + A_{12}^2$.

- **symmetry**
  $(A_{11}A_{11} + A_{12}A_{12})^\intercal = (A_{11}^\intercal A_{11}^\intercal + A_{12}^\intercal A_{12}^\intercal) = A_{11}A_{11} + A_{12}A_{12}$
- **psd**
  $x^\intercal(A_{11}A_{11} + A_{12}A_{12})x = x^\intercal A_{11}^\intercal A_{11}x + x^\intercal A_{12}^\intercal A_{12}x = (A_{11}x)^\intercal A_{11}x + (A_{12}x)^\intercal A_{12}x$
  $= ||A_{11}x||^2 + ||A_{12}x||^2 \geq 0$

Therefore it follows that the permanent is a psd matrix. □

It seems like the proof for higher dimensions is more difficult. Perhaps we can gain something from well known facts of linear algebra that will help us with the proof.

Lemma 7.4. *If $A, B \in \mathbb{R}^{n\times n}$ are psd matrices so is $A + B$.*

Proof. For any arbitrary $x \in \mathbb{R}^n$ it holds that

$$x^\intercal(A + B)x = x^\intercal Ax + x^\intercal Bx \geq 0$$

□

So if each summand that appears in the permanent is psd we have solved our problem. Unfortunately that is not true.

Lemma 7.5. *If $A, B \in \mathbb{R}^{n\times n}$ are psd then AB is psd if and only if AB is normal i.e. $(AB)^\intercal AB = AB(AB)^\intercal$.*

Proof. For a matrix X, we denote $\sigma(X)$ for the set of all eigenvalues of X. First note that for two arbitrary matrices X and Y it holds that $\sigma(XY) = \sigma(YX)$. By a known result of linear algebra we can write $AB = A^{\frac{1}{2}}A^{\frac{1}{2}}B$ where $A^{\frac{1}{2}}$ is psd. Thus $\sigma(AB) = \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}})$. Now choose any arbitrary $x \in \mathbb{R}^n$ and it holds that $x^\intercal A^{\frac{1}{2}}BA^{\frac{1}{2}}x = (A^{\frac{1}{2}}x)^\intercal B(A^{\frac{1}{2}}x) \geq 0$. Therefore the eigenvalues of AB must be nonnegative. Finally, since AB is normal we can use the spectral theorem and it follows that $AB = UDU^\intercal$ for some diagonal matrix D and unitary matrix U, therefore AB is psd. □

One can also use Sylvester's law of inertia to prove that if A is spd and B is symmetric then the eigenvalues of AB have the same sign as the eigenvalues of B. So if we would demand all entries of the block matrix to be spd we could achieve that all summands of the permanent have positive eigenvalues. But this also doesn't work because the sum of two matrices with positive eigenvalues can have negative eigenvalues.
Since we couldn't prove that our definition holds true for all block matrices we try to find a counterexample to our definition and proof that the permanent cannot be defined in that way for block matrices.

It is easy to find two psd matrices such that their product is not symmetric anymore and therefore not psd, but we also know by the proof of Lemma 7.5 that the eigenvalues are always nonnegative.

Example 7.6.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad AB = \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix}$$

If we consider products of 3 psd matrices we easily find A, B and C psd such that the product ABC has negative eigenvalues even with the condition $A + B + C = I$. As a matter of fact we can also look at the symmetric product $ABC + ACB + BAC + BCA + CAB + CBA$ and still find 3 matrices with the conditions from above such that the symmetric product has negative eigenvalues. Thus the summands of the permanent don't have to be psd. To achieve this just run this python code and it will find a counter example rather fast.

```
import numpy as np

def psd(dim):
    erg = np.eye(dim)
    while all(i >= 0 for i in np.linalg.eig(erg)[0]): #Eigenvalues ≥ 0
        A = np.random.uniform(-1,1,(dim,dim))
        B = np.random.uniform(-1,1,(dim,dim))
        A = np.matmul(A,np.transpose(A))  #A is psd
        B = np.matmul(B,np.transpose(B))  #B is psd
        C = np.eye(dim) - (A+B) #Symmetric by construction
        if all (i>=0 for i in np.linalg.eig(C)[0]): #Check if C is psd
            erg = A@B@C + A@C@B + B@A@C + B@C@A + C@A@B + C@B@A
    return np.linalg.eig(erg)[0], A, B, C
```

It is easy to see how this code works. At first we define two random matrices A and B and multiply them with their transpose. This generates two random psd matrices. The proof that $AA^{\intercal}$ is psd for any arbitrary matrix A is trivial. Choose an arbitrary matrix $A \in \mathbb{R}^{n \times n}$ and a vector $x \in \mathbb{R}^{n \times n}$

$$x^{\intercal} A A^{\intercal} x = (A^{\intercal} x)^{\intercal} A^{\intercal} x = y^{\intercal} y = \sum_{i}^{n} y_i^2 \geq 0$$

Thus the result is psd. Then we create the matrix C in such a way that $A + B + C = I$ holds true. By doing this C is always symmetric by construction. In the next step we check if C has nonnegative eigenvalues and therefore is psd. If that is true, we finally compute the symmetric product. This is done until we find a matrix with a negative eigenvalue.

But we not only want to find a summand with negative eigenvalues, we want to find matrices such that the permanent has negative eigenvalues. The first approach one cold make to achieve this is to set up an arbitrary 3×3 block matrix and the corresponding equations.

$$P = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix}$$

$$\begin{aligned} A+B+C=I \quad & A+D+G=I \\ D+E+F=I \quad & B+E+H=I \\ G+H+J=I \quad & C+F+J=I \end{aligned}$$

This is a system with 6 equations and 9 unknowns and therefore 3 degrees of freedom. To solve this we choose A, B and D (psd) and then solve the system of equations to get a unique solution for all entries of P. If we set up a program to run through many possibilities for psd matrices A, B and D it always finds solutions for the system of equations but the corresponding matrices are not all psd and the code gets stuck in a permanent loop.

In order to set up a code that finds suiting block matrices and checks if their permanent has negative eigenvalues we are going to look at latin [2] and semi classical magic squares [3].

Definiton 7.7. *A $n \times n$ block matrix A is called latin magic square if there are n different psd matrices of dimension $m \times m$ such that each matrix appears exactly once in each column and row. Furthermore it must hold true that $\sum_i A_{ij} = \sum_j A_{ij} = I$.*

Example 7.8. *Let $X, Y, Z \in \mathbb{R}^{m \times m}$ be psd such that $X + Y + Z = I$.*

$$A = \begin{pmatrix} X & Y & Z \\ Y & Z & X \\ Z & X & Y \end{pmatrix}$$

Then A is a latin magic square.

Definiton 7.9. *A $n \times n$ block matrix A is called semiclassical magic square if it can be constructed in the following way.*

$$A = \sum_{\pi} P_\pi \otimes q_\pi \tag{7.2}$$

where q_π are $m \times m$ psd matrices with $\sum_\pi q_\pi = I$, $P_\pi \in \mathbb{R}^{n \times n}$ are permutation matrices and $\otimes$ is the Kronecker product.

By this definition we get $\sum_i A_{ij} = I$ automatically. This can be obtained in the following way. If we have a $n \times n$ block matrix, then there are $n!$ permutation matrices. For an arbitrary entry i, j there are $(n-1)!$ permutation matrices with a 1 at that entry. So we have $(n-1)!$ of the existing $n!$ coefficients in that entry. The same holds for any other entry in that row i. Therefore we have $n * (n-1)! = n!$ coefficients (q_π) in row i. And because the permutation matrices that had a 1 at i, j cannot have a 1 at any other entry in the row i, every q_π appears exactly once in that row. Therefore it follows that $\sum_i A_{ij} = I$.
Once again we can look at an instructive example to make the definition more clear.

Example 7.10.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2/3 & 1/4 \\ 1/4 & 1/2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1/3 & -1/4 \\ -1/4 & 1/2 \end{pmatrix} = \left(\begin{array}{cc|cc} 2/3 & 1/4 & 1/3 & -1/4 \\ 1/4 & 1/2 & -1/4 & 1/2 \\ \hline 1/3 & -1/4 & 2/3 & 1/4 \\ -1/4 & 1/2 & 1/4 & 1/2 \end{array}\right)$$

To try and find a counterexample for the psd property of the permanent of 3×3 latin magic squares we can look at this short python code.

```
import numpy as np

def latin3(dim):
    erg = np.eye(dim)
    while all(i >= 0 for i in np.linalg.eig(erg)[0]): #Eigenvalues ≥ 0
        A = np.random.uniform(-0.5,0.5,(dim,dim))
        B = np.random.uniform(-0.5,0.5,(dim,dim))
        A = np.matmul(A,np.transpose(A))   #A is psd
        B = np.matmul(B,np.transpose(B))   #B is psd
        C = np.eye(dim) - (A+B) #Symmetric by construction
        if all(i >= 0 for i in np.linalg.eig(C)[0]):  #Check if C is psd
            erg = A@A@A + B@B@B + C@C@C + 1/2*(A@B@C + A@C@B + B@A@C +
                B@C@A + C@A@B + C@B@A)
    return np.linalg.eig(erg), A,B,C
```

This code works analogous to the code above but instead of the symmetric product of A, B and C we calculate the whole permanent of the matrix. The code does not find any counter examples so it seems logic to try again in higher dimensions. The next code is for 4×4 latin magic squares and does the same thing.

```
import numpy as np

def latin4(dim):
    erg = np.eye(dim)
    while all(i >= 0 for i in np.linalg.eig(erg)[0]): #Eigenvalues ≥ 0
        A = np.random.uniform(-0.5,0.5,(dim,dim))
        B = np.random.uniform(-0.5,0.5,(dim,dim))
        C = np.random.uniform(-0.5,0.5,(dim,dim))
        A = np.matmul(A,np.transpose(A)) #A is psd
        B = np.matmul(B,np.transpose(B)) #B is psd
        C = np.matmul(C,np.transpose(C)) #C is psd
        D = np.eye(dim) - (A+B+C) #Symmetric by construction
        if all(i >= 0 for i in np.linalg.eig(D)[0]): #Check if D isd psd
            erg = (A@A@A@A + B@B@B@B + C@C@C@C + D@D@D@D) + 1/24*8*(
                A@B@C@D + A@B@D@C + A@C@B@D + A@C@D@B + A@D@B@C + A@D@C@B
                + B@A@C@D + B@A@D@C + B@C@A@D + B@C@D@A + B@D@A@C +
                B@D@C@A + C@A@B@D + C@A@D@B + C@B@A@D + C@B@D@A + C@D@A@B
                + C@D@B@A + D@A@B@C + D@A@C@B + D@B@A@C + D@B@C@A +
                D@C@A@B + D@C@B@A)\ + 1/24*2*4*(A@A@B@B + A@B@A@B +
                A@B@B@A + B@A@A@B + B@B@A@A + B@A@B@A + A@A@D@D + A@D@A@D
                + A@D@D@A + D@A@A@D + D@D@A@A + D@A@D@A + A@A@C@C +
                A@C@A@C + A@C@C@A + C@A@A@C + C@C@A@A + C@A@C@A + D@D@B@B
                + D@B@D@B + D@B@B@D + B@D@D@B + B@B@D@D + B@D@B@D +
                C@C@B@B + C@B@C@B + C@B@B@C + B@C@C@B + B@B@C@C + B@C@B@C
                + D@D@C@C + D@C@D@C + D@C@C@D + C@D@D@C + C@C@D@D +
                C@D@C@D)
    return np.linalg.eig(erg), A,B,C
```

Just as the code for 3×3 latin magic squares this one also finds no counter example. Since we cannot find a counter example we write a program for semiclassical magic squares which include more different block matrices than just the latin magic squares. To understand this code we need to look at an arbitrary 3×3 semiclassical magic square.

$$A = \begin{pmatrix} q_1 + q_2 & q_3 + q_4 & q_5 + q_6 \\ q_3 + q_6 & q_1 + q_5 & q_2 + q_4 \\ q_4 + q_5 & q_2 + q_6 & q_1 + q_3 \end{pmatrix} \tag{7.3}$$

where $\sum_i q_i = I$ must hold true. We proceed as follows, we define $Q_1, \dots Q_5$ and multiply with their transpose to once again get random psd matrices. After that we generate Q_6 such that $\sum_i Q_i = I$ holds true and check if it's eigenvalues are nonnegative. If all of that checks out we can compute the permanent as before according to our definition and Equation 7.3.

```
import numpy as np

def semi(dim):
    erg = np.eye(dim)
    while all(i >= 0 for i in np.linalg.eig(erg)[0]):
        Q1 = np.random.uniform(-0.5,0.5,(dim,dim))
        Q2 = np.random.uniform(-0.5,0.5,(dim,dim))
        Q3 = np.random.uniform(-0.5,0.5,(dim,dim))
        Q4 = np.random.uniform(-0.5,0.5,(dim,dim))
        Q5 = np.random.uniform(-0.5,0.5,(dim,dim))
        Q1 = np.matmul(Q1,np.transpose(Q1))
        Q2 = np.matmul(Q2,np.transpose(Q2))
        Q3 = np.matmul(Q3,np.transpose(Q3))
        Q4 = np.matmul(Q4,np.transpose(Q4))
        Q5 = np.matmul(Q5,np.transpose(Q5)) #Q_1,...,Q_5 are all psd
        Q6 = np.eye(dim) - (Q1 + Q2 + Q3 + Q4 + Q5)
        A = Q1 + Q2; B = Q3 + Q4; C = Q5 + Q6
        D = Q3 + Q6; E = Q1 + Q5; F = Q2 + Q4
        G = Q2 + Q5; H = Q4 + Q6; J = Q1 + Q3
        #A,...,J are all psd if Q_6 is psd
        if all(i >= 0 for i in np.linalg.eig(Q6)[0]): #Check if Q_6 is psd
            erg = 1/6*(A@E@J + A@J@E + E@A@J + E@J@A + J@A@E + J@E@A\
                      + A@F@H + A@H@F + F@A@H + F@H@A + H@A@F + H@F@A\
                      + B@D@J + B@J@D + J@B@D + J@D@B + D@B@J + D@J@B\
                      + B@F@G + B@G@F + F@G@B + F@B@G + G@B@F + G@F@B\
                      + C@E@G + C@G@E + E@C@G + E@G@C + G@C@E + G@E@C\
                      + C@D@H + C@H@D + D@C@H + D@H@C + H@C@D + H@D@C)
    return np.linalg.eig(erg)[0], Q1,Q2,Q3,Q4,Q5,Q6
```

It turns out that alike the codes above this one also fails to find a counterexample. By this outcome one could assume that Definition 7.1 might work...

References

[1] Donald E. Knuth, *A Permanent Inequality*, Stanford University, december 1981

[2] Gemma De las Cuevas, Tom Drescher, Tim Netzer, *Quantum magic squares: Dilations and their limitations*, Universität Innsbruck, november 2020

[3] Gemma De las Cuevas, Tim Netzer, Inga Valentiner-Branth, *Magic squares: Latin, semiclassical, and quantum*, Universität Innsbruck, february 2023

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