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Algebraic Topology

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Contents

Introduction

Topology is a branch of mathematics that deals with certain spatial structures. However, the concept of spatiality is very loosely defined here. Unlike in geometry, one does not speak of size, angles, distances, etc., but rather uses only a qualitative concept of spatial *proximity*. This is sufficient, however, to discuss, for example, convergence and continuity. Since the concept of a topological space is so general, it is used in almost all areas of mathematics.

As is common in mathematics, one tries to classify the objects of interest as completely as possible, up to isomorphism. In topology, an isomorphism is also called a *homeomorphism*; it is a bijective mapping that is continuous in both directions. This term expresses that spaces can be transformed into one another by a deformation that may change the size and angles, but not the proximity of points defined by the topology. Thus, a space (to put it simply) must not be torn open or glued together anywhere. A weaker concept than homeomorphism is *homotopy equivalence*of spaces. Here, material may also be destroyed during the deformation (again, to put it very simply). For example, a ball is homotopy equivalent to a point.

If two spaces are homeomorphic or homotopy equivalent, this can often be proven by explicitly giving a feasible transformation (e.g., a homeomorphism). If they are not, the matter is often more difficult. One cannot usually check all possible mappings to see whether they are homeomorphisms. Instead, one must extract properties of the spaces that exclude homeomorphism. In algebraic topology, one assigns certain algebraic structures to the spaces, for example groups, vector spaces, or modules. Isomorphic structures are assigned to homeomorphic spaces. Interestingly, one can sometimes actually show that the assigned structures are not isomorphic. This disproves the homeomorphism.

This approach will be demonstrated in the lecture. In the first chapter, we review the most important topological concepts and learn about some interesting constructions. In the second chapter, we discuss the *fundamental group* of a topolog-

ical space. This intuitively understandable concept is also historically the beginning of algebraic topology; later, much more abstract methods developed from it. We will discuss one such example, the so-called *homology theory*, in Chapter 3, first purely combinatorially for simplicial complexes and then more generally.
This lecture is partly based on a lecture given by Prof. Volker Puppe at the University of Konstanz. Any errors are, of course, my own. A list of some books on the subject of the lecture can be found in the appendix.

Chapter 1

Fundamentals of Topology

1.1 Basic Definitions

First, we briefly review some basic concepts of set-theoretic topology.

Definition 1.1.1. Let X be a nonempty set. A subset $\mathcal{O} \subseteq \mathcal{P}(X)$ of the power set of X is called a **topology**, if:

(i) $\emptyset, X \in \mathcal{O}$

(ii) $A_i \in \mathcal{O}$ for $i \in I \Rightarrow \bigcup_{i\in I} A_i \in \mathcal{O}$

(iii) $A, B \in \mathcal{O} \ \Rightarrow A \cap B \in \mathcal{O}$.

The pair $(X, \mathcal{O})$ is then called a **topological space**, an element $A \in \mathcal{O}$ is called an **open set**, and $X \setminus A$ is called a **closed set**. △

Example 1.1.2. (i) $\mathcal{O} = \mathcal{P}(X)$ is always a topology on X, it is the *finest possible topology*. Likewise, $\mathcal{O} = \{\emptyset, X\}$ is always a topology, the *coarsest possible*.
(ii) Let (X, d) be a metric space. We define

$$A \subseteq X \text{ open } :\Leftrightarrow \ \forall a \in A\, \exists \epsilon > 0 : B_\epsilon(a) \subseteq A.$$

Then the set

$$\mathcal{O} = \{A \subseteq X \mid A \text{ open}\}$$

forms a topology on X, the **metric-induced topology**. △

Definition 1.1.3. Let $(X, \mathcal{O})$ be a topological space, and $x \in U \subseteq X$.

(i) U is called a **neighborhood** of x in X if tehre exists $A \in \mathcal{O}$ with $x \in A \subseteq U$. In this case, x is called an **interior point** of U.

(ii) The set $\mathring{U}$ of all interior points of U is called the *interior* of U. It is the largest open subset of X contained in U.

(iii) The set $\overline{U} = \{x \in X \mid \text{every neighborhood of } x \text{ intersects } U\}$ is called the *closure* of U. It is the smallest closed superset of U.

(iv) The set $\partial U := \overline{U} \setminus \mathring{U}$ is called the **boundary of** U.

(v) A sequence $(x_n)_{n\in\mathbb{N}}$ in X **converges** to $x \in X$, if for every neighborhood U of x there exists an n_0 such that $x_n \in U$ for all $n \geqslant n_0$. △

Definition 1.1.4. The topological space $(X, \mathcal{O})$ is called **Hausdorff** if for any two distinct points $x, y \in X$ there exist open sets $A, B \in \mathcal{O}$ with

$$x \in A, y \in B \text{ and } A \cap B = \emptyset. \qquad \triangle$$

Example 1.1.5. Metric spaces induce Hausdorff topologies. For $x \neq y$ we have

$$B_{d(x,y)/3}(x) \cap B_{d(x,y)/3}(y) = \emptyset. \qquad \triangle$$

In this lecture we are primarily interested in metric spaces. However, different metrics often define the same topology, and almost all of our statements depend solely on the topology. The concept of abstract topology also allows us to perform constructions much simpler than with metric spaces.

Definition 1.1.6. Let $(X, \mathcal{O})$ and $(Y, \mathcal{O}')$ be topological spaces, and let $f\colon X \to Y$ be a map.

(i) f is called **continuous,** if for every $B \in \mathcal{O}'$ we have $f^{-1}(B) \in \mathcal{O}$.

(ii) f is called a **homeomorphism,** if there exists a continuous map $g\colon Y \to X$ with

$$g \circ f = \mathrm{id}_X,\ f \circ g = \mathrm{id}_Y.$$

(iii) X and Y are called **homeomorphic** (written $X \cong Y$), if a homeomorphism $f\colon X \to Y$ exists. △

From now on, we will often not even mention the topology $\mathcal{O}$ on X, but will only speak of the topological space X. Of course, a topology is still fixed in the background. Usually, it is the one induced by a canonical metric. Furthermore, practically all the mappings we consider are continuous. We will therefore not mention it again either.

Remark/Example 1.1.7. (i) If X, Y are metric spaces, then the definition of continuity of a map is equivalent to the well-known ϵ-δ–definition from analysis (Exercise 1).
(ii) Homeomorphism of topological spaces is an equivalence relation.
(iii) For example, $(-1, 1) \cong \mathbb{R}$ and $[0, 1] \cong [0, 2]$. As a homeomorphism, one can use a suitably scaled tangent function in the first case, and a linear function in the second case.
(iv) We have $[0, 1] \ncong \mathbb{R}$ (Exercise 1).
(v) The map

$$\begin{aligned} f\colon [0, 2\pi) &\to S^1 \subseteq \mathbb{C} \\ r &\mapsto e^{ir} \end{aligned}$$

is continuous and bijective, but not a homeomorphism (Exercise 1). △

One of the key questions of this lecture is how to determine whether two spaces are homeomorphic. If a homeomorphism exists, it can usually be found explicitly. If none exists, the question is much harder, and one has to resort to other arguments. For example, it is not easy to show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.

1.2 Constructions

We now discuss some interesting constructions with topological spaces, which will be taken up again later.

Definition 1.2.1. Let $(X, \mathcal{O})$ be a topological space and $Y \subseteq X$. Then

$$\mathcal{O}' := \{A \cap Y \mid A \in \mathcal{O}\}$$

is a topology on Y, called the **subspace topology** or **induced topology**. △

Remark 1.2.2. (*i*) The subspace topology is the coarsest topology which makes the inclusion $\iota\colon Y \hookrightarrow X$ continuous.
(*ii*) A map $f\colon Z \to Y$ is continuous if and only if $\iota \circ f\colon Z \to X$ is continuous.
(*iii*) When we speak of subsets of $\mathbb{R}^n$, such as $[0,1]$ or

$$S^{n-1} = \{a \in \mathbb{R}^n \mid \|a\|_2 = 1\}$$

$$D^n = \{a \in \mathbb{R}^n \mid \|a\|_2 \leqslant 1\},$$

we always understand them as having the subspace topology of the canonical topology of $\mathbb{R}^n$. △

Definition 1.2.3. Let $(X, \mathcal{O})$ be a topological space and $\sim$ an equivalence relation on X. Let

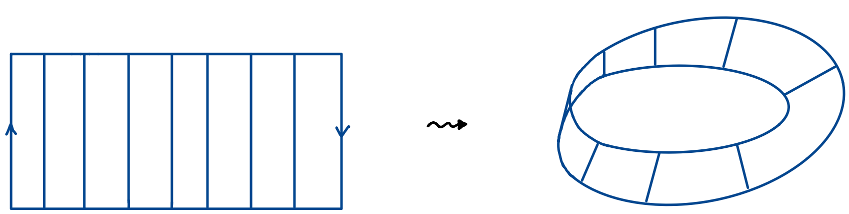
$$X/_\sim = \{[x] \mid x \in X\}$$

be the set of equivalence classes and $p\colon X \to X/_\sim$ be the canonical projection. The **quotient topology** $\mathcal{O}'$ on $X/_\sim$ is defined by

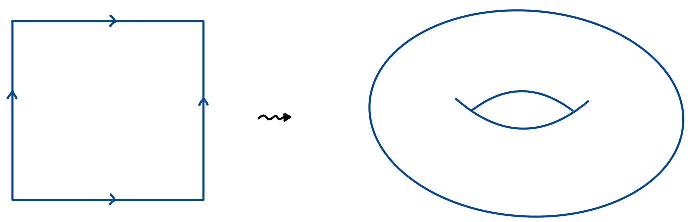
$$\mathcal{O}' := \left\{B \subseteq X/_\sim \mid p^{-1}(B) \in \mathcal{O}\right\}. \qquad \triangle$$

Remark 1.2.4. (*i*) The quotient topology is the finest topology on $X/_\sim$ which makes the projection p continuous.
(*ii*) A map $f\colon X/_\sim \to Y$ is continuous if and only if the composition $f \circ p\colon X \to Y$ is continuous. △

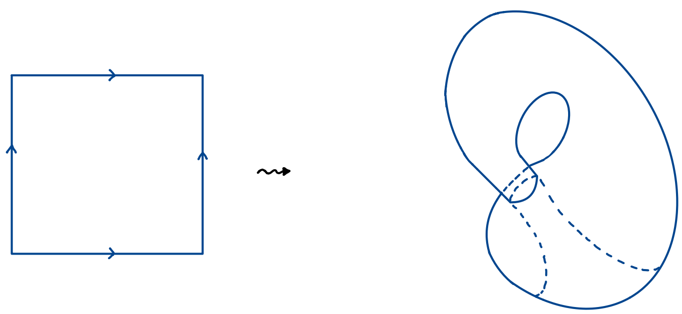
Example 1.2.5. (*i*) The **Möbius strip** can, for example, be defined as a quotient. Here, the points on one side of a rectangle are identified with points on the opposite side, according to the following orientation:



(*ii*) By identifying opposite sides using the following orientation, we obtain a **torus**:



(iii) If one identifies opposite side w.r.t. the following orientation, the **Klein bottle** is created, which, however, cannot be embedded in three-dimensional space without self-intersections:



(iv) If one identifies each point in $S^n \subseteq \mathbb{R}^{n+1}$ with its opposite point ($x \sim -x$), the quotient is the **real projective space:**

$$\mathbb{P}^n(\mathbb{R}) := S^n/_\sim. \qquad \triangle$$

Definition 1.2.6. Let $(X, \mathcal{O})$ and $(Y, \mathcal{O}')$ be topological spaces. On the disjoint union $X \sqcup Y$, one defines a topology

$$\mathcal{O}'' := \{A \subseteq X \sqcup Y \mid A \cap X \in \mathcal{O}, A \cap Y \in \mathcal{O}'\},$$

the so-called **direct sum topology**. $\triangle$

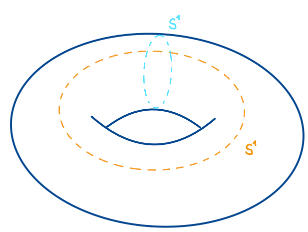
Remark 1.2.7. (i) The sum topology is the finest topology on $X \sqcup Y$, which makes both inclusions $\iota_1 \colon X \hookrightarrow X \sqcup Y$, $\iota_2 \colon Y \hookrightarrow X \sqcup Y$ continuous.
(ii) A map $f \colon X \sqcup Y \to Z$ is continuous if and only if $f \circ \iota_1 \colon X \to Z$ and $f \circ \iota_2 \colon Y \to Z$ are continuous. $\triangle$

Example 1.2.8. (i) If $[0, 1]$ and $[2, 3]$ are each given the subspace topology of $\mathbb{R}$, then the direct sum topology is again exactly the subspace topology that $[0, 1] \cup [2, 3]$ inherits from $\mathbb{R}$.
(ii) This is not true for $[0, 1]$ and $(1, 2]$. For example, the set $[0, 1]$ is open in the direct sum topology of $[0, 1] \cup (1, 2]$, but not in the subspace topology on $[0, 2]$. $\triangle$

Definition 1.2.9. Let $(X, \mathcal{O})$ and $(Y, \mathcal{O}')$ be topological spaces. On the Cartesian product $X \times Y$, we consider the topology $\mathcal{O}''$, which is generated by all sets of the form $A \times B$ with $A \in \mathcal{O}, B \in \mathcal{O}'$. Thus, $\mathcal{O}''$ consists of arbitrary unions of such sets. It is called the **product topology**. $\triangle$

Remark 1.2.10. (i) The product topology is the coarsest topology that makes the two canonical projections $p_1 \colon X \times Y \to X, p_2 \colon X \times Y \to Y$ continuous.
(ii) A map $f \colon Z \to X \times Y$ is continuous if and only if $p_1 \circ f \colon Z \to X$ and $p_2 \circ f \colon Z \to Y$ are continuous. $\triangle$

Example 1.2.11. (i) The product topology on $\mathbb{R}^n \times \mathbb{R}^m$ corresponds exactly to the standard topology on $\mathbb{R}^{n+m}$.
(ii) For example, the torus can also be defined as a product: $T := S^1 \times S^1$:



$\triangle$

Definition 1.2.12. Let the following diagram of topological spaces and continuous maps be given:

$$\begin{tikzcd} & X_1 \\ X_0 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & \\ & X_2 \end{tikzcd}$$

Then

$$X := (X_1 \sqcup X_2) /_{j_1(x) \sim j_2(x)}$$

is called the **pushout** or **fibered sum** of the diagram. Embedding in the sum and projection onto the quotient yield the following commuting diagram:

$$\begin{tikzcd} & X_1 \arrow[dr, "i_1"] & \\ X_0 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & X \\ & X_2 \arrow[ur, "i_2"'] & \end{tikzcd}$$

△

Remark 1.2.13. The pushout has the following universal property:

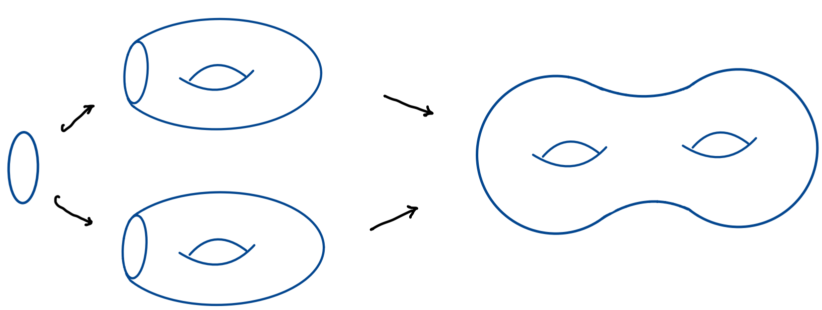
$$\begin{tikzcd} & X_1 \arrow[dr, "i_1"] \arrow[drr, bend left, "f_1"] & & \\ X_0 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & X \arrow[r, "f"] & Y \\ & X_2 \arrow[ur, "i_2"'] \arrow[urr, bend right, "f_2"'] & & \end{tikzcd}$$

Continuous maps $f\colon X \to Y$ are in one-to-one correspondence to pairs (f_1, f_2) of continuous maps $f_i\colon X_i \to Y$ with $f_1 \circ j_1 = f_2 \circ j_2$, using the assignment

$$f \mapsto (f \circ i_1, f \circ i_2).$$

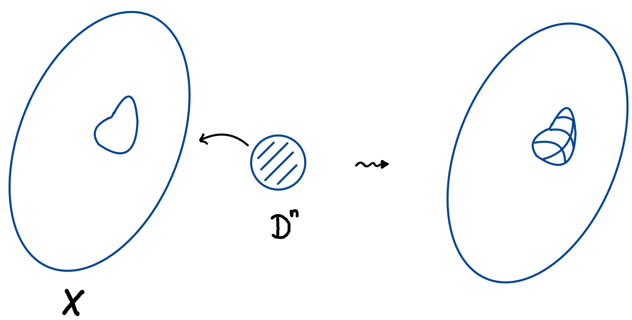
△

Example 1.2.14. (*i*) *Cutting and gluing spaces* can be realized as a pushout. For example, a double torus is created by gluing two cut tori X_1, X_2 along the cutting line X_0:



(ii) *Attaching of cells* is also implemented as a pushout:

$$\begin{tikzcd}
 & X \arrow[dr] & \\
S^{n-1} \arrow[ur, "\varphi"] \arrow[dr, "\iota"'] & & (X \sqcup D^n)/_\sim =: X \cup_\varphi D^n. \\
 & D^n \arrow[ur] &
\end{tikzcd}$$



△

Definition 1.2.15. A space that arises by attaching a finite number of cells to a finite set of points is called a **finite CW complex**. △

Example 1.2.16. (i) For example, the sphere S^{n-1} is formed by attaching a cell D^{n-1} to a single point. Thus, all spheres are finite CW complexes.
(ii) The torus is created by attaching two cells D^1 (i.e. intervals) to a point, and then attaching a D^2 in a suitable manner (try to visualize this!). △

Definition 1.2.17. Let the following diagram of topological spaces and continuous maps be given:

$$\begin{tikzcd}
X_1 \arrow[dr, "q_1"] & \\
 & X_0 \\
X_2 \arrow[ur, "q_2"'] &
\end{tikzcd}$$

Then

$$X := \{(x_1, x_2) \in X_1 \times X_2 \mid q_1(x_1) = q_2(x_2)\}$$

is called the **pullback** or **fibred product** of the diagram. Embedding in the product and projecting onto the components creates a commutative diagram:

$$\begin{tikzcd} & X_1 \arrow[dr, "q_1"] & \\ X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & X_0 \\ & X_2 \arrow[ur, "q_2"'] & \end{tikzcd}$$

△

Remark 1.2.18. The pullback has the following universal property:

$$\begin{tikzcd} & & X_1 \arrow[dr, "q_1"] & \\ Y \arrow[r, "f"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & X_0 \\ & & X_2 \arrow[ur, "q_2"'] & \end{tikzcd}$$

Continuous maps $f\colon Y \to X$ are in one-to-one correspondence to pairs (f_1, f_2) of continuous maps $f_i\colon Y \to X_i$ with $q_1 \circ f_1 = q_2 \circ f_2$, via the assignment

$$f \mapsto (p_1 \circ f, p_2 \circ f).$$ △

1.3 Compactness

Definition 1.3.1. A topological space $(X, \mathcal{O})$ is called **compact**, if every cover of X with open sets has a finite subcover. In formulas:

$$X = \bigcup_{i \in I} A_i,\ A_i \in \mathcal{O} \ \Rightarrow \exists i_1, \ldots, i_n \in I\colon X = A_{i_1} \cup \cdots \cup A_{i_n}.$$ △

Example 1.3.2. (i) $\mathbb{R}^n$ is not compact. The cover

$$\mathbb{R}^n = \bigcup_{n=1}^{\infty} \mathring{B}_n(0)$$

by open balls of increasing radius has no finite subcover.
(ii) The interval $[0,1]$ is compact. Suppose $[0,1] = \bigcup_{i\in I} A_i$ is an open cover. Then there exists an $\epsilon > 0$, such that for all $x \in [0,1]$ there exists an $i \in I$ with $B_\epsilon(x) \subseteq U_i$. Otherwise, one could choose a sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$ such that

$$\left(x_n - \frac{1}{n}, x_n + \frac{1}{n}\right) \nsubseteq U_i \text{ for all } i \in I.$$

This sequence, however, would have a convergent subsequence (for this one does not have to use compactness of the interval; one can show it with interval nesting and completeness of $\mathbb{R}$!). So if without loss of generality $x_n \to x \in [0,1]$ holds, then there is a $\delta > 0$ with $B_\delta(x) \subseteq U_i$ for some $i \in I$, and this leads to a contradiction.
From the existence of $\epsilon > 0$ as above, one immediately obtains a finite subcover for $[0,1]$. Such a ϵ is also called a *Lebesgue constant* for the given cover. △

Theorem 1.3.3. *Let X, Y be topological spaces, $A \subseteq X$ and $f\colon X \to Y$ continuous. Then*

- *(i) If X is compact and $A \subseteq X$ is closed, then A is compact.*
- *(ii) If X is Hausdorff and A is compact, then $A \subseteq X$ is closed.*
- *(iii) If X is compact, then so is $f(X)$. In particular, $X \cong Y$ implies that X is compact if and only if Y is compact.*
- *(iv) X, Y compact $\Leftrightarrow$ $X \sqcup Y$ compact.*
- *(v) For $X, Y \neq \emptyset$ we have: X, Y compact $\Leftrightarrow$ $X \times Y$ compact.*

Proof. Exercise 7. □

Theorem 1.3.4 (Heine-Borel)**.** *For $X \subseteq \mathbb{R}^n$ we have*

$$X \textit{ compact } \Leftrightarrow X \textit{ bounded and closed}.$$

Proof. Exercise 8. □

Example 1.3.5. D^n and S^{n-1} are compact. In particular, D^n and S^{n-1} are not homeomorphic to $\mathbb{R}^m$. △

Theorem 1.3.6. *Let X be compact, Y Hausdorff, and $f\colon X \hookrightarrow Y$ continuous and injective. Then $X \cong f(X)$.*

Proof. Exercise 9. □

1.4 Contectedness

Definition 1.4.1. X is called **connected**, if X cannot be written as the disjoint union of two non-empty open (or closed) subsets. $\triangle$

Example 1.4.2. (i) $[0,1]$ is connected. Indeed, if $[0,1] = A \sqcup B$ is a disjoint union of open sets, then the characteristic function $\mathbb{1}_A$ is a continuous map from $[0,1]$ to $[0,1]$, which takes at most 0 and 1 as its values. The intermediate value theorem implies that either A or B must be empty.
(ii) If $f\colon X \to Y$ is continuous and X is connected, then $f(X)$ is connected. In particular, it follows from $X \cong Y$ that X is connected if and only if Y is connected.
(iii) If $A \subseteq X$ is connected, then so is $\overline{A}$ (Exercise 10). $\triangle$

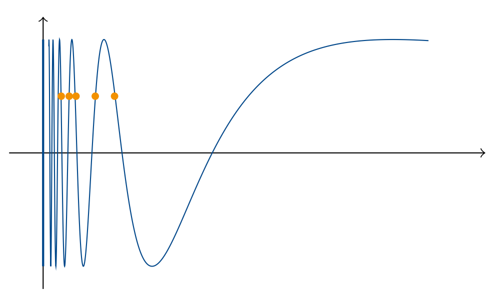
Definition 1.4.3. X is called **path connected**, if for all $x, y \in X$ there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$. Such a γ is called a *path* from x to y. $\triangle$

Proposition 1.4.4. *Path-connected spaces are connected. The converse is not true in general.*

Proof. Suppose $X = A \sqcup B$ is a partition into nonempty open sets. Choose $x \in A, y \in B$ and a path γ from x to y. Then $\gamma^{-1}(A) \sqcup \gamma^{-1}(B)$ is a partition of $[0,1]$ into nonempty open sets, a contradiction to Example 1.4.2 (i). Now consider the set

$$X = \{(x, \sin(1/x)) \mid x \in (0, 1/(2\pi)]\} \subseteq \mathbb{R}^2,$$

which is obviously path connected and thus connected.



Thus, according to Example 1.4.2 (iii), $\overline{X} \subseteq \mathbb{R}^2$ is also connected. However, in $\overline{X}$ there is no continuous path γ from $(1/(2\pi), 0)$ to $(0,1)$. Otherwise there would be a sequence $x_n \to 1$ with $\gamma(x_n) = (*, 1/2)$, and thus $(0,1) = \gamma(1) = (*, 1/2)$, a contradiction. $\square$

Example 1.4.5. (i) $\mathbb{R}^n$ is path connected. The same also holds for $\mathbb{R}^n \setminus \{a_1, \ldots, a_m\}$, if $n \geqslant 2$.
(ii) If $f\colon X \to Y$ is continuous and X is path connected, then $f(X)$ is path connected. In particular, it follows from $X \cong Y$ that X is path connected if and only if Y is path connected.
(iii) $\mathbb{R}^1 \setminus \{a\}$ is not (path) connected, but $\mathbb{R}^n \setminus \{a\}$ is, for $n \geqslant 2$. In particular, $\mathbb{R}^1 \ncong \mathbb{R}^n$ holds. The same is true for D^1 and D^n. Similarly, one obtains $S^1 \ncong S^n$, $S^1 \ncong D^m$ for all m, and that S^1 is not homeomorphic to the Möbius strip. △

Definition/Lemma 1.4.6. Let X be a topological space.
(i) An equivalence relation is defined on X by the following statement:

$$a \sim b \;:\Leftrightarrow\; \text{there exists a continuous path from } a \text{ to } b \text{ in } X.$$

An equivalence class is also called a **path-connected component**. The set of equivalence classes is denoted by

$$\pi_0(X).$$

(ii) For every continuous function $f\colon X \to Y$ the following holds:

$$a \sim b \;\Rightarrow\; f(a) \sim f(b).$$

Thus, f induces a well-defined map

$$\begin{aligned} \pi_0(f)\colon \pi_0(X) &\to \pi_0(Y) \\ [a] &\mapsto [f(a)]. \end{aligned}$$

(iii) For mappings $f\colon X \to Y, g\colon Y \to Z$ we have

$$\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$$

and

$$\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}.$$

(iv) For homeomorphic spaces $X \cong Y$ there exists a bijection between $\pi_0(X)$ and $\pi_0(Y)$.

Proof. Easy exercise. □

The properties (i)-(iii) from the last statement define a so-called *functor* between *categories*. Because it significantly improves the understanding of many mathematical concepts, we will briefly introduce these terms in the next section.

1.5 Some Basic Category Theory

Definition 1.5.1. (i) A **category** $\mathcal{C}$ consists of a class

$$\mathrm{Obj}(\mathcal{C})$$

(of so-called **objects**), and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a set

$$\mathcal{C}(X, Y)$$

(of so-called **morphisms**), as well as a composition of morphisms

$$\begin{aligned} \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) &\to \mathcal{C}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

which satisfies the following two conditions:

(1) $\forall X \in \mathrm{Obj}(\mathcal{C})\ \exists \mathrm{id}_X \in \mathcal{C}(X, X)$ with

$$\mathrm{id}_X \circ f = f,\ g \circ \mathrm{id}_X = g$$

for all $f \in \mathcal{C}(Y, X), g \in \mathcal{C}(X, Y)$.

(2) For all $f \in \mathcal{C}(W, X), g \in \mathcal{C}(X, Y), h \in \mathcal{C}(Y, Z)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(ii) A morphism $f \in \mathcal{C}(X, Y)$ is called an **isomorphism**, if there exists $g \in \mathcal{C}(Y, X)$ with

$$g \circ f = \mathrm{id}_X,\ f \circ g = \mathrm{id}_Y.$$

(iii) Two objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ are called **isomorphic** (in symbols $X \cong Y$), if there exists an isomorphism in $\mathcal{C}(X, Y)$. △

Graphically, parts of categories are usually represented by (commutative) arrow diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Example 1.5.2. (i) The category $\mathcal{S}et$ of sets has sets as objects and mappings as morphisms, with the usual composition. An isomorphism is an invertible (i.e. bijective) map, and two sets are isomorphic if they have the same cardinality.
(ii) The category $\mathcal{T}\mathrm{op}$ of topological spaces has topological spaces as objects and continuous maps as morphisms. An isomorphism is precisely a homeomorphism, and isomorphic objects are homeomorphic spaces.
(iii) The category $\mathcal{G}r$ of groups has as objects groups and as morphisms group homomorphisms.
In a similar way, one defines the categories of rings, fields, vector spaces over a fixed field, modules over a fixed ring, etc.
(iv) Morphisms do not always have to be maps. Let $(G, \cdot)$ be a fixed group. We define a category $\mathcal{C}_G$ by

$$\mathrm{Obj}(\mathcal{C}_G) := \{*\}$$

and

$$\mathcal{C}_G(*, *) := G.$$

The group operation $\cdot$ serves as composition of morphisms

$$\cdot\colon \mathcal{C}_G(*, *) \times \mathcal{C}_G(*, *) \to \mathcal{C}_G(*, *)$$

and one easily checks conditions (1) and (2). In this way, G can be regarded as a separate category. Every morphism here is an isomorphism. $\triangle$

Definition 1.5.3. A **covariant** (resp. **contravariant**) **functor** from the category $\mathcal{C}$ to the category $\mathcal{D}$ consists of a map

$$\mathcal{F}\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

as well as maps

$$\mathcal{F}\colon \mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$$

$$(\text{resp. } \mathcal{F}\colon \mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}(Y), \mathcal{F}(X)))$$

with

(i) $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$

(ii) $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ (resp. $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$). $\triangle$

Example 1.5.4. (i) $\pi_0\colon \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ is a covariant functor. We have verified the corresponding properties in Definition/Lemma 1.4.6.
(ii) Let k be a field and denote by k-$\mathcal{V}ek$ the category of k-vector spaces and linear maps. Forming the dual space establishes a contravariant functor from k-$\mathcal{V}ek$ into itself.
(iii) Let $\mathcal{C}$ be a category in which the objects really consist of set, and the morphisms really consist of maps between them (e.g., $\mathcal{T}\mathrm{op}, \mathcal{G}r, \ldots$). The *forgetful functor* is a covariant functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$, that simply forgets any additional structure on the objects of the category (e.g. a topology, a group structure,...). It also forgets the fact that morphisms may be very special maps and simply considers them as maps.
(iv) Forming the coordinate algebra is a contravariant functor from the category of affine k-varieties to the category of finitely generated reduced k-algebras.
(v) Let G, H be groups and $\mathcal{C}_G, \mathcal{C}_H$ the corresponding categories (see Example 1.5.2 (iv)). Every group homomorphism $\varphi\colon G \to H$ induces a canonical covariant functor $\mathcal{F}_\varphi\colon \mathcal{C}_G \to \mathcal{C}_H$, and every such functor comes from a group homomorphism. △

Lemma 1.5.5. *Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor and assume $X \cong Y$ holds in $\mathcal{C}$. Then*

$$\mathcal{F}(X) \cong \mathcal{F}(Y)$$

holds in $\mathcal{D}$.

Proof. Let $\mathcal{F}$ be covariant, without loss of generality. Let $f \in \mathcal{C}(X, Y)$ be an isomorphism with inverse morphism $g \in \mathcal{C}(Y, X)$. Then $g \circ f = \mathrm{id}_X$ and after applying $\mathcal{F}$ we have

$$\mathrm{id}_{\mathcal{F}(X)} = \mathcal{F}(\mathrm{id}_X) = \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Analogously we get $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathrm{id}_{\mathcal{F}(Y)}$ and thus $\mathcal{F}(X) \cong \mathcal{F}(Y)$. □

To conclude the first chapter, a brief reflection. Our main goal is to decide homeomorphism of topological spaces. If two spaces are homeomorphic, one usually finds a homeomorphism sooner or later, thus settling the question. The matter is more difficult if the spaces are not homeomorphic. One must then find properties in which the spaces differ and which exclude homeomorphism. An example of this is compactness, see Theorem 1.3.3 (iii). Another criterion is (path)-connectedness. We have formulated this somewhat more conceptually with the functor π_0:

$$X \cong Y \;\Rightarrow\; \pi_0(X) \cong \pi_0(Y).$$

Conversely, if $\pi_0(X)$ and $\pi_0(Y)$ are not isomorphic (in the category of sets), then X and Y cannot have been homeomorphic. In this case, one can also remove a finite number of points in X and Y, because if $f\colon X \to Y$ is a homeomorphism, then

$$f\colon X \setminus \{x_1, \dots, x_n\} \to Y \setminus \{f(x_1), \dots, f(x_n)\}$$

is also a homeomorphism. Using π_0, we were able to show, for example,

$$\mathbb{R}^1 \ncong \mathbb{R}^n \text{ for } n \geqslant 2.$$

But with this approach we can't get much further. The idea is now to construct a higher-dimensional analogue to π_0. This is first the so-called *fundamental group* $\pi_1(X)$, which then allows us to prove $\mathbb{R}^2 \ncong \mathbb{R}^n$ for $n \geqslant 3$. This is the subject of Chapter 2. Building on this further, *homology theory* was developed, which generalizes the constructions to all higher dimensions. This will be the content of Chapter 3.

Chapter 2

Homotopy and Fundamental Group

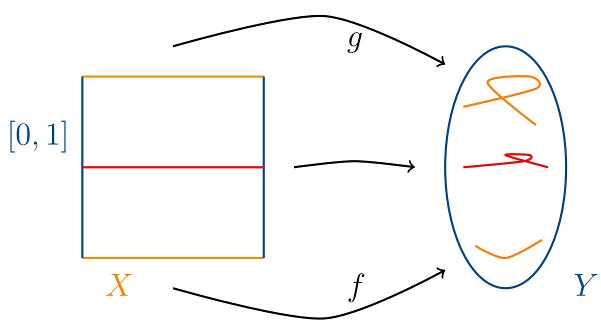
2.1 Homotopy

A homotopy is a continuous deformation from one continuous mapping to another:

Definition 2.1.1. (i) Let $f, g\colon X \to Y$ be two continuous maps. A **homotopy** from f to g is a continuous map

$$h\colon X \times [0,1] \to Y$$

with $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$.



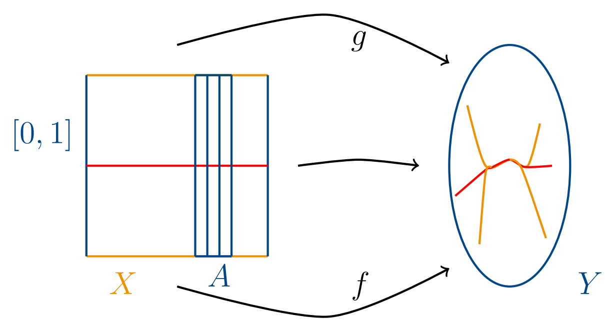
(ii) Two continuous maps $f, g\colon X \to X$ are called **homotopic,** if there exists a homotopy h between them. We denote this by

$$f \simeq g \quad \text{or} \quad f \underset{h}{\simeq} g.$$

(iii) If $f_{|A} = g_{|A}$ already holds for a subset $A \subseteq X$, then f and g are called **homotopic relative** A**,** if a homotopy h exists with

$$h(a,t) = f(a) = g(a) \quad \forall a \in A, t \in [0,1].$$

We denote this by $f \overset{A}{\underset{h}{\simeq}} g$.



△

Remark 2.1.2. (i) It is easy to see that homotopy of maps is an equivalence relation on $\mathcal{T}\mathrm{op}(X,Y)$.
(ii) If $f_1, g_1 \colon X \to Y$ and $f_2, g_2 \colon Y \to Z$ are continuous maps with $f_1 \simeq g_1$, $f_2 \simeq g_2$, then

$$(f_2 \circ f_1) \simeq (g_2 \circ g_1),$$

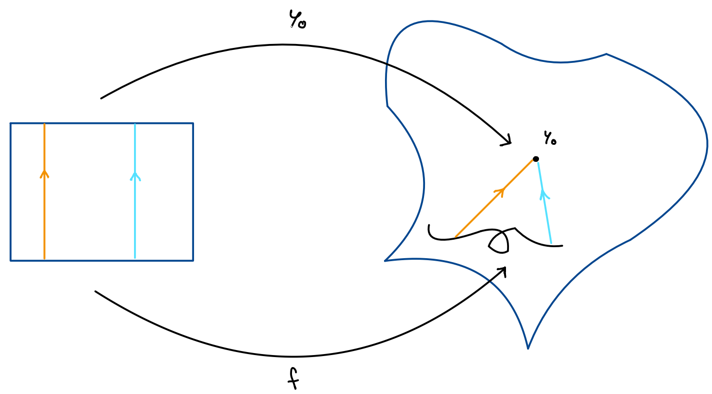
see Exercise 13. △

Example 2.1.3. (i) A subset $Y \subseteq \mathbb{R}^n$ is called *star-shaped*, if there exists a $y_0 \in Y$ such that

$$ty_0 + (1-t)y \in Y \quad \forall y \in Y, t \in [0,1].$$

A special case of this is, of course, a convex set. If Y is star-shaped, then every continuous map $f \colon X \to Y$ is homotopic to the constant map $x \mapsto y_0$. One obtains a homotopy by the formula

$$h(x,t) = ty_0 + (1-t)f(x).$$



In particular, any two continuous maps $f, g\colon X \to Y$ are homotopic.
(ii) Let $x, y \in X$ and

$$f\colon \{*\} \to X \qquad\qquad g\colon \{*\} \mapsto X$$
$$* \mapsto x \qquad\qquad * \mapsto y.$$

Then a homotopy of f and g is nothing other than a continuous path in X from x to y. In particular, one could define $\pi_0(X)$ as the set of homotopy classes of continuous maps $f\colon \{*\} \to X$.
(iii) The following should be clear (an exact proof follows later): If a subset $X \subseteq \mathbb{R}^2$ has a hole, then the map that wraps the interval $[0,1]$ once around the hole is not homotopic to a constant map. $\triangle$

Definition 2.1.4. Let X, Y be topological spaces.
(i) X and Y are called **homotopy equivalent,** if there exist continuous maps $f\colon X \to Y, g\colon Y \to X$ with

$$g \circ f \simeq \mathrm{id}_X, \quad f \circ g \simeq \mathrm{id}_Y .$$

We denote this by

$$X \simeq Y.$$

The maps f and g are then called **homotopy equivalences**.
(ii) X is called **contractible**, if $X \simeq \{*\}$.
(iii) A subset $A \subseteq X$ is called **retract** if there exists a continuous map $r\colon X \to A$ with $r_{|_A} = \mathrm{id}_A$. Such an r is called a **retraction**.
(iv) $A \subseteq X$ is called **deformation retract**, if a retraction r exists with

$$\iota \circ r \simeq \mathrm{id}_X,$$

where ι denotes the inclusion of A into X. A is called a **strong deformation retract**, if even

$$\iota \circ r \overset{A}{\simeq} \mathrm{id}_X$$

holds. △

Remark 2.1.5. Intuitively, two spaces X and Y are homotopy equivalent if they can be continuously transformed into each other. A is a deformation retract of X if X can be continuously retracted to A. If the points of A are not moved during the retraction, it is a strong deformation retract. We illustrate this with the following examples. △

Remark/Example 2.1.6. (i) Homotopy equivalence is an equivalence relation on the class of topological spaces.
(ii) Homeomorphic spaces are homotopy equivalent:

$$X \cong Y \;\Rightarrow\; X \simeq Y.$$

(iii) If $A \subseteq X$ is a deformation retract, then in particular $A \simeq X$ holds.
(iv) Let $X \subseteq \mathbb{R}^n$ be star-shaped with respect to x_0. Then $\{x_0\}$ is a strong deformation retract. The (only) retraction $r\colon X \to \{x_0\}$ yields

$$\iota \circ r \overset{\{x_0\}}{\simeq} \mathrm{id}_X$$

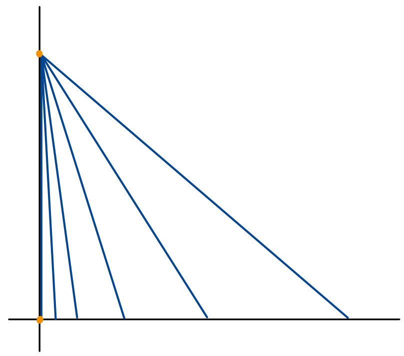
by means of the homotopy from Example 2.1.3 (i). In particular, X is contractible. In particular,

$$\mathbb{R}^m \simeq \mathbb{R}^n$$

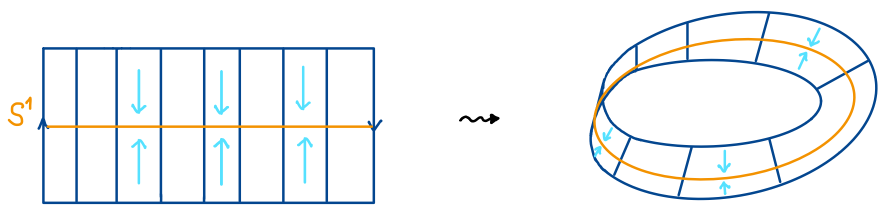
holds for all values of m, n.
(v) We consider the set

$$X = \mathrm{conv}\,\{(0,1),(0,0)\} \cup \bigcup_{n \geqslant 1} \mathrm{conv}\,\{(0,1),(1/n,0)\} \subseteq \mathbb{R}^2.$$

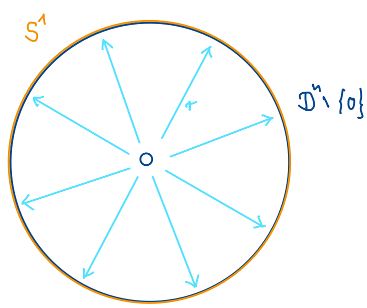
Then the point $(0,1)$ is a strong deformation retract. The point $(0,0)$ is a deformation retract, but not a strong deformation retract!



(vi) Let M be the Möbius strip and $S^1 \subseteq M$ be embedded in the middle. Then S^1 is a strong deformation retract, in particular $S^1 \simeq M$. However, $S^1 \not\cong M$ holds, as we saw in Example 1.4.5 (iii).



(vii) $S^{n-1} \subseteq D^n \setminus \{0\}$ is a strong deformation retract.



However, S^{n-1} is not homotopy equivalent to D^n, and there does not exist a retraction $r\colon D^n \to S^{n-1}$. We will prove this later, and the famous Brouwer's Fixed Point Theorem is based on it. $\triangle$

Lemma 2.1.7. *The functor* $\pi_0\colon \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ *is homotopy invariant:*

$$f, g \in \mathcal{T}\mathrm{op}(X, Y),\ f \simeq g \ \Rightarrow\ \pi_0(f) = \pi_0(g).$$

In particular, $X \simeq Y$ *implies* $\pi_0(X) \cong \pi_0(Y)$*. That means, in case of homotopy equivalence, the number of path-connected components is the same.*

Proof. If $h\colon X \times [0,1] \to Y$ is a homotopy from f to g, then for every $x \in X$ the map

$$h(x,\cdot)\colon [0,1] \to Y$$

is a continuous path in Y from $f(x)$ to $g(x)$. This implies

$$\pi_0(f)([x]) = [f(x)] = [g(x)] = \pi_0(g)([x]),$$

so $\pi_0(f) = \pi_0(g)$. The functorial properties therefore immediately implies that homotopy equivalences become isomorphisms through π_0. □

2.2 The Fundamental Group

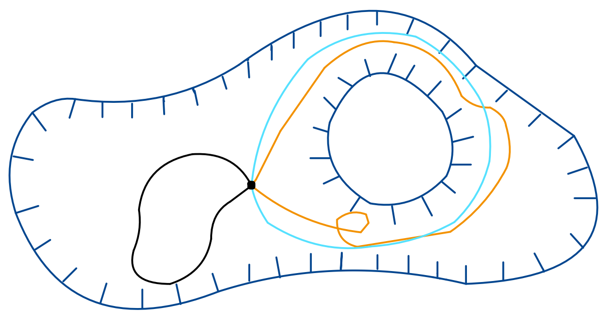
We will now define and study a first higher-dimensional generalization of the functor π_0.

Definition 2.2.1. Let (X, x_0) be a topological space with a distinguished point (a so-called *pointed topological space*). We define

$$\pi_1(X, x_0) := \{\gamma\colon [0,1] \to X \text{ continuous} \mid \gamma(0) = \gamma(1) = x_0\} \Big/ \begin{matrix}\text{homotopy} \\ \text{relative } \{0,1\}\end{matrix}$$

and call it the **fundamental group** of (X, x_0). △

The fundamental group thus consists of homotopy classes of closed paths starting at x_0, where the homotopy is understood relative to the starting and ending points of the path. In the following image, the light-blue and orange paths are homotopic to each other, but not to the black path.



Remark 2.2.2. If $\gamma, \delta\colon [0,1] \to X$ are two paths with $\gamma(1) = \delta(0)$, then one can concatenate γ and δ, thus obtaining a new path $\gamma\delta$ from $\gamma(0)$ to $\delta(1)$. Formally, this is defined as:

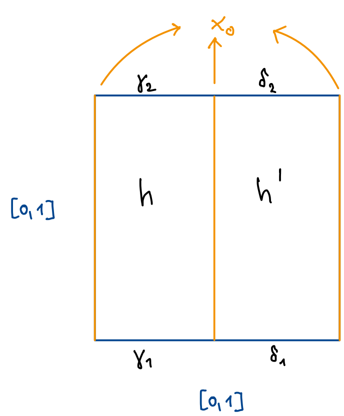
$$(\gamma\delta)(s) := \begin{cases} \gamma(2s) & s \leqslant 1/2 \\ \delta(2s-1) & s \geqslant 1/2. \end{cases}$$ △

Theorem 2.2.3. *The concatenation of paths turns $\pi_1(X, x_0)$ into a group.*

Proof. First, we show well-definedness, i.e. that concatenation is compatible with homotopy. Let

$$\gamma_1 \underset{h}{\overset{\{0,1\}}{\simeq}} \gamma_2 \text{ and } \delta_1 \underset{h'}{\overset{\{0,1\}}{\simeq}} \delta_2$$

be homotopic paths. From this, we construct a homotopy h'' relative $\{0,1\}$ from $\gamma_1\delta_1$ to $\gamma_2\delta_2$ as follows:



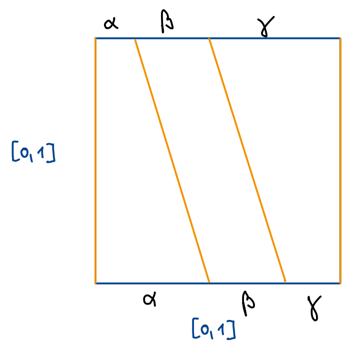
As a formula:

$$h''(s,t) := \begin{cases} h(2s,t) & s \leqslant 1/2 \\ h'(2s-1,t) & s \geqslant 1/2. \end{cases}$$

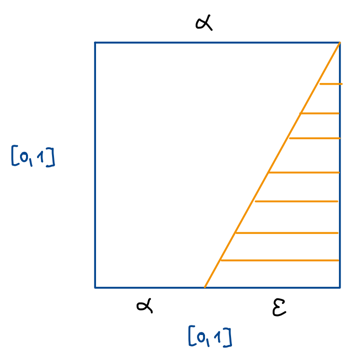
For the associative law, it must be shown:

$$\alpha(\beta\gamma) \overset{\{0,1\}}{\simeq} (\alpha\beta)\gamma.$$

This is obtained by a homotopy as in the following sketch, where the horizontal paths are each traveled with a linearly scaled speed:



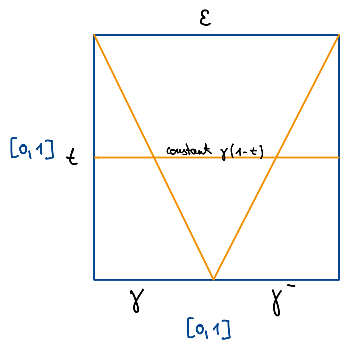
The identity element of the group is the constant path $\epsilon = x_0$:



For a path γ we denote the backward path with γ^- :

$$\gamma^-(s) := \gamma(1-s).$$

Then in $\pi_1(X, x_0)$ we obviously have $\gamma\gamma^- = \epsilon = \gamma^-\gamma$, as can be seen with the following homotopy:



Here, the paths are traveled at time t with constant (double) speed, but only up to the point $\gamma(1-t) = \gamma^-(t)$. In between, the path remains constant at this point. This completes the proof. $\square$

Example 2.2.4. (i) If $X \subseteq \mathbb{R}^n$ is star-shaped, then $\pi(X, x) = \{\epsilon\}$ holds for all $x \in X$ (Exercise 14).
(ii) We have $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ (Exercise 15).
(iii) We will show in the next section that $\pi_1(S^1, x_0) \cong \mathbb{Z}$ holds. This implies $\pi_1(T, x_0) \cong \mathbb{Z}^2$ for the torus.
(iv) We have $\pi_1(S^n, x_0) = \{\epsilon\}$ for $n \geqslant 2$ (Exercise 16).
(v) Fundamental groups are in general not abelian (see Exercise 18). $\triangle$

Theorem 2.2.5. *π_1 is a covariant functor from the category of pointed topological spaces (with point-preserving continuous maps) to the category of groups. It is invariant under homotopy relative to the ground point. In particular, homotopy equivalent punctured spaces have isomorphic fundamental groups.*

Proof. Let $(X, x_0), (Y, y_0)$ be pointed spaces and $f\colon X \to Y$ continuous with $f(x_0) = y_0$. For every path $\gamma\colon [0,1] \to X$ from x_0 to x_0,

$$(f \circ \gamma)\colon [0,1] \to Y$$

is a continuous path from y_0 to y_0, and a homotopy

$$\gamma \overset{\{0,1\}}{\underset{h}{\simeq}} \delta$$

obviously induces a homotopy

$$(f \circ \gamma) \overset{\{0,1\}}{\underset{f \circ h}{\simeq}} (f \circ \delta).$$

Thus f induces a well-defined map

$$\begin{aligned}\pi_1(f)\colon \pi_1(X,x_0) &\to \pi_1(Y,y_0)\\ [\gamma] &\mapsto [f\circ\gamma].\end{aligned}$$

Clearly also

$$f\circ(\gamma\delta) = (f\circ\gamma)(f\circ\delta)$$

holds, and thus $\pi_1(f)$ is a group homomorphism. Furthermore,

$$\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)} \ \text{ and } \ \pi_1(g\circ f) = \pi_1(g)\circ\pi_1(f)$$

hold, proving the properties of a covariant functor.
Now assume that $f \overset{x_0}{\simeq} g$ holds for two maps $f,g\colon (X,x_0)\to(Y,y_0)$. This implies

$$f\circ\gamma \overset{\{0,1\}}{\simeq} g\circ\gamma$$

for all paths $\gamma\colon [0,1]\to(X,x_0)$, see Remark 2.1.2 ($ii$). This means that $\pi_1(f) = \pi_1(g)$. □

Proposition 2.2.6. *Let $x_0, x_1 \in X$ be connected by a continuous path. Then*

$$\pi_1(X,x_0) \cong \pi_1(X,x_1).$$

Proof. Let $\delta\colon [0,1]\to X$ be a path from x_0 to x_1. The following mapping then yields a well-defined group homomorphism, as can be seen analogously to the proof of Theorem 2.2.3:

$$\begin{aligned}\pi_1(X,x_0) &\to \pi_1(X,x_1)\\ [\gamma] &\mapsto [\delta^-\gamma\delta].\end{aligned}$$

Obviously, one obtains an inverse homomorphism by starting with the inverse path δ^-. □

Remark 2.2.7. (i) For path-connected spaces X we can simply use the notation $\pi_1(X)$.
(ii) A slightly generalized version of homotopy invariance of π_1 with respect to Proposition 2.2.6 can be found in Exercise 19. △

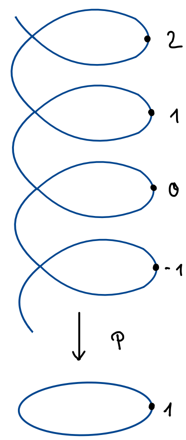
Definition 2.2.8. A topological space X is called **simply connected**, if $\pi_0(X)$ and $\pi_1(X)$ are trivial. This means that X is path connected and every closed path can be contracted relative to the base-point. △

Example 2.2.9. Every star-shaped space is simply connected, for example, $\mathbb{R}^n$ and D^n. For $n \geqslant 2$, S^n is simply connected (Exercise 16), but S^1, the Möbius strip, and the torus are not. △

2.3 The Fundamental Group of the Circle

We now want to compute the fundamental group of S^1. This is not entirely trivial, but the method can be adapted to calculate other fundamental groups (see for example Exercise 17 and Exercise 18). We consider so-called *coverings* of the respective spaces and attempt to *lift* continuous paths.
We consider the map

$$\begin{aligned} p\colon \mathbb{R} &\to S^1 \subseteq \mathbb{C} \\ r &\mapsto e^{2\pi i r}. \end{aligned}$$



Proposition 2.3.1. *(i) For every continuous path $\gamma\colon [0,1] \to S^1$ with $\gamma(0) = 1$ there is exactly one continuous map $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}$ with $\widetilde{\gamma}(0) = 0$ and $p \circ \widetilde{\gamma} = \gamma$.*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

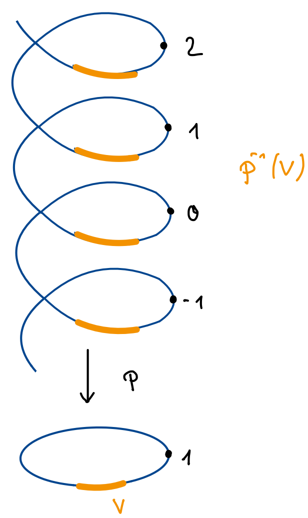
(ii) For $\gamma, \delta\colon [0,1] \to S^1$ with $\gamma(0) = \gamma(1) = \delta(0) = \delta(1) = 1$ we have

$$\gamma \overset{\{0,1\}}{\simeq} \delta \Rightarrow \widetilde{\gamma} \overset{\{0,1\}}{\simeq} \widetilde{\delta}.$$

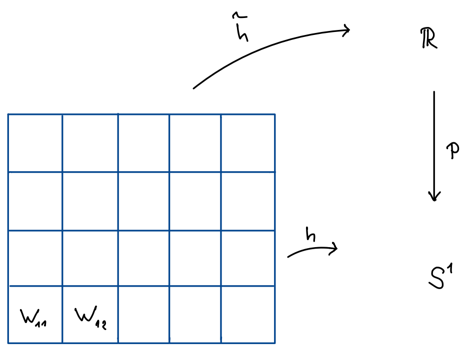
Proof. (i) *Uniqueness*: Let $\widetilde{\gamma}$ and $\widehat{\gamma}$ be two such liftings of γ. The set

$$G = \{s \in [0,1] \mid \widetilde{\gamma}(s) = \widehat{\gamma}(s)\}$$

is not empty, since $0 \in G$. Furthermore, G is both open and closed. Let first $s \in G$ be arbitrary. Then p is restricted to a small open neighborhood V of $\widetilde{\gamma}(s) = \widehat{\gamma}(s)$ is injective. From continuity it follows that there exists an open neighborhood U of s which is mapped to V by $\widetilde{\gamma}$ and $\widehat{\gamma}$. Since both are lifts of γ, injectivity of p on V implies $U \subseteq G$. Thus, G is open. Closedness follows quite elementary from continuity of $\widetilde{\gamma}, \widehat{\gamma}$ and the Hausdorff property of $\mathbb{R}$. Since $[0,1]$ is connected, it follows that $G = [0,1]$ holds so $\widetilde{\gamma} = \widehat{\gamma}$.
Existence: For every point in S^1, there exists an open neighborhood V, such that $p^{-1}(V)$ decomposes into countably infinite many connected components, each of which is homeomorphic to V via p. In particular, one can lift any path with image in such a V to $\mathbb{R}$ if one specifies the connected component in the preimage.



We now cover S^1 with such open sets and, by pulling back using γ, obtain an open cover of $[0,1]$. A Lebesque constant now exists for this cover (compare to Example 1.3.2 (*ii*)). Thus, we obtain a decomposition $0 = s_0 < s_1 < \cdots < s_n = 1$ such that γ can be lifted on any subinterval $[s_{i-1}, s_i]$ if we specify the connected component of $p^{-1}\left(\gamma\left([s_{i-1}, s_i]\right)\right)$. With the initial setting $\widetilde{\gamma}(0) = 0$ we iteratively obtain a continuous lifting of γ, where in the i-th step we aleays chose the connected component of $\widetilde{\gamma}(s_{i-1})$.
(*ii*) Let $h\colon [0,1] \times [0,1] \to S^1$ be a homotopy of γ and δ relative to $\{0,1\}$. As in part (*i*), we see that $[0,1] \times [0,1]$ can be partitioned into small squares W_{ij} such that h restricted to each W_{ij} has a unique continuous lift to $\mathbb{R}$, given the connected component of $p^{-1}\left(h(W_{ij})\right)$.



We now lift h on W_{11} into the connected component of 0. Then, due to uniqueness, the lift on the lower boundary of W_{11} coincides with $\widetilde{\gamma}$. After that, we lift h on W_{12} so that it coincides with the already defined lift on the intersection with W_{11} (which is automatically the case after choosing the correct connected component). We continue in this way until we have continuously lifted h on the entire lower row of the square. On the lower boundary of the row, it must then coincide with $\widetilde{\gamma}$ due to uniqueness from (*i*). The left boundary of the row is constantly mapped to 0, the right boundary is constantly mapped to $\widetilde{\gamma}(1)$.
Now we replace γ with the path obtained on the upper edge of the lower row and iterate the process. Finally, we obtain a complete continuous lifting, where the left edge of the square is constantly mapped to 0, the right edge is constantly mapped to $\widetilde{\gamma}(1)$, and on the upper edge of the square we obtain, again with uniqueness from (*i*), exactly $\widetilde{\delta}$. This proves the claim. □

Theorem 2.3.2. *We have* $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. We choose $x_0 = 1 \in S^1 \subseteq \mathbb{C}$ and denote the unique lifting of a path γ from the last theorem by $\widetilde{\gamma}$. Then we consider the following map:

$$\begin{aligned} \varphi\colon \pi_1(S^1, 1) &\to \mathbb{Z} \\ [\gamma] &\mapsto \widetilde{\gamma}(1). \end{aligned}$$

Independence from the homotopy class follows from Proposition 2.3.1 (*ii*), in particular from the liftability of the homotopy relative to 1. Since $\widetilde{\gamma}$ is a lift of γ, $\widetilde{\gamma}(1) \in \mathbb{Z}$ follows from $\gamma(1) = 1 \in S^1$. Now for two paths γ, δ,

$$\widetilde{\gamma\delta} = \widetilde{\gamma}(\widetilde{\delta} + \widetilde{\gamma}(1))$$

holds and thus φ is a group homomorphism. Obviously, 1 lies in the image of φ, so φ is surjective. For injectivity, we assume $\widetilde{\gamma}(1) = 0$. Since $\mathbb{R}$ is star-shaped, $\widetilde{\gamma}$

is homotopic to the constant zero map, relative to $\{0,1\}$ (see Example 2.2.4 (i)). By composing the homotopy with p we immediately obtain $\gamma \overset{\{0,1\}}{\simeq} \varepsilon$, so $[\gamma] = [\epsilon]$ in $\pi_1(S^1, 1)$. □

Example 2.3.3. (i) $\pi_1(S^n) = \{\epsilon\}$ holds for $n \geqslant 2$ (Exercise 16). This implies $S^1 \not\simeq S^n$ for $n \geqslant 2$. Likewise, $S^1 \not\simeq \mathbb{R}^n$ and $S^1 \not\simeq S^1 \times S^1$ follows.
(ii) Suppose $\mathbb{R}^2 \cong \mathbb{R}^n$. Then

$$S^1 \simeq \mathbb{R}^2 \setminus \{a\} \cong \mathbb{R}^n \setminus \{b\} \simeq S^{n-1}$$

and from this it follows that $1 = n - 1$, so $n = 2$.
(iii) $S^1 \subseteq D^2$ is not a retract. By applying π_1 to a possible diagram

$$\begin{tikzcd} S^1 \arrow[r, "\iota"] \arrow[rr, bend right, "\mathrm{id}"'] & D^2 \arrow[r, "r"] & S^1 \end{tikzcd}$$

one would obtain

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\pi_1(\iota)"] \arrow[rr, bend right, "\mathrm{id}"'] & \{0\} \arrow[r, "\pi_1(r)"] & \mathbb{Z}, \end{tikzcd}$$

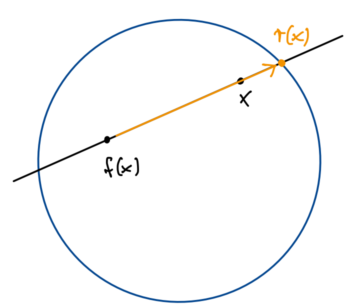
an obvious contradiction. △

Corollary 2.3.4 (Brouwer's Fixed Point Theorem)**.** *Every continuous map*

$$f \colon D^2 \to D^2$$

has a fixed point.

Proof. If f had no fixed point, one would get a retraction $r \colon D^2 \to S^1$ by mapping $x \in D^2$ to the intersection point of the line through x and $f(x)$ with $\partial D^2 = S^1$, on the side of x:



□

Corollary 2.3.5 (Fundamental Theorem of Algebra)**.** *$\mathbb{C}$ is algebraically closed, i.e. every non-constant polynomial in $\mathbb{C}[x]$ has a zero in $\mathbb{C}$.*

Proof. Suppose the statement is false. Then there exists some $d \geqslant 1$ and

$$p = x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{C}[x]$$

with $p(z) \neq 0$ for all $z \in \mathbb{C}$. On $S^1 \subseteq \mathbb{C}$ we now consider the following two well-defined continuous maps $f_i \colon S^1 \to S^1$:

$$f_0 \colon z \mapsto z^d$$
$$f_1 \colon z \mapsto \frac{p(z)}{|p(z)|}$$

We have $f_0 \simeq f_1$ by means of the following homotopy:

$$h(z,t) := \frac{z^d + t a_1 z^{d-1} + t^2 a_2 z^{d-2} + \cdots + t^d a_d}{|z^d + t a_1 z^{d-1} + t^2 a_2 z^{d-2} + \cdots + t^d a_d|}.$$

Note that the enumerator is non-zero for all $t \in [0,1]$ and $z \in S^1$,because for $t > 0$ it is identical to $t^d p(z/t)$. It follows that f_0 and f_1 define the same mapping on the fundamental groups using π_1 (for an exact statement, see Exercise 19). However, it is obvious that

$$\begin{aligned} \pi_1(f_0) \colon \pi_1(S^1) = \mathbb{Z} &\to \mathbb{Z} = \pi_1(S^1) \\ n &\mapsto d \cdot n, \end{aligned}$$

as can be seen by applying f_0 to the simple circular path. However, the mapping $f_1 \colon S^1 \to S^1$ can be continuously extended to a mapping $f_1 \colon D^2 \to S^1$, simply using the same formula:

$$\begin{tikzcd} S^1 \arrow[r, "\iota"] \arrow[dr, "f_1"'] & D^2 \arrow[d, "f_1"] \\ & S^1 \end{tikzcd}$$

We now apply the functor π_1:

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "\pi_1(\iota)"] \arrow[dr, "\pi_1(f_1)"'] & \pi_1(D^2) \arrow[d, "\pi_1(f_1)"] \\ & \pi_1(S^1). \end{tikzcd}$$

Since $\pi_1(D^2) = \{\epsilon\}$, we find that $\pi_1(f_1) \colon \mathbb{Z} \to \mathbb{Z}$ must be the constant zero map. This contradicts $d \geqslant 1$. $\square$

Chapter 3

Homology Theory

Homology theory is a method to systematically generalize the functors π_0 and π_1 from the previous chapters to higher dimensions. We first discuss the construction of homology for abstract simplicial complexes, which is the easiest to understand and compute. Then we develop homology theory axiomatically for arbitrary topological spaces. From the existence of a homology theory, one can subsequently derive many interesting statements. We then finally provide an explicit construction of a general homology theory in the last section.

3.1 Homology for Abstract Simplicial Complexes

We set $[n] := \{0, 1, \ldots, n\}$ and denote by $\mathcal{P}([n])$ its power set.

Definition 3.1.1. (i) An **(abstract) simplicial complex** is a subset

$$\mathcal{K} \subseteq \mathcal{P}\left([n]\right)$$

with the following properties:

(1) $S \in \mathcal{K},\, \emptyset \neq T \subseteq S \;\Rightarrow\; T \in \mathcal{K}$.

(2) $\bigcup_{S \in \mathcal{K}} S = [n]$.

(ii) An element $S \in \mathcal{K}$ is called an (abstract) **simplex**, a subset $T \subseteq S$ is called a **face** of S, and a face of cardinality 1 is called a **vertex** of S. Faces of simplices of $\mathcal{K}$ are themselves simplices of $\mathcal{K}$, by axiom (1).

(iii) The **dimension** of a simplex $S \in \mathcal{K}$ is defined as

$$\dim(S) := |S| - 1,$$

the dimension $\dim(\mathcal{K})$ of $\mathcal{K}$ is the maximum dimension of its simplices.
(iv) For two (abstract) simplicial complexes

$$\mathcal{K} \subseteq \mathcal{P}([m]) \text{ and } \mathcal{L} \subseteq \mathcal{P}([n]),$$

a **morphism of (abstract) simplicial complexes** $\varphi\colon \mathcal{K} \to \mathcal{L}$ is a map $\varphi\colon [m] \to [n]$ with

$$S \in \mathcal{K} \Rightarrow \varphi(S) \in \mathcal{L}. \qquad \triangle$$

Remark/Example 3.1.2. (i) The sets

$$\mathcal{G} = \{\{0\}, \{1\}, \dots, \{n\}\} \text{ and } \mathcal{F} = \mathcal{P}([n]) \setminus \{\emptyset\}$$

are simplicial complexes. Here, $\mathcal{G}$ is of dimension 0 and $\mathcal{F}$ is of dimension n. The identity map $\mathrm{id}\colon [n] \to [n]$ is a morphism from $\mathcal{G}$ to $\mathcal{F}$, but not in the other direction.
(ii) A one-dimensional simplicial complex is simply a graph, the one-dimensional simplices are the edges.
(iii) Abstract simplicial complexes with morphisms form a category $\mathcal{SC}$. In particular, we also obtain a canonical notion of isomorphism. By $\mathcal{SC}_{\text{inj}}$ we denote the category in which we only admit injective morphisms $f\colon [n] \hookrightarrow [m]$. The notion of isomorphism is the same in both categories. $\triangle$

Definition 3.1.3. (i) A subset $S \subseteq \mathbb{R}^m$ is called a d-**dimensional (concrete) simplex** if S is the convex hull of $d+1$ affine independent points. $d+1$ points are called affine independent if they do not jointly lie in a $d-1$-dimensional affine subspace.
(ii) A **geometric realization** of an abstract simplicial complex $\mathcal{K} \subseteq \mathcal{P}([n])$ is an injective map

$$f\colon [n] \to \mathbb{R}^m,$$

such that for all $S \in \mathcal{K}$ the set $f(S)$ is affine independent, and for all $S, T \in \mathcal{K}$ the following holds:

$$\operatorname{conv} f(S) \cap \operatorname{conv} f(T) = \operatorname{conv} f(S \cap T).$$

Often only the set

$$\mathcal{R}(\mathcal{K}, f) := \bigcup_{S \in \mathcal{K}} \operatorname{conv} f(S)$$

is then called a geometric realization, where of course one only has to use the maximal simplices. Intuitively, a geometric realization is a union of concrete simplices in $\mathbb{R}^m$, that intersect in the same way as the abstract simplices do. $\triangle$

Remark/Example 3.1.4. (i) The set

$$\Delta_n := \operatorname{conv}\{0, e_1, \ldots, e_n\} \subseteq \mathbb{R}^n$$

is an n-dimensional (concrete) simplex, called the **n-dimensional unit simplex**.
(ii) For every abstract simplicial complex $\mathcal{K} \subseteq \mathcal{P}([n])$, the following map provides a geometric realization:

$$\begin{aligned} f \colon [n] &\to \mathbb{R}^n \\ 0 &\mapsto 0 \\ i &\mapsto e_i. \end{aligned}$$

For $\mathcal{G} = \{\{0\}, \ldots, \{n\}\}$ and $\mathcal{F} = \mathcal{P}([n])$, we thereby obtain

$$\mathcal{R}(\mathcal{G}, f) = \{0, e_1, \ldots, e_n\} \text{ and } \mathcal{R}(\mathcal{F}, f) = \Delta_n.$$

(iii) If $\mathcal{K} \subseteq \mathcal{P}([n])$ is an abstract simplicial complex of dimension d, then $\mathcal{K}$ can at best be geometrically realized in $\mathbb{R}^d$, but not in smaller dimension (simply due to the definition of affine independence). However, a realization in $\mathbb{R}^d$ need not necessarily exist. As we saw in (ii), however, there is always a realization in $\mathbb{R}^n$.
(iv) If $\mathcal{K}$ is one-dimensional (i.e. a graph), then it can be realized in $\mathbb{R}^1$ if and only if it is a straight line. If it can be realized in $\mathbb{R}^2$, then the graph must be planar, i.e. it must be possible to draw it in the plane without any intersection of edges (however, this is only necessary and not sufficient, since we only allow straight edges in realizations). In the picture below we see a one-dimensional realization of

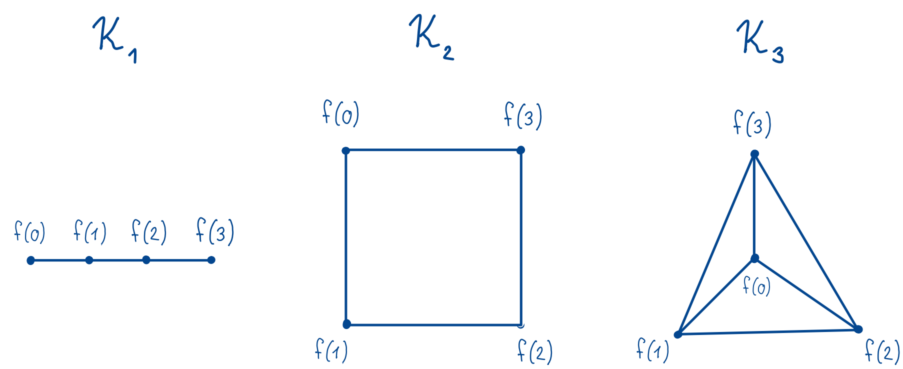
$$\mathcal{K}_1 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{2,3\}\}$$

and a two-dimensional one of

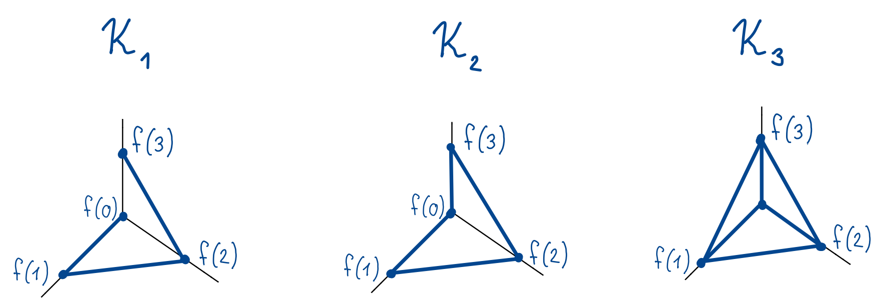
$$\mathcal{K}_2 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{2,3\}, \{0,3\}\}$$

as well as

$$\mathcal{K}_3 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}\}.$$



Next we see the respective realization in $\mathbb{R}^3$ from (ii), which exists for all $\mathcal{K}_i$:

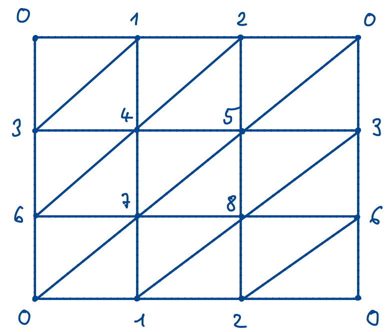


(v) For arbitrary n we consider

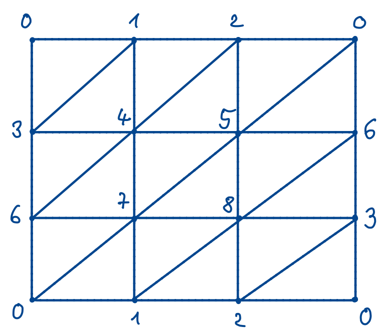
$$\mathcal{S} = \{S \subseteq [n] \mid S \neq \emptyset, |S| \leqslant n\}.$$

This is an $n-1$-dimensional simplicial complex, whose geometric realization from (ii) is exactly the boundary of the n-dimensional unit simplex in $\mathbb{R}^n$. It is homeomorphic to S^{n-1}.

(vi) We consider the simplicial complex $\mathcal{T} \subseteq \mathcal{P}([8])$, which is schematically shown in the following figure:



Each segment represents a 1-dimensional simplex, and each triangle a two-dimensional one. It should be clear that the geometric realization is a space that is homeomorphic to the torus. Analogously, a simplicial complex $\mathcal{KB} \subseteq \mathcal{P}([8])$ for the Klein bottle is obtained using the following diagram:



△

Lemma 3.1.5. *Any two geometric realizations f, g of the same abstract simplicial complex $\mathcal{K}$ yield homeomorphic topological spaces:*

$$\mathcal{R}(\mathcal{K}, f) \cong \mathcal{R}(\mathcal{K}, g).$$

Proof. Exercise 22. □

Definition 3.1.6. Let $\mathcal{K}$ be an abstract simplicial complex.
(i) For $k \in \mathbb{Z}$ let

$$C_k(\mathcal{K}) := \bigoplus_{\substack{S \in \mathcal{K} \\ \dim(S) = k}} \mathbb{Z}$$

be the free $\mathbb{Z}$-module generated by the simplices of $\mathcal{K}$ of dimension k. Instead of writing elements as tuples of integers, we often write them as formal sums with integer coefficients:

$$\sum_{\substack{S \in \mathcal{K} \\ \dim(S) = k}} c_S \cdot S.$$

(ii) Let $S \in \mathcal{K}$ with $\dim(S) = k$. We write

$$S = \{i_0, i_1, \dots, i_k\} \text{ with } i_0 < i_1 < \cdots < i_k$$

and define

$$\partial_k(S) := \sum_{j=0}^{k} (-1)^j \cdot (S \setminus \{i_j\}) \in C_{k-1}(\mathcal{K}).$$

In this way we obtain a module-homomorphism

$$\partial_k \colon C_k(\mathcal{K}) \to C_{k-1}(\mathcal{K}),$$

called the **boundary map**. The resulting chain of mappings

$$\cdots \to C_{k+1}(\mathcal{K}) \overset{\partial_{k+1}}{\to} C_k(\mathcal{K}) \overset{\partial_k}{\to} C_{k-1}(\mathcal{K}) \to \cdots$$

is called the **chain complex** associated with $\mathcal{K}$.
(iii) We define

$$Z_k(\mathcal{K}) := \ker(\partial_k) \text{ and } B_k(\mathcal{K}) := \operatorname{im}(\partial_{k+1}).$$

Elements of $Z_k(\mathcal{K})$ are called **cycles** and elements of $B_k(\mathcal{K})$ are called **boundaries**. The notion of *boundaries* is relatively easy to motivate: Every simplex is mapped under ∂_k to the alternating sum of its highest-dimensional sides; that is, just to its boundary. The image of ∂_k is thus generated by boundaries of simplices. We motivate the notion of *cycles* in Example 3.1.7 (iii) below. $\triangle$

Example 3.1.7. (i) We consider $\mathcal{G} = \{\{0\}, \dots, \{n\}\}$ and see

$$C_0(\mathcal{G}) = \mathbb{Z}^{n+1} \text{ and } C_k(\mathcal{G}) = \{0\} \text{ for } k \neq 0.$$

Thus, the boundary maps ∂_k are already uniquely determined, and the relevant part of the chain complex is

$$\{0\} \overset{\partial_1}{\to} \mathbb{Z}^{n+1} \overset{\partial_0}{\to} \{0\}.$$

We see
$$B_0(\mathcal{G}) = \{0\} \text{ and } Z_0(\mathcal{G}) = \mathbb{Z}^{n+1}.$$
(ii) We consider $\mathcal{F} = \mathcal{P}([n]) \setminus \{\emptyset\}$ and obtain

$$C_k(\mathcal{F}) = \mathbb{Z}^{\binom{n+1}{k+1}}$$

for $k = 0, \ldots, n$ and the following chain complex

$$\{0\} \to \mathbb{Z} \xrightarrow{\partial_n} \mathbb{Z}^{n+1} \xrightarrow{\partial_{n-1}} \mathbb{Z}^{\binom{n+1}{n-1}} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} \mathbb{Z}^{n+1} \xrightarrow{\partial_0} \{0\}.$$

Here the boundary maps are a bit more difficult to describe explicitly. For example, in the case $n = 2$ we obtain

$$\begin{aligned} \partial_2 \colon \mathbb{Z} = C_2(\mathcal{F}) &\to C_1(\mathcal{F}) = \mathbb{Z}^3 \\ 1 &\mapsto \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

where we have sorted the 1-simplices as follows:

$$e_1 = \{1,2\},\ e_2 = \{0,2\},\ e_3 = \{0,1\}.$$

If we sort the 0-simplices as $e_1 = \{0\}, e_2 = \{1\}, e_3 = \{2\}$ we obtain

$$\begin{aligned} \partial_1 \colon \mathbb{Z}^3 = C_1(\mathcal{F}) &\to C_0(\mathcal{F}) = \mathbb{Z}^3 \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto \begin{pmatrix} -b-c \\ c-a \\ b+a \end{pmatrix}. \end{aligned}$$

We thus have
$$B_2(\mathcal{F}) = Z_2(\mathcal{F}) = \{0\}$$
$$B_1(\mathcal{F}) = Z_1(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$
$$B_0(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } Z_0(\mathcal{F}) = \mathbb{Z}^3.$$

(*iii*) We consider $\mathcal{K}_2 = \{\{0\},\{1\},\{2\},\{3\},\{0,1\},\{1,2\},\{2,3\},\{0,3\}\}$ from Remark/Example 3.1.4 (*iv*). Here we get the chain complex

$$\{0\} \to \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z}^4 \to \{0\}$$

with

$$\begin{aligned} \partial_1 \colon \mathbb{Z}^4 &\to \mathbb{Z}^4 \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &\mapsto \begin{pmatrix} -a-d \\ a-b \\ b-c \\ c+d \end{pmatrix}. \end{aligned}$$

So

$$B_1(\mathcal{K}_2) = \{0\} \text{ and } Z_1(\mathcal{K}_2) = \operatorname{span}_{\mathbb{Z}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

as well as

$$B_0(\mathcal{K}_2) = \operatorname{span}_{\mathbb{Z}} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ and } Z_0(\mathcal{K}_2) = \mathbb{Z}^4.$$

From $Z_1(\mathcal{K}_2)$ you can see why the term *cycle* is well chosen. There is an element in the kernel of ∂_1, namely

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix},$$

which, in other notation, is simply $\{0,1\}+\{1,2\}+\{2,3\}-\{0,3\}$. It is in the kernel of ∂_1 precisely because it geometrically corresponds to a cycle/circle between the numbers $0, 1, 2, 3$ (which can be clearly seen in the geometric realization).
(*iv*) We consider $\mathcal{S} = \{S \subseteq [n] \mid S \neq \emptyset, |S| \leqslant n\}$, whose realization is homeomorphic to S^{n-1}. The resulting chain complex is almost identical to that in (*ii*), except that $C_n(\mathcal{S}) = \{0\}$ holds, since the full simplex $[n]$ does not belong to $\mathcal{S}$.
(*v*) The chain complexes, boundaries and cycles of the simplicial complexes of the torus and Klein's bottle from Remark/Example 3.1.4 (*vi*) can also be determined, with considerable computational effort. △

Lemma 3.1.8. *For $k \in \mathbb{Z}$ we have $\partial_k \circ \partial_{k+1} = 0$, so*

$$B_k(\mathcal{K}) \subseteq Z_k(\mathcal{K}).$$

Proof. It suffices to show that $\partial_k(\partial_{k+1}(S)) = 0$ for all simplices S of dimension $k+1$. However, $\partial_k(\partial_{k+1}(S))$ is a sum over all sides of S of dimension $k-1$, where each of these sides appears exactly twice (one can delete two vertices of S in different orders). This, however, results in different signs, as one easily checks. □

Definition 3.1.9. For an abstract simplicial complex $\mathcal{K}$ and $k \in \mathbb{Z}$ we define

$$H_k(\mathcal{K}) := Z_k(\mathcal{K})/B_k(\mathcal{K})$$

and call it is the **k-th homology module** of $\mathcal{K}$. △

Remark 3.1.10. Let $\mathcal{K} \subseteq \mathcal{P}([n])$ be a simplicial complex with $\emptyset \notin \mathcal{K}$.
(i) We have $H_k(\mathcal{K}) = \{0\}$ for $k < 0$ and for $k > \dim(\mathcal{K})$.
(ii) It always holds that

$$H_0(\mathcal{K}) \cong \mathbb{Z}^s$$

where s is the number of (path) connected components of the geometric realization of $\mathcal{K}$. Obviously,

$$Z_0(\mathcal{K}) = C_0(\mathcal{K}) = \mathbb{Z}^{n+1}.$$

Assume that for two vertices $i < j$, $\{i, j\} \in \mathcal{K}$ holds. Then from

$$\partial_1(\{i, j\}) = \{j\} - \{i\} \in B_0(\mathcal{K})$$

it follows that

$$\overline{\{i\}} = \overline{\{j\}} \in H_0(\mathcal{K}).$$

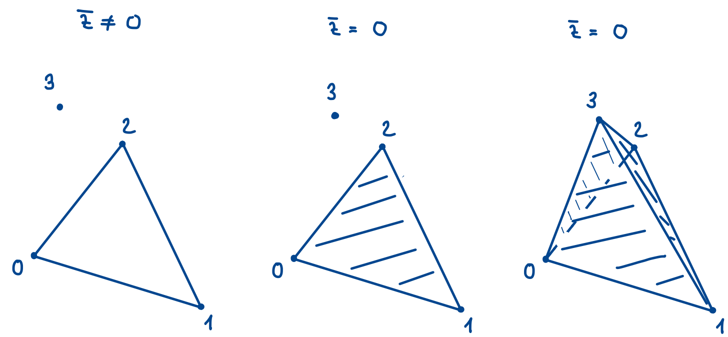
If two vertices lie in the same connected component, they can be connected by a finite sequence of such edges. This yields the statement.
(iii) We must imagine a k-cycle as a configuration of simplices that completely enclose a region of dimension $k+1$, for example,

$$z = \{0, 1\} + \{1, 2\} - \{0, 2\}.$$

Whether a cycle $\overline{z}$ becomes trivial in homology depends on whether there exists a configuration of higher simplices whose boundary is precisely the cycle, and which thus fills the resulting spatial region. For example,

$$\partial_2 (\{0, 1, 2\}) = z = \partial_2 (\{0, 1, 3\} + \{1, 2, 3\} - \{0, 2, 3\})$$

and the corresponding simplices fill the region enclosed by z:



In this respect, the k-th homology module measures the number holes of dimension $k+1$ in the simplicial complex. △

Example 3.1.11. We compute all homology modules of the simplicial complexes from Example 3.1.7.
(i) For $\mathcal{G} = \{\{0\}, \ldots, \{n\}\}$ we obtain

$$H_0(\mathcal{G}) = \mathbb{Z}^{n+1}$$

and all other homology modules are trivial.
(ii) For the complex $\mathcal{F} = \mathcal{P}([n]) \setminus \{\emptyset\}$ we obtain

$$H_0(\mathcal{F}) \cong \mathbb{Z}$$

and all other homology modules are trivial (Exercise 23).
(iii) For $\mathcal{K}_2 = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{2,3\}, \{0,3\}\}$ one directly sees

$$H_0(\mathcal{K}_2) \cong \mathbb{Z},\ H_1(\mathcal{K}_2) \cong \mathbb{Z}$$

and all other modules are trivial.
(iv) For $\mathcal{S} = \{S \subseteq [n] \mid S \neq \emptyset, |S| \leqslant n\}$ we obtain almost the same chain complex as in (ii), only $C_n(\mathcal{S}) = \{0\}$. From this we see

$$H_0(\mathcal{S}) \cong \mathbb{Z},\ H_{n-1}(\mathcal{S}) \cong \mathbb{Z}$$

and all other modules are trivial (Exercise 23).
(v) After long and tedious calculations, with a bit of luck, one obtains:

$$H_0(\mathcal{T}) \cong \mathbb{Z},\ H_1(\mathcal{T}) \cong \mathbb{Z} \oplus \mathbb{Z},\ H_2(\mathcal{T}) \cong \mathbb{Z}$$

as well as

$$H_0(\mathcal{KB}) \cong \mathbb{Z},\ H_1(\mathcal{KB}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

and all other modules are trivial. △

Theorem 3.1.12. *For every $k \in \mathbb{Z}$,*

$$H_k \colon \mathcal{SC} \to \mathbb{Z} - \mathcal{M}od$$

is a covariant functor. In particular,

$$\mathcal{K} \cong \mathcal{L} \;\Rightarrow\; H_k(\mathcal{K}) \cong H_k(\mathcal{L})$$

holds for all $k \in \mathbb{Z}$.

Proof. We only need to define the induced maps and prove functoriality. Let

$$\mathcal{K} \subseteq \mathcal{P}([m]) \text{ and } \mathcal{L} \subseteq \mathcal{P}([n])$$

be simplicial complexes and $\varphi \colon [m] \to [n]$ a morphism from $\mathcal{K}$ to $\mathcal{L}$. For $S \in \mathcal{K}$ with $\dim(S) = k$ we define

$$\varphi_k(S) := \begin{cases} \operatorname{sgn}\left(\varphi_{|S}\right) \cdot \varphi(S) & \text{if} \dim(\varphi(S)) = k \\ 0 & \text{otherwise.} \end{cases}$$

This gives us morphisms

$$\varphi_k \colon C_k(\mathcal{K}) \to C_k(\mathcal{L})$$

which make the following diagram commutative (Exercise 24):

$$\begin{tikzcd} C_{k+1}(\mathcal{K}) \arrow[r, "\partial_{k+1}"] \arrow[d, "\varphi_{k+1}"] & C_k(\mathcal{K}) \arrow[r, "\partial_k"] \arrow[d, "\varphi_k"] & C_{k-1}(\mathcal{K}) \arrow[d, "\varphi_{k-1}"] \\ C_{k+1}(\mathcal{L}) \arrow[r, "\partial_{k+1}"] & C_k(\mathcal{L}) \arrow[r, "\partial_k"] & C_{k-1}(\mathcal{L}) \end{tikzcd} .$$

This immediately implies

$$\varphi_k(Z_k(\mathcal{K})) \subseteq Z_k(\mathcal{L}) \text{ and } \varphi_k(B_k(\mathcal{K})) \subseteq B_k(\mathcal{L}).$$

So φ_k induces a well-defined morphism

$$H_k(\varphi) \colon H_k(\mathcal{K}) \to H_k(\mathcal{L}).$$

The functoriality properties are checked easily. □

Remark/Example 3.1.13. The simplicial complexes $\mathcal{T}$ and $\mathcal{KB}$ have different homology modules, so they are not isomorphic. But note that we have not shown that the torus and the Klein bottle are not homeomorphic.

3.2 Axiomatic Homology Theory

We now want to develop a homology theory not only for abstract simplicial complexes, but for arbitrary topological spaces. Such a theory can be constructed explicitly; we'll do that later. However, unlike simplicial homology, homology modules usually cannot be computed directly. Therefore, certain properties of homology are used to reduce the computation to simpler spaces. We will therefore introduce homology theory axiomatically, using its properties, and prove many strong statements from its existence alone. Of course, this existence will also have to be demonstrated later.

Definition 3.2.1. A **homology theory** h_* is a sequence $(h_k)_{k\in\mathbb{Z}}$ of covariant functors

$$h_k\colon \mathcal{T}op \to R - \mathcal{M}od$$

where R is a commutative ring. In addition, for all X with an open cover $X = X_1 \cup X_2$ there are maps

$$\partial_k\colon h_k(X_1 \cup X_2) \to h_{k-1}(X_1 \cap X_2)$$

which make the following diagram commutative for all $f\colon (X_1 \cup X_2) \to (Y_1 \cup Y_2)$ with $f(X_i) \subseteq Y_i$:

$$\begin{tikzcd} h_k(X_1 \cup X_2) \arrow[r, "\partial_k"] \arrow[d, "h_k(f)"] & h_{k-1}(X_1 \cap X_2) \arrow[d, "h_{k-1}(f)"] \\ h_k(Y_1 \cup Y_2) \arrow[r, "\partial_k"] & h_{k-1}(Y_1 \cap Y_2). \end{tikzcd}$$

The following properties should apply:

(H) All h_k are homotopy invariant.

(MV) For every open cover $X = X_1 \cup X_2$, the following (infinite) *Mayer-Vietoris sequence* of R-modules is exact:

$$\cdots \to h_k(X_1 \cap X_2) \stackrel{(h_k(i_1),h_k(i_2))}{\to} h_k(X_1) \oplus h_k(X_2) \stackrel{h_k(j_1)+h_k(j_2)}{\to} h_k(X) \stackrel{\partial_k}{\to} h_{k-1}(X_1 \cap X_2) \to \cdots$$

where the mappings are from the canonical diagram (which is a pushout):

$$\begin{tikzcd} & X_1 \arrow[dr, "j_1"] & \\ X_1 \cap X_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X \\ & X_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

(N) We have $h_k(\emptyset) = \{0\}$ for all $k \in \mathbb{Z}$.

If the following axiom is also satisfied, then h_* is called an **ordinary homology theory**:

(D) We have $h_k(\{*\}) = \{0\}$ for all $k \neq 0$. △

Remark 3.2.2. Let h_* be a homology theory.
(i) For all $k \in \mathbb{Z}$, $X \simeq Y \Rightarrow h_k(X) \cong h_k(Y)$ holds. This follows, as usual, directly from the required homotopy invariance of the functors h_k.
(ii) For all $k \in \mathbb{Z}$ we have $h_k(X \sqcup Y) \cong h_k(X) \oplus h_k(Y)$. This is obtained from the exact Mayer-Vietoris sequence for the open cover $X \sqcup Y = X \cup Y$:

$$\{0\} = h_k(X \cap Y) \to h_k(X) \oplus h_k(Y) \to h_k(X \sqcup Y) \to h_{k-1}(X \cap Y) = \{0\}.$$

Since the two modules on the very left and right are trivial, the map in the middle must be both injective and surjective, i.e. an isomorphism.
(iii) For $f\colon X = X_1 \cup X_2 \to Y = Y_1 \cup Y_2$ with $f(X_i) \subseteq Y_i$, the entire following diagram is obviously commutative:

$$\begin{tikzcd} h_k(X_1 \cap X_2) \arrow[r] \arrow[d, "h_k(f)"] & h_k(X_1) \oplus h_k(X_2) \arrow[r] \arrow[d, "h_k(f) \oplus h_k(f)"] & h_k(X_1 \cup X_2) \arrow[r, "\partial_k"] \arrow[d, "h_k(f)"] & h_{k-1}(X_1 \cap X_2) \arrow[d, "h_{k-1}(f)"] \\ h_k(Y_1 \cap Y_2) \arrow[r] & h_k(Y_1) \oplus h_k(Y_2) \arrow[r] & h_k(Y_1 \cup Y_2) \arrow[r, "\partial_k"] & h_{k-1}(Y_1 \cap Y_2). \end{tikzcd}$$

For the first two blocks it simply follows from the functor property, for the last block we have explicitly required it. △

When we want to compute homology modules, we usually use the exact Mayer-Vietoris sequence. It's best if the trivial module appears in many places. To achieve this, we often first proceed to the *reduced homology* and only then compute the actual homology.

Definition 3.2.3. For $X \neq \emptyset$ and $p\colon X \to \{*\}$ we call

$$\tilde{h}_k(X) := \ker\left(h_k(X) \overset{h_k(p)}{\twoheadrightarrow} h_k(\{*\})\right)$$

the **reduced homology** corresponding to h. $\triangle$

Remark 3.2.4. (i) The projection $p\colon X \to \{*\}$ has a right inverse, so the map $h_k(p)$ is surjective.
(ii) We have $\tilde{h}_k(\{*\}) = \{0\}$ for *all* $k \in \mathbb{Z}$.
(iii) $\tilde{h}_k$ is also a functor. Applying h_k to the left commutative diagram, one sees that $h_k(f)$ maps the kernel of $h_k(p)$ to the kernel of $h_k(p)$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "p"] \\
\{*\} \arrow[r] & \{*\}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
h_k(X) \arrow[r, "h_k(f)"] \arrow[d, "h_k(p)"] & h_k(Y) \arrow[d, "h_k(p)"] \\
h_k(\{*\}) \arrow[r] & h_k(\{*\}).
\end{tikzcd}$$

By restriction, one obtains the morphism

$$\tilde{h}_k(f)\colon \tilde{h}_k(X) \to \tilde{h}_k(Y).$$

This also shows homotopy invariance.
(iv) If $X = X_1 \cup X_2$ with $X_1 \cap X_2 \neq \emptyset$, then the Mayer-Vietoris sequence is also exact for $\tilde{h}$. To do this, consider the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & \tilde{h}_k(X_1 \cap X_2) \arrow[r] \arrow[d, hook] & \tilde{h}_k(X_1) \oplus \tilde{h}_k(X_2) \arrow[r] \arrow[d, hook] & \tilde{h}_k(X) \arrow[r] \arrow[d, hook] & \tilde{h}_{k-1}(X_1 \cap X_2) \arrow[r] \arrow[d, hook] & {} \\
{} \arrow[r] & h_k(X_1 \cap X_2) \arrow[r] \arrow[d] & h_k(X_1) \oplus h_k(X_2) \arrow[r] \arrow[d] & h_k(X) \arrow[r] \arrow[d] & h_{k-1}(X_1 \cap X_2) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & h_k(\{*\}) \arrow[r] \arrow[d] & h_k(\{*\}) \oplus h_k(\{*\}) \arrow[r] \arrow[d] & h_k(\{*\}) \arrow[r] \arrow[d] & h_{k-1}(\{*\}) \arrow[r] \arrow[d] & {} \\
& 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Here the bottom two lines are exact, as are all columns. This implies that also the first line is exact (Exercise 26).

(v) For nonempty X we have

$$h_k(X) \cong \tilde{h}_k(X) \oplus h_k(\{*\}).$$

To see this, consider the exact sequence

$$0 \to \tilde{h}_k(X) \hookrightarrow h_k(X) \overset{h_k(p)}{\twoheadrightarrow} h_k(\{*\}) \to 0,$$

in which $h_k(p)$ has a right inverse. This yields the statement (Exercise 27). In particular, for an ordinary homology theory h_* with $M := h_0(\{*\})$ we always have

$$h_k(X) = \begin{cases} \tilde{h}_k(X) \oplus M & k = 0 \\ \tilde{h}_k(X) & \text{otherwise.} \end{cases}$$

Furthermore we have

$$\tilde{h}_k(X \sqcup \{*\}) \cong h_k(X),$$

see Exercise 28. △

Proposition 3.2.5. *Let h_* be an ordinary homology theory with $M := h_0(\{*\})$. Then for all $n \geqslant 1$ we have*

$$h_k(D^n) = \begin{cases} M & k = 0 \\ \{0\} & \textit{otherwise} \end{cases}$$

and

$$h_k(S^n) = \begin{cases} M & k = 0, n \\ \{0\} & \textit{otherwise.} \end{cases}$$

For the torus and Klein bottle we have

$$\begin{aligned} h_0(\mathcal{T}) &= M,\ h_1(\mathcal{T}) = M \oplus M,\ h_2(\mathcal{T}) = M \\ h_k(\mathcal{T}) &= \{0\} \textit{ otherwise} \\ h_0(\mathcal{KB}) &= M,\ h_1(\mathcal{KB}) = M \oplus M/2M \\ h_2(\mathcal{KB}) &= \{m \in M \mid 2m = 0\} \\ h_k(\mathcal{KB}) &= \{0\} \textit{ otherwise.} \end{aligned}$$

Proof. Since $D^n \simeq \{*\}$, the first statement follows directly from homotopy invariance. We now cover the sphere S^n openly by an upper and a lower hemisphere, which slightly overlap at the equator. Both hemispheres are homotopy equivalent to a point, the intersection to S^{n-1}. We therefore obtain the following exact Mayer-Vietoris sequence for $\tilde{h}_*$:

$$\underbrace{\tilde{h}_k(\{*\}) \oplus \tilde{h}_k(\{*\})}_{=\{0\}} \to \tilde{h}_k(S^n) \to \tilde{h}_{k-1}(S^{n-1}) \to \underbrace{\tilde{h}_{k-1}(\{*\}) \oplus \tilde{h}_{k-1}(\{*\})}_{=\{0\}}$$

This gives $\tilde{h}_k(S^n) \cong \tilde{h}_{k-1}(S^{n-1})$ and iteratively

$$\tilde{h}_k(S^n) \cong \tilde{h}_{k-n}(S^0) = h_{k-n}(\{*\}) = \begin{cases} M & k = n \\ \{0\} & \text{otherwise.} \end{cases}$$

By going from $\tilde{h}_*$ to h_*, the additional factor M arises for $k = 0$. Computing the homology modules of the torus and the Klein bottle is Exercise 30. □

Proposition 3.2.6. *Let $Y = X \cup_\varphi D^n$ be formed by attaching an n-cell to X. Then*

$$\tilde{h}_k(X) \cong \tilde{h}_k(Y) \text{ for } k \neq n, n-1.$$

Proof. Let $0 \in D^n$ be the center of the disk. We cover Y by the two open sets

$$Y_1 = Y \setminus \{0\} \simeq X \text{ and } Y_2 = Y \setminus X = \mathring{D}^n \simeq \{*\}.$$

We have

$$Y_1 \cap Y_2 = \mathring{D}^n \setminus \{0\} \simeq S^{n-1},$$

and thus obtain for arbitrary k the Mayer-Vietoris sequence

$$\tilde{h}_k(S^{n-1}) \to \underbrace{\tilde{h}_k(Y_1)}_{\cong \tilde{h}_k(X)} \oplus \underbrace{\tilde{h}_k(Y_2)}_{=\{0\}} \to \tilde{h}_k(Y) \to \tilde{h}_{k-1}(S^{n-1}).$$

For $k \neq n, n-1$ the module on the left and right is trivial, and this provides the desired isomorphism. □

In the following we see how h_0 generalizes the functor π_0:

Theorem 3.2.7. *Let R be a principal ideal domain and $h_*\colon \mathcal{T}op \to R-\mathcal{M}od$ an ordinary homology theory with $h_0(\{*\}) = R$. Then for every space X that is homotopy equivalent to a finite CW-complex, $h_0(X)$ is a free R-module with*

$$\dim_R h_0(X) = |\pi_0(X)|.$$

Proof. Note that for CW complexes, connected and path-connected components coincide, and thus X is the topological sum of its finitely many connected components.
We prove the statement by induction on the structure of the CW complex X. If X consists only of zero cells, then the statement follows from Remark 3.2.2 (ii). If an n-cell is attached to X for $n \geqslant 2$, then by Proposition 3.2.6 $\tilde{h}_0(X)$ and thus

also $h_0(X)$ do not change. On the other hand, the number of path-connected components cannot change either, since S^{n-1} is path-connected. So let's assume we attach a 1 cell to X. There are two possibilities.
1. Case: The number of path-connected components remains the same. We have thus attached both ends of D^1 to the same path-connected component. We now partition the new space Y again by removing an unattached point from D^1 for Y_1 and choosing $Y_2 = Y \setminus X$. Then $Y_1 \simeq X, Y_2 \simeq \{*\}$ and $Y_1 \cap Y_2 \simeq S^0$. We obtain

$$R = \tilde{h}_0(S^0) \to \tilde{h}_0(Y_1) \oplus \underbrace{\tilde{h}_0(Y_2)}_{=\{0\}} \to \tilde{h}_0(Y) \to \{0\}.$$

The map on the left, however, comes from the inclusion $Y_1 \cap Y_2 \hookrightarrow Y_1$, and this is homotopic to a constant map, since the attachment points lie in the same path-connected component. Constant maps factor through a point and thus induce the zero map in reduced homology. This gives us

$$\tilde{h}_0(X) \cong \tilde{h}_0(Y_1) \cong \tilde{h}_0(Y)$$

and hence also $h_0(X) \cong h_0(Y)$.
2nd Case: The number of path-connected components decreases by 1. We have thus connected two components X_1, X_2 (we can assume that X consists only of these two components). We now choose Y_1 to be precisely X_1 including the attached D^1, but without its other endpoint. Analogously, we define Y_2. Then $Y_1 \simeq X_1, Y_2 \simeq X_2$ and $Y_1 \cap Y_2 \simeq \{*\}$. Using the exact Mayer-Vietoris sequence we obtain

$$\tilde{h}_0(Y) \cong \tilde{h}_0(X_1) \oplus \tilde{h}_0(X_2).$$

We now have

$$\begin{aligned} h_0(Y) \oplus R &\cong \tilde{h}_0(Y) \oplus R \oplus R \\ &\cong \tilde{h}_0(X_1) \oplus R \oplus \tilde{h}_0(X_2) \oplus R \\ &\cong h_0(X_1) \oplus h_0(X_2) \\ &\cong R \oplus R, \end{aligned}$$

where we have used the induction hypothesis for X_1 and X_2. Using the elementary divisor theorem, $h_0(Y)$ is now also a free R-module with $\dim h_0(Y) + 1 = 2$, so $\dim h_0(Y) = 1$. □

Next we obtain a connection between π_1 and h_1:

Theorem 3.2.8. *Let (X, x_0) be a pointed space and h_* an ordinary homology theory with $h_0(\{*\}) = \mathbb{Z}$. A continuous path $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$ can be regarded as a map $\gamma\colon S^1 \to X$ with $\gamma(1) = x_0$. Because $h_1(S^1) \cong \mathbb{Z}$,*

$$h_1(\gamma)\colon h_1(S^1) \to h_1(X)$$

yields a well-defined group homomorphism

$$\begin{aligned} \pi_1(X, x_0) &\to h_1(X) \\ \gamma &\mapsto h_1(\gamma)(1). \end{aligned}$$

Proof. We know that homotopic paths $\gamma, \delta\colon S^1 \to X$ yield identical maps $h_1(\gamma) = h_1(\delta)$. The map is thus well-defined. We can realize a concatenation of paths γ, δ using the following diagram:

$$\begin{tikzcd} S^1 \arrow[r, "z"] \arrow[rr, bend right, "\gamma\delta"'] & S^1 \vee S^1 \arrow[r, "\gamma \vee \delta"] & X \end{tikzcd}$$

Here, z wraps the circle once around both copies of S^1, and $\gamma \vee \delta$ uses γ on the first copy of S^1, and δ on the second copy. By applying h_1 we obtain

$$\begin{tikzcd} \mathbb{Z} = h_1(S^1) \arrow[r, "h_1(z)"] \arrow[rr, bend right, "h_1(\gamma\delta)"'] & h_1(S^1 \vee S^1) \arrow[r, "h_1(\gamma \vee \delta)"] & h_1(X) \end{tikzcd}.$$

But

$$h_1(S^1 \vee S^1) \cong h_1(S^1) \oplus h_1(S^1)$$

and under this isomorphism, $h_1(z)$ corresponds to $(\mathrm{id}, \mathrm{id})$ and $h_1(\gamma \vee \delta)$ corresponds to $h_1(\gamma)+h_1(\delta)$. This results in the desired homomorphism property. $\square$

Remark/Example 3.2.9. (i) For $X = S^1$, one thus obtains the identity as an isomorphism between $\pi_1(S^1) = \mathbb{Z}$ and $h_1(S^1) = \mathbb{Z}$. The simple circular path γ generates $\pi_1(S^1)$ and, because $h_1(\gamma) = \mathrm{id}$, is mapped to 1 using the given homomorphism.
(ii) The homomorphism $\pi_1(X, x_0) \to h_1(X)$ cannot, in general, be injective. Homology groups are always abelian, while fundamental groups are generally not. $\triangle$

Theorem 3.2.10. (1) *From the existence of a homology theory with $h_k(\{*\}) \neq \{0\}$ for some k follows:*

(i) $S^{n-1} \subseteq D^n$ *is not a retract.*

(ii) *Brouwer's fixed point theorem: Every continuous map $f\colon D^n \to D^n$ has a fixed point.*

(2) *From the existence of an ordinary homology theory with $h_0(\{*\}) \neq \{0\}$ follows*

(i) $S^m \simeq S^n \Rightarrow m = n.$

(ii) $U \subseteq R^m$ *open,* $V \subseteq R^n$ *open,* $V \cong W \Rightarrow m = n.$

Proof. (1) (i): Suppose we have a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\iota"] \arrow[dr, "\mathrm{id}"'] & D^n \arrow[d, "r"] \\ & S^{n-1}. \end{tikzcd}$$

We apply $\tilde{h}_k$ to it and obtain

$$\begin{tikzcd} \tilde{h}_k(S^{n-1}) \arrow[r, "\tilde{h}_k(\iota)"] \arrow[dr, "\tilde{h}_k(\mathrm{id})=\mathrm{id}"'] & \tilde{h}_k(D^n) = \{0\} \arrow[d, "\tilde{h}_k(r)"] \\ & \tilde{h}_k(S^{n-1}). \end{tikzcd}$$

But

$$\tilde{h}_k(S^{n-1}) \cong \tilde{h}_{k-n+1}(S^0) \cong h_{k-n+1}(\{*\}) \neq \{0\}$$

for suitable k. This is a contradiction, since the identity does not factor through the trivial module.

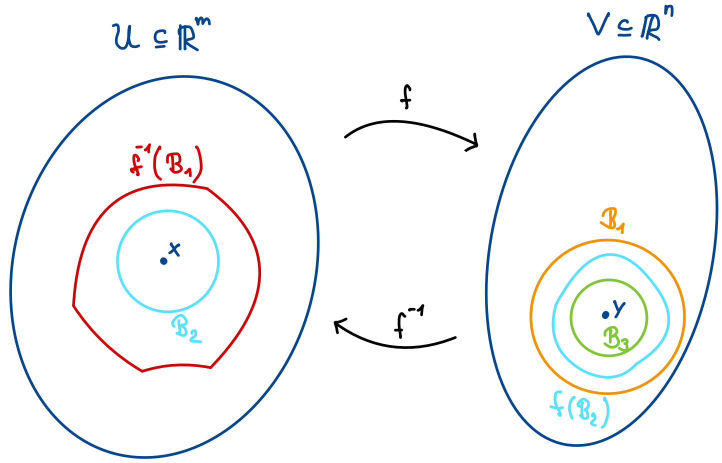
(ii): Brouwer's fixed point theorem is proved in the same way as in Corollary 2.3.4; from a map without a fixed point one could directly construct a retraction $r\colon D^n \to S^{n-1}$.

(2) (i): From $S^m \simeq S^n$ it follows

$$\{0\} \neq h_0(\{*\}) \cong \tilde{h}_m(S^m) \cong \tilde{h}_m(S^n) \cong h_{m-n}(\{*\})$$

and from this $m = n$, since h_* is ordinary.

(ii): Let $f\colon U \to V$ be a homeomorphism and $x \in U, y = f(x) \in V$. Choose an open ball B_1 in V around y, an open ball B_2 in U around x with $B_2 \subseteq f^{-1}(B_1)$ and finally an open ball B_3 in V around y with $B_3 \subseteq f(B_2)$:



We obtain the following commutative diagram, in which ι is a homotopy equivalence:

$$\begin{tikzcd} B_3 \setminus \{y\} \arrow[r, "f^{-1}"] \arrow[rr, bend right, "\iota"'] & B_2 \setminus \{x\} \arrow[r, "f"] & B_1 \setminus \{y\}. \end{tikzcd}$$

We then apply $\tilde{h}_{n-1}$:

$$\begin{tikzcd} \tilde{h}_{n-1}(B_3 \setminus \{y\}) \arrow[r] \arrow[rr, bend right, "\cong"'] & \tilde{h}_{n-1}(B_2 \setminus \{x\}) \arrow[r] & \tilde{h}_{n-1}(B_1 \setminus \{y\}). \end{tikzcd}$$

But $B_3 \setminus \{y\} \cong B_1 \setminus \{y\} \simeq S^{n-1}$, so

$$\tilde{h}_{n-1}(B_3 \setminus \{y\}) \cong \tilde{h}_{n-1}(B_1 \setminus \{y\}) \cong \tilde{h}_{n-1}(S^{n-1}) \cong h_0(\{*\}) \neq \{0\}.$$

Because of $B_2 \setminus \{x\} \cong S^{m-1}$ we obtain

$$\{0\} \neq \tilde{h}_{n-1}(B_2 \setminus \{x\}) \cong \tilde{h}_{n-1}(S^{m-1}) \cong h_{n-m}(\{*\})$$

and from this finally $n = m$. □

Definition 3.2.11. Let h_* be an ordinary homology theory with $h_0(\{*\}) = \mathbb{Z}$. Every continuous map $f\colon S^n \to S^n$ induces a group homomorphism

$$h_n(f)\colon h_n(S^n) \cong \mathbb{Z} \to \mathbb{Z} \cong h_n(S^n).$$

We define the **degree** of f as $\deg(f) := h_n(f)(1)$. △

Remark/Example 3.2.12. (i) We have $\deg(\mathrm{id}) = 1$.
(ii) From the functorial properties of h_n it immediately follows that

$$\deg(f \circ g) = \deg(f) \cdot \deg(g).$$

(iii) Homotopic maps have the same degree. △

Lemma 3.2.13. *The reflection*

$$\begin{aligned} s\colon S^n &\to S^n \\ (x_1, \dots, x_n, x_{n+1}) &\mapsto (x_1, \dots, x_n, -x_{n+1}) \end{aligned}$$

has degree -1.

Proof. We consider the following sequence of maps

$$S^n \xrightarrow{g} S^n \vee S^n \xrightarrow{\mathrm{id} \vee s} S^n,$$

where g simply contracts the equator to a point. This sequence is homotopic to a constant mapping. Alternatively, one can first fold the lower hemisphere into the upper one (which is then already homotopic to a constant mapping) and then contract the lower one. In the n-th (reduced) homology we obtain

$$0\colon \mathbb{Z} \xrightarrow{\mathrm{id} \oplus \mathrm{id}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{id} + \tilde{h}_n(s)} \mathbb{Z}$$

and from this it follows that $h_n(s) = \tilde{h}_n(s) = -\mathrm{id}_{\mathbb{Z}}$. □

Example 3.2.14. The map

$$\begin{aligned} -\mathrm{id}\colon S^n &\to S^n \\ x &\mapsto -x \end{aligned}$$

has degree $(-1)^{n+1}$. It is the successive execution of $n + 1$ reflections. △

Corollary 3.2.15. *On* S^n*,* id *and* $-\mathrm{id}$ *are homotopic if and only if* n *is odd.*

Proof. Exercise 33. □

Definition 3.2.16. A **tangent vector field** on S^n is a continuous map $\nu\colon S^n \to \mathbb{R}^{n+1}$ with

$$\langle x, \nu(x) \rangle = 0$$

for all $x \in S^n$. △

The following theorem states that odd-dimensional hedgehogs cannot be combed continuously.

Theorem 3.2.17. *On S^n there exists a nowhere vanishing tangent vector field if and only if n is odd.*

Proof. Suppose $\nu\colon S^n \to \mathbb{R}^{n+1}$ is a tangent vector field with $\nu(x) \neq 0$ for all $x \in S^n$. Then we obtain a homotopy of id and $-\mathrm{id}$ on S^n by moving each point x along the circle given by $\nu(x)$ on S^n up to $-x$. By Corollary 3.2.15, this implies that n is odd.
For odd n, however, such a tangent vector field can be easily given explicitly by

$$\nu(x_1, x_2, \dots, x_n, x_{n+1}) := (x_2, -x_1, \dots, x_{n+1}, -x_n). \qquad \square$$

3.3 Construction of Singular Homology

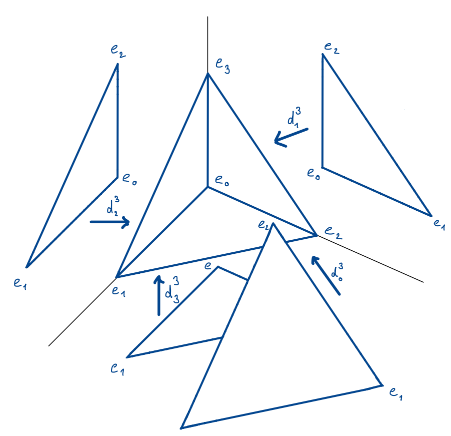
In this section, we outline the construction of an ordinary homology theory H_* with $H_0(\{*\}) = R$, for an arbitrary commutative ring R. Let

$$\Delta_n = \mathrm{conv}\{e_0 = 0, e_1, \dots, e_n\} \subseteq \mathbb{R}^n$$

be the n-dimensional unit simplex. Then for $i = 0, \dots, n$ there are canonical continuous embeddings

$$d_i^n\colon \Delta_{n-1} \hookrightarrow \Delta_n,$$

by simply mapping $\Delta_{n-1} = \mathrm{conv}\{e_0, \dots, e_{n-1}\}$ to $\mathrm{conv}\{e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n\}$ via an (index-order preserving) affine transformation.



For every topological space and every continuous map $\gamma\colon \Delta_n \to X$ one obtains continuous maps

$$\gamma^i := \gamma \circ d_i^n \colon \Delta_{n-1} \to X.$$

For $k \in \mathbb{Z}$ let

$$C_k(X) = \bigoplus_{\mathcal{T}op(\Delta_k, X)} R$$

be the R-module freely generated by all continuous functions $\gamma\colon \Delta_k \to X$, where we set $C_k(X) = \{0\}$ for $k < 0$. We also write elements of $C_k(X)$ as (finite) formal R-linear combinations of functions:

$$\sum_{\gamma\colon \Delta_k \to X} c_\gamma \cdot \gamma$$

with $c_\gamma \in R$. For $\gamma\colon \Delta_k \to X$ we set

$$\partial_k(\gamma) := \sum_{i=0}^{k} (-1)^i \gamma^i \in C_{k-1}(X)$$

and thus obtain morphisms $\partial_k\colon C_k(X) \to C_{k-1}(X)$ and a chain complex

$$\cdots \to C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \to \cdots$$

Again, one easily checks

$$\partial_k \circ \partial_{k+1} = 0$$

for all $k \in \mathbb{Z}$. Thus, for

$$B_k(X) := \operatorname{im}(\partial_{k+1}) \text{ and } Z_k(X) := \ker(\partial_k)$$

we always have $B_k(X) \subseteq Z_k(X)$ and we define the **k-th singular homology module** as

$$H_k(X) := Z_k(X)/B_k(X).$$

Every continuous map $f\colon X \to Y$ yields a map

$$\begin{aligned} f_k\colon \mathcal{T}op(\Delta_k, X) &\to \mathcal{T}op(\Delta_k, Y) \\ \gamma &\mapsto f \circ \gamma \end{aligned}$$

and therefore a morphism

$$f_k\colon C_k(X) \to C_k(Y)$$

which makes the following diagram commutative:

$$\begin{tikzcd} C_{k+1}(X) \arrow[r, "\partial_{k+1}"] \arrow[d, "f_{k+1}"] & C_k(X) \arrow[r, "\partial_k"] \arrow[d, "f_k"] & C_{k-1}(X) \arrow[d, "f_{k-1}"] \\ C_{k+1}(Y) \arrow[r, "\partial_{k+1}"] & C_k(Y) \arrow[r, "\partial_k"] & C_{k-1}(Y) \end{tikzcd} .$$

So we obtain a well-defined homomorphism

$$H_k(f)\colon H_k(X) \to H_k(Y)$$

and the functoriality properties are satisfied.
The axiom (N) is obviously satisfied, since there are no continuous maps $\Delta_k \to \emptyset$, and thus $C_k(\emptyset) = \{0\}$ already holds for all k.
Axiom (D) is also satisfied. For $k \geqslant 0$ there is exactly one continuous map $\gamma\colon \Delta_k \to \{*\}$ and the chain complex looks like this:

$$\cdots \to R \overset{\text{id}}{\to} R \overset{0}{\to} R \overset{\text{id}}{\to} R \overset{0}{\to} R \to \{0\} \to \cdots$$

Thus,

$$H_0(\{*\}) = R \text{ and } H_k(\{*\}) = \{0\} \text{ for } k \neq 0.$$

Homotopy invariance, the construction of the boundary map and axiom (MV) are much more difficult to prove, so we omit them here.

Here we illustrate the connection to π_0 and π_1. A continuous map $\gamma\colon \Delta_0 \to X$ simply corresponds to a point of X, so $C_0(X)$ is simply generated free of the points of X. A continuous map $\gamma\colon \Delta_1 \to X$ is simply a continuous path, and $\partial_1(\gamma) = \gamma(1) - \gamma(0)$. This directly shows the statement of Theorem 3.2.7 again. Furthermore, in the case $R = \mathbb{Z}$ one obtains the following well-defined group homomorphism (Exercise 36):

$$\begin{aligned} \pi_1(X, x_0) &\to H_1(X) \\ [\gamma] &\mapsto \overline{\gamma}. \end{aligned}$$

If X is path connected, it is surjective.

Finally, we want to prove the Borsuk-Ulam theorem, and for this we must return to the degree of a map. In the following proof, we resort to the explicit construction of simplicial homology. A map $f\colon S^n \to S^n$ is called **antipodal** if $f(-x) = -f(x)$ for all $x \in S^n$.

Theorem 3.3.1. *Every antipodal map $f\colon S^n \to S^n$ has odd degree. In particular, an antipodal map is never homotopic to a constant map.*

Proof sketch. We recall the definition of projective space as

$$\mathbb{P}^n := \mathbb{P}^n(\mathbb{R}) := S^n/(x \sim -x),$$

and the canonical projection $p\colon S^n \to \mathbb{P}^n$. With respect to p, every continuous map $\gamma\colon \Delta_k \to \mathbb{P}^n$ can be lifted to a continuous map $\tilde{\gamma}\colon \Delta_k \to S^n$ in exactly two ways. This can be seen with an argument as in the proof of Proposition 2.3.1. We denote these two lifts by γ_1, γ_2.
We now use singular homology with $R = \mathbb{Z}/2\mathbb{Z}$. We obtain a short exact sequence

$$0 \to C_k(\mathbb{P}^n) \stackrel{\tau_k}{\to} C_k(S^n) \stackrel{p_k}{\to} C_k(\mathbb{P}^n) \to 0,$$

with $\tau_k(\gamma) := \gamma_1 + \gamma_2$. Injectivity of τ is clear, surjectivity of p_k follows from the existence of a lift of γ, and exactness in the middle from the fact that we use $R = \mathbb{Z}/2\mathbb{Z}$. In the following diagram, everything commutes (the rows are chain complexes of singular homology, and the columns are exact):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{k+1}(\mathbb{P}^n) \arrow[r, "\partial_{k+1}"] \arrow[d, "\tau_{k+1}"'] & C_k(\mathbb{P}^n) \arrow[r, "\partial_k"] \arrow[d, "\tau_k"'] & C_{k-1}(\mathbb{P}^n) \arrow[r] \arrow[d, "\tau_{k-1}"'] & \cdots \\
\cdots \arrow[r] & C_{k+1}(S^n) \arrow[r, "\partial_{k+1}"] \arrow[d, "p_{k+1}"'] & C_k(S^n) \arrow[r, "\partial_k"] \arrow[d, "p_k"'] & C_{k-1}(S^n) \arrow[r] \arrow[d, "p_{k-1}"'] & \cdots \\
\cdots \arrow[r] & C_{k+1}(\mathbb{P}^n) \arrow[r, "\partial_{k+1}"] \arrow[d] & C_k(\mathbb{P}^n) \arrow[r, "\partial_k"] \arrow[d] & C_{k-1}(\mathbb{P}^n) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This gives a long exact sequence in homology:

$$\cdots \to H_k(\mathbb{P}^n) \stackrel{\tau_k}{\to} H_k(S^n) \stackrel{p_k}{\to} H_k(\mathbb{P}^n) \stackrel{d_k}{\to} H_{k-1}(\mathbb{P}^n) \to \cdots$$

The mapping d_k arises from the Snake Lemma (Exercise 37), the exactness is proven by diagram chase.

Now let $f\colon S^n \to S^n$ be antipodal. Then f induces a continuous map

$$\begin{aligned}\overline{f}\colon \mathbb{P}^n &\to \mathbb{P}^n\\ [x] &\mapsto [f(x)],\end{aligned}$$

which gives us the following commutative diagram in homology

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(\mathbb{P}^n) \arrow[r, "\tau_k"] \arrow[d, "H_k(\overline{f})"] & H_k(S^n) \arrow[r, "p_k"] \arrow[d, "H_k(f)"] & H_k(\mathbb{P}^n) \arrow[r, "d_k"] \arrow[d, "H_k(\overline{f})"] & H_{k-1}(\mathbb{P}^n) \arrow[r] \arrow[d, "H_{k-1}(\overline{f})"] & \cdots \\
\cdots \arrow[r] & H_k(\mathbb{P}^n) \arrow[r, "\tau_k"] & H_k(S^n) \arrow[r, "p_k"] & H_k(\mathbb{P}^n) \arrow[r, "d_k"] & H_{k-1}(\mathbb{P}^n) \arrow[r] & \cdots
\end{tikzcd}$$

The middle rectangle commutes because it already commutes at the level of continuous maps. The left rectangle commutes because $f\circ\gamma_1$ and $f\circ\gamma_2$ are precisely the two lifts of $\overline{f}\circ\gamma$. The commutativity of the right rectangle follows from a three-dimensional diagram hunt by adding a second layer to the diagram above, containing the maps induced by f and $\overline{f}$, respectively.
However, most homology modules of S^n are trivial, so we get the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{P}^n) \arrow[r] \arrow[d] & H_n(S^n) \arrow[r] \arrow[d] & H_n(\mathbb{P}^n) \arrow[r] \arrow[d] & H_{n-1}(\mathbb{P}^n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_n(\mathbb{P}^n) \arrow[r] & H_n(S^n) \arrow[r] & H_n(\mathbb{P}^n) \arrow[r] & H_{n-1}(\mathbb{P}^n) \arrow[r] & 0
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
0 \arrow[r] & H_k(\mathbb{P}^n) \arrow[r] \arrow[d] & H_{k-1}(\mathbb{P}^n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_k(\mathbb{P}^n) \arrow[r] & H_{k-1}(\mathbb{P}^n) \arrow[r] & 0
\end{tikzcd}$$

for $k = n-1, \dots, 2$, and

$$\begin{tikzcd}
0 \arrow[r] & H_1(\mathbb{P}^n) \arrow[r] \arrow[d] & H_0(\mathbb{P}^n) \arrow[r] \arrow[d] & H_0(S^n) \arrow[r] \arrow[d] & H_0(\mathbb{P}^n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1(\mathbb{P}^n) \arrow[r] & H_0(\mathbb{P}^n) \arrow[r] & H_0(S^n) \arrow[r] & H_0(\mathbb{P}^n) \arrow[r] & 0.
\end{tikzcd}$$

By Theorem 3.2.7, $H_0(\mathbb{P}^n) = \mathbb{Z}/2\mathbb{Z}$ holds. Likewise, $H_0(S^n) = H_n(S^n) = \mathbb{Z}/2\mathbb{Z}$ holds. This allows us to iteratively work our way up and obtain $H_k(\mathbb{P}^n) = \mathbb{Z}/2\mathbb{Z}$ for $k = 0, \dots, n$. All occurring mappings are either 0 or id. In the same way, one obtains iteratively that all $H_k(f)$ and $H_k(\overline{f})$ are isomorphisms, where this is clear for $k = 0$. In particular,

$$H_n(f)\colon H_n(S^n) \to H_n(S^n)$$

is an isomorphism, i.e., the identity on $\mathbb{Z}/2\mathbb{Z}$. However, $H_k(f)$ arises simply from the reduction of all coefficients modulo 2 of the corresponding construction with $R = \mathbb{Z}$. Thus, f must have an odd degree. □

Corollary 3.3.2 (Borsuk-Ulam)**.** *For every continuous map*

$$f\colon S^n \to \mathbb{R}^n$$

there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Proof. Suppose the statement is not true for some f. Define

$$\begin{aligned} h\colon S^n &\to S^{n-1} \\ x &\mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \end{aligned}$$

Here, h is obviously continuous and antipodal. If we embed S^{n-1} as the equator in S^n, we obtain an antipodal mapping of S^{n-1} into itself by composition with h, which, however, factors through S^n and therefore becomes trivial in $n-1$-th homology. It therefore has degree 0, and this contradicts Theorem 3.3.1. □

Corollary 3.3.3 (Tofu Sandwich Theorem)**.** *Let $A_1, \dots, A_n$ be bounded Lebesgue measurable subsets of $\mathbb{R}^n$. Then there exists an affine hyperplane that bisects all A_i simultaneously.*

Proof. For $v \in S^n \subseteq \mathbb{R}^{n+1}$ we define

$$P_v := \{a \in \mathbb{R}^n \mid \langle v, (a, 1)\rangle \geqslant 0\}.$$

Every affine half-space in $\mathbb{R}^n$ corresponds to exactly one such $v \in S^n$. Now we obtain a continuous map

$$\begin{aligned} \mathrm{vol}\colon S^n &\to \mathbb{R}^n \\ v &\mapsto (\mathrm{vol}(P_v \cap A_1)), \dots, \mathrm{vol}(P_v \cap A_n)). \end{aligned}$$

By the Borsuk-Ulam Theorem, there exists a $v \in S^n$ with $\mathrm{vol}(v) = \mathrm{vol}(-v)$. The hyperplane defined by v thus bisects all A_i simultaneously. □

Corollary 3.3.4 (Lusternik-Schnirelmann)**.** *For every cover of S^n with $n+1$ closed subsets, one of the subsets must contain a pair of antipodal points.*

Proof. Let a closed cover of S^n be given by subsets $A_0, \dots, A_n$. Suppose

$$A_i \cap -A_i = \emptyset$$

for $i = 1, \dots, n$. We choose continuous functions $f_i \colon S^n \to \mathbb{R}$ with

$$f_{i|_{A_i}} \equiv 1 \text{ and } f_{i|_{-A_i}} \equiv 0.$$

Then

$$f = (f_1, \dots, f_n) \colon S^n \to \mathbb{R}^n$$

is continuous, and thus there is a point $x \in S^n$ with $f_i(x) = f_i(-x)$ for $i = 1, \dots, n$. But this implies $x, -x \notin A_i$ for $i = 1, \dots, n$ and thus $x, -x \in A_0$. □

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Exercises

Exercise 1. Show:
(i) For metric spaces, the definition of continuity agrees with the well-known ε-δ–definition.
(ii) $[0, 1]$ is not homeomorphic to $\mathbb{R}$.
(iii) The map

$$\begin{aligned} f\colon [0, 2\pi) &\to S^1 \subseteq \mathbb{C} \\ r &\mapsto e^{ir} \end{aligned}$$

is continuous and bijective, but not a homeomorphism.

Exercise 2. Show that continuity is a local property: A map $f\colon X \to Y$ is continuous if and only if for every $x \in X$ there exists a neighborhood $U \subseteq X$ such that $f_{|U} : U \to Y$ is continuous.

Exercise 3. (i) Show that there exists a Hausdorff topology on the space $\mathcal{F}([0, 1], \mathbb{R})$ of functions, such that a sequence $(f_n)_n$ converges to f if and only if it converges pointwise on $[0, 1]$.
(ii) There is no metric on $\mathcal{F}([0, 1], \mathbb{R})$ that induces the topology from (i).

Exercise 4. We provide the set $X = \mathrm{Mat}_{m,n}(\mathbb{R})$ of real $m \times n$-matrices with the Euclidean topology and define for $A, B \in X$

$$A \sim B \;:\Leftrightarrow \exists S \in \mathrm{GL}_m(\mathbb{R}), T \in \mathrm{GL}_n(\mathbb{R})\colon\, SAT = B.$$

Describe $X/\sim$ with the quotient topology as explicitly as possible.

Exercise 5. Let X be a topological space, $X \times X$ be endowed with the product topology, and let $\Delta_X := \{(x, x) \mid x \in X\}$ be the diagonal in $X \times X$. Show that X is Hausdorff if and only if Δ_X is closed in $X \times X$.

Exercise 6. Let the following diagram be a pushout:

$$\begin{tikzcd} & X_1 \arrow[dr, "i_1"] & \\ X_0 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & X \\ & X_2 \arrow[ur, "i_2"'] & \end{tikzcd}$$

Show that if j_1 is a homeomorphism, then so is i_2.

Exercise 7. Prove Theorem 1.3.3.

Exercise 8. Prove the Heine-Borel Theorem.

Exercise 9. Prove Theorem 1.3.6.

Exercise 10. Show that if $A \subseteq X$ is connected, then so is $\overline{A}$. Is the converse also true?

Exercise 11. Let $X \subseteq \mathbb{R}^n$ be open. Show:

(i) Every path-connected component of X is open.

(ii) X is connected $\Leftrightarrow$ X is path connected.

(iii) Is the same statement also true for closed sets $X \subseteq \mathbb{R}^n$?

Exercise 12. Let X be a topological space. Show:

(i) The following statement defines an equivalence relation on X:

$$x \sim y \;:\Leftrightarrow\; \exists A \subseteq X \text{ connected with } x, y \in A.$$

(ii) The subsets of X that consist of mutually equivalent points are called connected components of X. Connected components are connected and closed, but not necessarily open. Is the quotient topology induced on the set of equivalence classes discrete?

(iii) Every continuous map $f\colon X \to Y$ induces a well-defined map between the respective sets of equivalence classes.

Exercise 13. Let $f_1, g_1 \colon X \to Y$ and $f_2, g_2 \colon Y \to Z$ be continuous maps with $f_1 \simeq g_1$, $f_2 \simeq g_2$. Then

$$(f_2 \circ f_1) \simeq (g_2 \circ g_1).$$

Exercise 14. Show that $\pi_1(X, x) = \{\epsilon\}$ for all $x \in X$, if $X \subseteq \mathbb{R}^n$ is star-shaped.

Exercise 15. Show $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Exercise 16. Show that $\pi_1(S^n) = \{\epsilon\}$ holds for $n \geqslant 2$.

Exercise 17. Try to adapt the computation of the fundamental group of S^1 so that you can directly calculate the fundamental group of the torus.

Hint: Find a suitable covering of the torus (defined as in Example 1.2.5 (ii)) onto which one can lift paths. Technical details need not be proven.

Exercise 18. Compute $\pi_1\left(\mathbb{R}^2 \setminus \{a, b\}\right)$, where $a \neq b$ are points from $\mathbb{R}^2$.

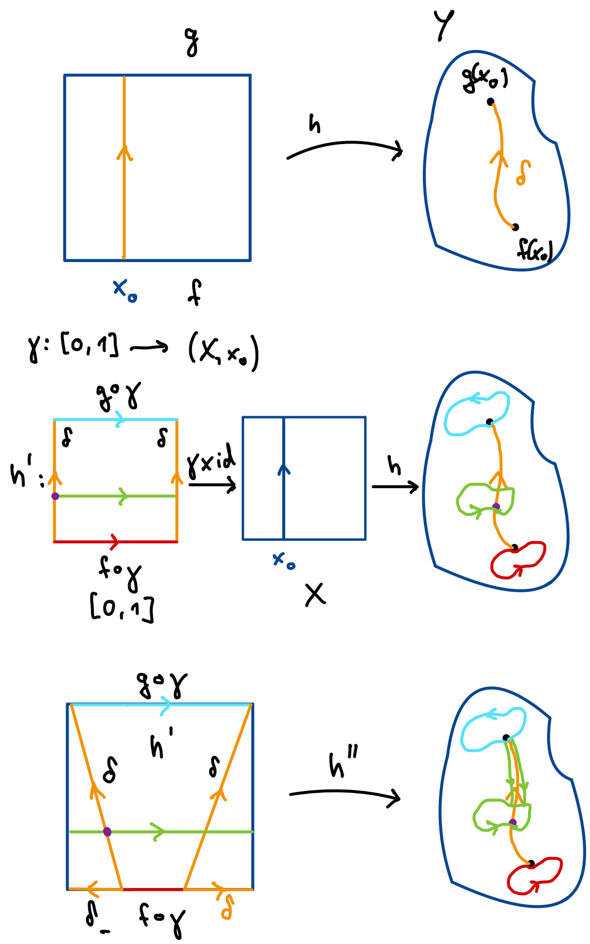
Hint: $\mathbb{R}^2 \setminus \{a, b\}$ is homotopy equivalent to the union of two circles touching at a point. Find a suitable covering of these circles. If necessary, google *Cayley graph of the free group*. Technical details need not be proven.

Exercise 19. Let $f, g \colon X \to Y$ be continuous and h a homotopy from f to g. Let $x_0 \in X$ be fixed and $\delta(\cdot) = h(x_0, \cdot)$ be the continuous path from $f(x_0)$ to $g(x_0)$ induced by h. We also denote by δ the isomorphism induced by δ from Proposition 2.2.6:

$$\begin{aligned} \delta \colon \pi_1(Y, f(x_0)) &\to \pi_1(Y, g(x_0)) \\ [\gamma] &\mapsto [\delta^- \gamma \delta]. \end{aligned}$$

Show that the following diagram is commutative:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\pi_1(f)"] \arrow[dr, "\pi_1(g)"'] & \pi_1(Y, f(x_0)) \arrow[d, "\delta"] \\ & \pi_1(Y, g(x_0)). \end{tikzcd}$$



Exercise 20. A continuous map $h\colon S^1 \to S^1$ is called antipodal, if $h(-x) = -h(x)$ for all $x \in S^1$. Show that an antipodal map h never induces the zero map between the fundamental groups. In particular, h is not homotopic to a constant map.

Exercise 21. (i) Show that there is no antipodal continuous map $h\colon S^2 \to S^1$.
(ii) Show that for every continuous map $f\colon S^2 \to \mathbb{R}^2$ there exists a point $x \in S^2$ such that $f(x) = f(-x)$.

Exercise 22. Prove Lemma 3.1.5, i.e. show that any two geometric realizations of an abstract simplicial complex yield homeomorphic spaces.

Exercise 23. Compute the homology modules of $\mathcal{F} = \mathcal{P}([n]) \setminus \{\emptyset\}$ and $\mathcal{S} = \{S \subseteq [n] \mid S \neq \emptyset, |S| \leqslant n\}$.

Exercise 24. Show that the diagram

$$\begin{tikzcd} C_k(\mathcal{K}) \arrow[r, "\partial_k"] \arrow[d, "\varphi_k"] & C_{k-1}(\mathcal{K}) \arrow[d, "\varphi_{k-1}"] \\ C_k(\mathcal{L}) \arrow[r, "\partial_k"] & C_{k-1}(\mathcal{L}) \end{tikzcd}$$

from the proof of Theorem 3.1.12 is commutative.

Exercise 25. Given a finite sequence of vector spaces and linear maps

$$0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n \to 0$$

with $f_{i+1} \circ f_i = 0$ for all i. We define

$$H_i = \ker(f_{i+1})/\operatorname{im}(f_i).$$

Show

$$\sum_i (-1)^i \dim H_i = \sum_i (-1)^i \dim V_i.$$

Exercise 26. Given a commutative diagram of R-modules and morphisms, in which all columns and the bottom two rows are exact:

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[d] \\ E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[d] \\ I \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & L \arrow[d] \\ 0 & 0 & 0 & 0 \end{tikzcd}$$

Show that the top row is then exact at C.

Exercise 27. Assume that in the following exact sequence, f has a right inverse:

$$0 \to A \to B \xrightarrow{f} C \to 0.$$

Show that $B \cong A \oplus C$ holds.

Exercise 28. Show $\tilde{h}_k(X \sqcup \{*\}) \cong h_k(X)$ for all $k \in \mathbb{Z}$.

Exercise 29. Let $S^n \vee S^n$ be the union of two spheres that touch at a point. Show

$$\tilde{h}_k(S^n \vee S^n) \cong \tilde{h}_k(S^n) \oplus \tilde{h}_k(S^n)$$

and state isomorphisms in both directions as explicitly as possible.

Exercise 30. Compute the homology modules of the torus and the Klein bottle for an ordinary homology theory h_*.

Exercise 31. Compute the homology modules of the space $S^2 \vee S^1 \vee S^1$ for an ordinary homology theory and compare with the torus. Are the two spaces homotopy equivalent?

Exercise 32. Let $\mathbb{H}^n := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_n \geqslant 0\}$ and $U, V \subseteq \mathbb{H}^n$ be open (in the subspace topology of $\mathbb{R}^n$). Let $f: U \to V$ be a homeomorphism and $p \in U \cap \partial\mathbb{H}^n$. Show that then $f(p) \in V \cap \partial\mathbb{H}^n$ must hold. You may assume the existence of a homology theory with $h_k(\{*\}) \neq 0$ for some k.

Exercise 33. Show that id and $-\mathrm{id}$ are homotopic on S^n if and only if n is odd.

Exercise 34. Let $f\colon S^{n-1} \to S^{n-1}$ be continuous. Show:

(i) If f is a homotopy equivalence, then $\deg(f) = \pm 1$.

(ii) From $\deg(f) \neq 0$ it follows that f is surjective.

(iii) If f has no fixed point, f is homotopic to $-\mathrm{id}$.

(iv) If f does not map any point to its antipode, f is homotopic to id.

Exercise 35. An *operation of a group G on a topological space X* is a group homomorphism

$$\varphi\colon G \to \mathrm{Homeo}(X),$$

where the set of all homeomorphisms of X is understood as a group w.r.t. composition of functions. The operation is called *free* if $\varphi(g)$ has no fixed point, for all $g \neq e$.

Show that for odd n only the groups $\{e\}$ and $\mathbb{Z}/2\mathbb{Z}$ can operate freely on S^{n-1}, and give an example of such a free operation.

Exercise 36. Show that the following map is a well-defined group homomorphism (in the case $R = \mathbb{Z}$):

$$\begin{aligned} \pi_1(X, x_0) &\to H_1(X) \\ [\gamma] &\mapsto \overline{\gamma}. \end{aligned}$$

If X is path connected, it is surjective.

Exercise 37. Given the following commutative diagram of R-modules with exact columns:

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ B \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ C \arrow[r, "h"] \arrow[d] & F \\ 0 & \end{tikzcd}$$

Show that there exists a canonical map $d\colon \ker(h) \to D/f(A)$ such that the following sequence is exact:

$$\ker(f) \to \ker(g) \to \ker(h) \to D/f(A) \to E/g(B) \to F/h(C).$$

Exercise 38. In the following commutative diagram of R-modules, the rows are exact:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'. \end{tikzcd}$$

Show that if g, i are isomorphisms, f is surjective and j is injective, then h is an isomorphism.

Exercise 39. Let $p_1, \dots, p_n \in \mathbb{R}[x_0, \dots, x_n]$ be homogeneous polynomials of odd degree. Then there exists a $0 \neq a \in \mathbb{R}^{n+1}$ with

$$0 = p_1(a) = \cdots = p_n(a).$$

In other words: The projective variety of $p_1, \dots, p_n$ has a real point.