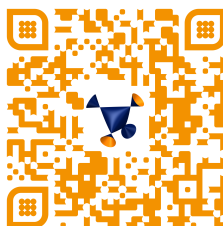
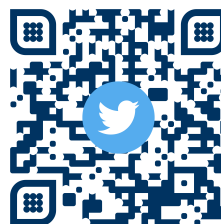
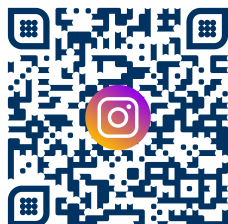


Introduction to Classical Algebraic Geometry

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Introduction: What is Algebraic Geometry?

Algebraic geometry emerged from the quest of solving arbitrary systems of polynomial equations over a field.

Systems of linear equations are considered exhaustively in linear algebra. The algorithm of Gauß provides a method to systematically compute all solutions to such a system, by parameterizing them with only a finite amount of data. This is also due to the fact that the space of all solutions, an affine space, is geometrically easy to understand. The algebra behind all of this is the theory of vector spaces and linear maps, which is quite well understood.

Passing to systems of *non-linear polynomial equations* makes everything much more complicated. In general there is no explicit algorithm producing all solutions anymore, and the geometry of the set of solutions is also much more complicated. To simplify things, one thus only searches for solutions in an algebraically closed field. This avoids problems that would otherwise even arise in a one-dimensional setup, as is well-known from the algebra courses. Further, instead of solving systems completely, we must restrict ourselves to classifying the solution sets up to some suitable notion of isomorphism. Given an unknown system of equations, one can then at least try to reduce it isomorphically to a system that is already understood. This approach culminates in the classification of all solution sets via their associated *affine coordinate rings*. It completely translates the problem to algebra, where it is often easier to solve. This will be done in Section 1 of these lecture notes.

More unexpected irregularities can arise when solving systems of polynomial equations. For example, already a system of two equations in many variables can be unsolvable, although one would expect each equation to reduce the dimension of the solution set by at most one. This already happens in linear algebra when considering non-homogeneous systems, but not for homogeneous systems. In

general such phenomena can be avoided by homogenizing the problem, i.e. by passing from affine to *projective space*. When adding points at infinity, the sets of solutions suddenly behave much more regular. We will do this in Section 3.

On the other hand, the classification of solution sets up to isomorphism becomes more complicated this way. The approach via affine coordinate rings does not admit a direct generalization. One thus weakens the notion of isomorphism to some isomorphism *almost everywhere*. This is called *birational equivalence*. Solving one system completely then leads to an almost complete solution for any birationally equivalent system. For birational equivalence there again exists a purely algebraic classification, via *function fields* of systems. We will deal with this in Section 4

The name *algebraic geometry* comes from the fact that we will always switch between a geometric view on the solution sets, and an algebraic view on objects derived from the system of equations. Some problems can be solved easier on one side, some on the other. We will see many of such examples in the course.

In Section 2 we consider algorithmic aspects. For example, given an explicit system of polynomial equations, we would like to decide whether the system is solvable or not. Since the considered fields are always infinite, we cannot just search through all possible points until we find a solution. However, there do exist methods to answer such questions computationally, even without numerical errors. It's the theory of *Gröbner bases* that provides such methods, which are also implemented in many computer algebra systems.

The literature on algebraic geometry is almost uncountable, so we just cite a short selection here. An elementary approach as in these lecture notes can be found in the books of Harris [6], Hulek [8], Shafarevich [10, 11] (and Fulton [5] for curves). Hartshorne [7] contains a comprehensive account of the modern theory of schemes, Eisenbud & Harris [4] provide a slightly easier introduction. For results on commutative algebra we recommend Atiyah & Macdonald [1], Eisenbud [3], and Lang [9]. The algorithmic aspects are for example described in Becker & Weispfenning [2].

These lecture notes are mostly based on parts of the books of Shafarevich, and unpublished lecture notes of Claus Scheiderer at the University of Konstanz. However, I am responsible for errors, and I am more than happy for hints on such. I thank Martin Berger for error corrections in a first version of this script, and Tom

Drescher for providing many of the exercises, and for his help in translating the initial German version into English. I also thank Daniel Scharler for providing many interesting exercises on applications of algebraic geometry in kinematics.

Chapter 1

Affine Varieties

1.1 Reminder from Algebra

All rings appearing in these lecture notes are *commutative* and have a *multiplicative identity element* 1. Most of the time we denote rings by R or S . Ideals of rings are usually denoted by I or J .

Definition 1.1.1. Let $I \subseteq R$ be an ideal.

(i) The following set is also an ideal of R (Exercise 2), called the **radical of I** :

$$\sqrt{I} := \{a \in R \mid \exists n \in \mathbb{N} : a^n \in I\}.$$

If $I = \sqrt{I}$, then I is called a **radical ideal**.

(ii) An element $a \in R$ is called **nilpotent** if $a^n = 0$ holds for some $n \in \mathbb{N}$. The ideal

$$\text{Nil}(R) := \sqrt{(0)}$$

of all nilpotent elements is called **nilradical of R** .

(iii) R is called **reduced** if $\text{Nil}(R) = (0)$, i.e. if there are no nilpotent elements except 0. △

Remark 1.1.2. Obviously we have $I \subseteq \sqrt{I}$ for all ideals I . In general the inclusion is proper. For example let $R = \mathbb{Z}$ and $I = (n)$ with the prime decomposition $n = p_1^{e_1} \cdots p_r^{e_r}$. Then $\sqrt{I} = (p_1 \cdots p_r)$. △

Lemma 1.1.3. Let $I, J \subseteq R$ be ideals. Then

$$(i) \quad \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

$$(ii) \sqrt{I} = R \Leftrightarrow I = R.$$

(iii) If $I \subseteq J$, then $\sqrt{J/I} = \sqrt{J}/I$ in the quotient ring R/I .

Proof. Exercise 2. □

As usual we denote with $\text{Spec}(R)$ the set of all prime ideals of R .

Theorem 1.1.4. For every ideal $I \subseteq R$ we have

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \text{Spec}(R) \\ I \subseteq \mathfrak{p}}} \mathfrak{p}.$$

Proof. " \subseteq ": Let $s \in \sqrt{I}$, i.e. $s^n \in I$ for some $n \in \mathbb{N}$. For every prime ideal \mathfrak{p} with $I \subseteq \mathfrak{p}$ the prime ideal property yields that $s^n \in \mathfrak{p}$ implies $s \in \mathfrak{p}$.

" \supseteq ": Let $s \in R \setminus \sqrt{I}$, i.e. $s^n \notin I$ for all $n \in \mathbb{N}$. Consider the multiplicative set $S = \{1, s, s^2, \dots\}$ and the corresponding localization $R_s := S^{-1}R$ as well as the natural homomorphism

$$\begin{aligned} \varphi: R &\rightarrow R_s \\ a &\mapsto a/1. \end{aligned}$$

We have that $1 \notin IR_s$, otherwise $1/1 = a/s^n$ would hold for some $a \in I, n \in \mathbb{N}$, and this implies $s^n \in I$ for some n , a contradiction.

Hence there is a maximal ideal \mathfrak{m} in R_s over IR_s , and $\mathfrak{p} := \varphi^{-1}(\mathfrak{m})$ is then a prime ideal in R with $I \subseteq \mathfrak{p}$. Because $s/1$ is invertible in R_s , this implies $s \notin \mathfrak{p}$. □

Corollary 1.1.5. $\text{Nil}(R)$ is the intersection of all prime ideals in R .

Recall that a ring R is called **noetherian**, if every ideal in R is finitely generated.

Theorem 1.1.6. If R is a noetherian ring, then so is $R[t]$.

Proof. Assume $I \subseteq R[t]$ is an ideal that is not finitely generated. Iteratively we choose $p_1, p_2, \dots \in I$ such that p_{n+1} has minimal degree in $I \setminus (p_1, \dots, p_n)$. For $d_n = \deg(p_n)$ we have $d_1 \leq d_2 \leq \dots$. Now let $a_n \in R$ be the leading coefficient of the polynomial p_n . Consider the ideal

$$J = (a_n \mid n \in \mathbb{N}) \subseteq R.$$

Since J is finitely generated by the assumption on R , there exists an identity

$$a_{m+1} = \sum_{i=1}^m b_i a_i$$

with $b_i \in R$. Define

$$g := p_{m+1} - \sum_{i=1}^m b_i p_i t^{d_{m+1}-d_i}.$$

By our construction we have $\deg(g) < \deg(p_{m+1})$, since the leading coefficients cancel. On the other hand, $p_{m+1} \in I \setminus (p_1, \dots, p_m)$ implies

$$g \in I \setminus (p_1, \dots, p_m).$$

This contradicts the choice of p_{m+1} . \square

Corollary 1.1.7 (Hilbert's Basis Theorem). *Let k be a field. Then $k[x]$ is noetherian.*

Proof. A field only has the two ideals (0) and (1) , both are finitely generated. Now apply Theorem 1.1.6 iteratively on the adjunction of each individual variable. \square

The most important fundamental theorem in classical algebraic geometry is certainly Hilbert's Nullstellensatz. We are going to prove it in its field-theoretic form first and interpret it in a geometric fashion afterwards. Before we can prove it however, we have to study the so called *integrality* of ring extensions. This is a variant form of the notion of algebraic field extensions that is specifically suited for rings.

Definition 1.1.8. Let $R \subseteq S$ be a ring extension.

(i) An element $b \in S$ is called **integral over** R if there are $a_0, \dots, a_{n-1} \in R$ such that

$$a_0 + a_1 b + \dots + a_{n-1} b^{n-1} + b^n = 0.$$

Such an equation is called **integrality equation** for b over R .

(ii) S is called **integral over** R if every element $b \in S$ is integral over R . \triangle

Remark 1.1.9. (i) The important detail for the integrality of b is that the corresponding integrality equation must be *monic*. If R is a field, then obviously every nontrivial equation can be normalized. Hence, in this case the integral elements in S over R are just the algebraic elements.

(ii) For $\mathbb{Z} \subseteq \mathbb{Q}$ the only integral elements over \mathbb{Z} in \mathbb{Q} are the elements of \mathbb{Z} itself. More generally this is true for the inclusion $R \subseteq K$ of an unique factorization domain in its field of fractions (Exercise 5).

(iii) For $R \subseteq S$ and $b_1, \dots, b_m \in S$ we define $R[b_1, \dots, b_m]$ as the subring of S generated by b_1, \dots, b_m and R , i.e.

$$R[b_1, \dots, b_m] = \left\{ \sum_{e \in \mathbb{N}^m} a_e b_1^{e_1} \cdots b_m^{e_m} \mid a_e \in R \right\}.$$

If we regard S as R -module, then $R[b_1, \dots, b_m]$ is a submodule, and in particular an R -module by itself. \triangle

Theorem 1.1.10. *Let $R \subseteq S$ be a ring extension and $b_1, \dots, b_m \in S$. Then the following are equivalent:*

- (i) b_1, \dots, b_m are integral over R .
- (ii) $R[b_1, \dots, b_m]$ is finitely generated as R -module.
- (iii) $R[b_1, \dots, b_m]$ is integral over R .

Proof. (i) \Rightarrow (ii): By solving the integrality equation of b_1 for b_1^n we obtain

$$b_1^n = - (a_{n-1} b_1^{n-1} + \cdots + a_0)$$

for certain $a_i \in R$. So we can replace the n th power of b_1 with lower powers of b_1 and coefficients in R . By doing the analogue procedure with b_i we see that $R[b_1, \dots, b_m]$ is generated by finitely many products $b_1^{e_1} \cdots b_m^{e_m}$.

(ii) \Rightarrow (iii): Finitely many elements $1 = c_1, \dots, c_n$ generate the R -module $M := R[b_1, \dots, b_m]$. Now let $c \in M$ be arbitrary. Since M is also a ring, we have $c \cdot c_i \in M$ and thus there are $a_{ij} \in R$ such that

$$c \cdot c_i = \sum_{j=1}^n a_{ij} c_j.$$

For the matrix

$$A = (a_{ij})_{i,j} \in \text{Mat}_n(R)$$

we then have

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

So $(c_1, \dots, c_n)^t$ is in the kernel of

$$N := cI_n - A.$$

Since

$$\text{adj}(N) \cdot N = \det(N) \cdot I_n$$

we obtain

$$\det(N) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0.$$

Now $c_1 = 1$ implies $\det(N) = 0$. From the Leibniz formula for determinants we see

$$\det(N) = c^n + a_{n-1}c^{n-1} + \dots + a_m$$

for certain $a_i \in R$. This yields an integrality equation of c over R .

(iii) \Rightarrow (i) is trivial. □

1.2 Affine Algebraic Varieties

From now on let k always be an arbitrary field, and K an *algebraically closed extension field* of k . For example we can choose $K = \bar{k}$ to be the algebraic closure of k . But K can also be bigger, e.g. $k = \mathbb{Q}$ and $K = \mathbb{C}$. Essential will only be that K is algebraically closed! The field k is also called **coefficient field** and K is also called **coordinate field**. With \underline{x} we denote the n -tuple of variables (x_1, \dots, x_n) .

Definition 1.2.1. (i) Let $P \subseteq k[\underline{x}]$ be a set of polynomials. We define

$$\mathcal{V}(P) = \{a \in K^n \mid p(a) = 0 \forall p \in P\}$$

and call $\mathcal{V}(P)$ the **affine variety defined by P** . It is the solution set (over K) of the system of polynomial equations defined by P .

(ii) A subset $V \subseteq K^n$ is called **affine k -variety** if $V = \mathcal{V}(P)$ for a set $P \subseteq k[\underline{x}]$. Affine k -varieties are thus the solutions sets for systems of polynomial equations (with coefficients in k).

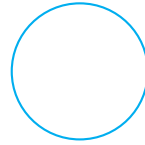
(iii) A **hypersurface** is a variety defined by a single polynomial, that is a variety of the form $\mathcal{V}(p)$ for some $p \in k[\underline{x}]$.

(iv) If $W \subseteq V$ are affine k -varieties, then W is called a **subvariety** of V .

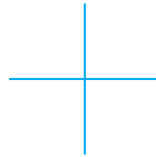
(v) $\mathbb{A}^n := K^n$ is called the **n -dimensional affine space**. △

Example 1.2.2. Notice: The following figures only show the *real points* of the respective affine variety in \mathbb{A}^2 . However as is well known, \mathbb{R} is not algebraically closed and we actually should consider the varieties over $K = \mathbb{C}$ for instance. Due to the dimension this is graphically hardly possible. But the real image usually (not always!) gives a good impression on the variety.

(i) Let $P = \{1 - x_1^2 - x_2^2\} \subseteq \mathbb{Q}[x_1, x_2]$. Then $\mathcal{V}(P)$ is a circle.



(ii) Let $P = \{x_1 x_2\} \subseteq \mathbb{Q}[x_1, x_2]$. Then $\mathcal{V}(P)$ is the union of the two coordinate axes.



(iii) Let $P = \{x_2^2 - x_1^2(x_1 + 1)\} \subseteq \mathbb{Q}[x_1, x_2]$. Then $\mathcal{V}(P)$ has the form of a ribbon.

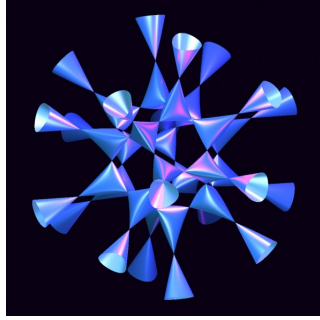


(iv) Let $P = \{x_2^2 - x_1^3\} \subseteq \mathbb{Q}[x_1, x_2]$. Then $\mathcal{V}(P)$ is a curve with a cusp.



(v) Let $P = \{x_1^2 + x_2^2\} \subseteq \mathbb{Q}[x_1, x_2]$. Here the real image only shows the point $(0, 0)$. But we also have for example $(1, i) \in \mathcal{V}(P)$. Even worse is the case $P = \{x_1^2 + x_2^2 + 1\}$, in which we don't see anything in the real image. But again the variety is not empty, for example we have $(0, i) \in \mathcal{V}(P)$.

(vi) The following image shows a part of an affine hypersurface in \mathbb{A}^3 . The defining equation has degree 6:



Author: Oliver Labs, www.imaginary.org, Project of the Mathematical Research Institute Oberwolfach, supported by the Klaus Tschira Foundation

(vii) Thus far the images have always shown hypersurfaces. But there are many more varieties besides hypersurfaces. For instance every point $a \in k^n \subseteq \mathbb{A}^n$ is an affine k -variety because

$$\{a\} = \mathcal{V}(x_1 - a_1, \dots, x_n - a_n)$$

and $x_1 - a_1, \dots, x_n - a_n \in k[x]$. For $a \in \mathbb{A}^n \setminus k^n$ this is not true anymore. For example $\{i\} \subseteq \mathbb{C} = \mathbb{A}^1$ is not an affine \mathbb{R} -variety. Every real polynomial that vanishes on i must also vanish on $-i$. But we have

$$i \in \{i, -i\} = \mathcal{V}(x_1^2 + 1) \subseteq \mathbb{C} = \mathbb{A}^1.$$

When we consider $k = \mathbb{Q}$, $K = \mathbb{C}$ we get examples where this is even more obvious. The only polynomial in $\mathbb{Q}[x]$ that vanishes on the point $a = \pi \in \mathbb{A}^1$ is the zero polynomial. In particular, the only affine \mathbb{Q} -variety in \mathbb{A}^1 that contains the point a is the entire affine space. \triangle

Lemma 1.2.3. *Let $P \subseteq k[x]$ and let $I = (P)$ be the ideal in $k[x]$ generated by P . Then*

$$\mathcal{V}(P) = \mathcal{V}(I) = \mathcal{V}(\sqrt{I}).$$

Proof. We have $P \subseteq I \subseteq \sqrt{I}$, and thus $\mathcal{V}(\sqrt{I}) \subseteq \mathcal{V}(I) \subseteq \mathcal{V}(P)$. Now let $p \in \sqrt{I}$, that is

$$p^m = \sum_i f_i p_i$$

where $p_i \in P$, $f_i \in k[x]$. From this we see that $a \in \mathcal{V}(P)$ implies $p^m(a) = 0$, and since fields don't have nontrivial zero divisors, we obtain $p(a) = 0$. Therefore we have $p = 0$ on $\mathcal{V}(P)$, and this shows $\mathcal{V}(P) \subseteq \mathcal{V}(\sqrt{I})$. \square

Corollary 1.2.4. *Every affine k -variety V is of the form*

$$V = \mathcal{V}(p_1, \dots, p_r)$$

with finitely many polynomials $p_1, \dots, p_r \in k[\underline{x}]$.

Proof. Let $V = \mathcal{V}(P)$ with $P \subseteq k[\underline{x}]$. Then we have $(P) = (p_1, \dots, p_r)$ by Corollary 1.1.7, and from Lemma 1.2.3 we get $V = \mathcal{V}(p_1, \dots, p_r)$. \square

Lemma 1.2.5. *(i) \emptyset, \mathbb{A}^n are affine k -varieties.*

(ii) $V_1, V_2 \subseteq \mathbb{A}^n$ affine k -varieties $\Rightarrow V_1 \cup V_2$ affine k -variety.

(iii) $V_\lambda \subseteq \mathbb{A}^n$ affine k -varieties ($\lambda \in \Lambda$) $\Rightarrow \bigcap_{\lambda \in \Lambda} V_\lambda$ affine k -variety.

(iv) $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ affine k -varieties $\Rightarrow V \times W \subseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ affine k -variety.

Proof. (i): $\emptyset = \mathcal{V}(1), \mathbb{A}^n = \mathcal{V}(0)$. (ii): For two ideals $I_1, I_2 \subseteq k[\underline{x}]$ we have

$$\mathcal{V}(I_1) \cup \mathcal{V}(I_2) = \mathcal{V}(I_1 \cap I_2) = \mathcal{V}(I_1 I_2).$$

Both inclusions " \subseteq " are clear because the ideals get smaller. Now let $a \in \mathcal{V}(I_1 I_2)$ and $a \notin \mathcal{V}(I_1)$. Then there is a $p \in I_1$ with $p(a) \neq 0$. For every $q \in I_2$ we have $pq \in I_1 I_2$, and thus $0 = (pq)(a) = p(a)q(a)$. Since fields don't have nontrivial zero divisors, it follows that $q(a) = 0$ holds for all $q \in I_2$, hence $a \in \mathcal{V}(I_2)$.

(iii): For $V_\lambda = \mathcal{V}(P_\lambda)$ with $P_\lambda \subseteq k[\underline{x}]$ we obviously have $\bigcap_\lambda V_\lambda = \mathcal{V}(\bigcup_\lambda P_\lambda)$.

(iv): Let us write $V = \mathcal{V}(P)$ and $W = \mathcal{V}(Q)$ for certain subsets $P \subseteq k[x_1, \dots, x_n]$ and $Q \subseteq k[y_1, \dots, y_m]$. Then $P \cup Q \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ and

$$V \times W = \mathcal{V}(P \cup Q). \quad \square$$

We want to put the constructions from the previous proof on record:

Corollary 1.2.6. *Let I_1, I_2, I_λ ($\lambda \in \Lambda$) be ideals in $k[\underline{x}]$. Then we have*

$$\mathcal{V}(I_1) \cup \mathcal{V}(I_2) = \mathcal{V}(I_1 \cap I_2) = \mathcal{V}(I_1 I_2)$$

and

$$\bigcap_{\lambda \in \Lambda} \mathcal{V}(I_\lambda) = \mathcal{V}\left(\sum I_\lambda\right). \quad \square$$

Definition 1.2.7. Let $V \subseteq \mathbb{A}^n$ be an arbitrary subset. Then

$$\mathcal{I}(V) := \{p \in k[\underline{x}] \mid p(a) = 0 \forall a \in V\}$$

is called the **vanishing ideal of V** . \triangle

Lemma 1.2.8. *Let $V, W \subseteq \mathbb{A}^n$ be subsets. Then*

(i) $\mathcal{I}(V)$ is a radical ideal.

(ii) $V \subseteq W \Rightarrow \mathcal{I}(W) \subseteq \mathcal{I}(V)$.

(iii) $\mathcal{I}(V \cup W) = \mathcal{I}(V) \cap \mathcal{I}(W)$.

(iv) In case V is an affine k -variety we have $\mathcal{V}(\mathcal{I}(V)) = V$.

(v) Every descending chain $V_1 \supseteq V_2 \supseteq \dots$ of affine k -varieties becomes stationary.

Proof. (i)-(iii) are clear. For (iv) let $V = \mathcal{V}(I)$ for some ideal I . Then we have $I \subseteq \mathcal{I}(V)$, and thus $V = \mathcal{V}(I) \supseteq \mathcal{V}(\mathcal{I}(V))$. The other inclusion " \subseteq " is clear. In (v) we obtain the ascending chain of ideals $\mathcal{I}(V_1) \subseteq \mathcal{I}(V_2) \subseteq \dots$, which becomes stationary by Hilbert's Basis Theorem. After applying $\mathcal{V}(\cdot)$ we get from (iv) that the chain of the V_i becomes stationary as well. \square

Theorem 1.2.9 (Hilbert's Nullstellensatz, field theoretic form). *Let F/k be a field extension such that F is finitely generated as k -algebra. Then F/k is finite (and thus algebraic).*

Proof. There are $\alpha_1, \dots, \alpha_n \in F$ with $F = k[\alpha_1, \dots, \alpha_n]$. We proof the claim by induction on n .

$n = 1$: Since $F = k[\alpha]$ is a field, there is a polynomial $p \in k[t]$ with $\alpha^{-1} = p(\alpha)$. This implies $\alpha \cdot p(\alpha) - 1 = 0$, and hence α is algebraic over k . But then the extension is finite.

$n - 1 \rightarrow n$: We have $F = k(\alpha_1)[\alpha_2, \dots, \alpha_n]$ because F is a field. By the induction hypothesis $\alpha_2, \dots, \alpha_n$ are algebraic over $k(\alpha_1)$. It is now enough to show that α_1 is algebraic over k . Then the entire extension F/k is algebraic and thus finite. Now $\alpha_2, \dots, \alpha_n$ being algebraic over $k(\alpha_1)$ means that there are identities

$$u_i \alpha_i^d + \sum_{j=0}^{d-1} r_{ij} \alpha_i^j = 0$$

with $u_i, r_{ij} \in k[\alpha_1]$ (possible denominators have been cleared). Consider $u := u_2 \cdots u_n \in k[\alpha_1]$. Then $\alpha_2, \dots, \alpha_n$ are integral over the ring $k[\alpha_1, 1/u]$, and by Theorem 1.1.10 the ring F is an integral extension of $k[\alpha_1, 1/u]$. Assume α_1 is transcendental over k , i.e. $k[\alpha_1]$ is isomorphic to a polynomial ring. Then we can choose an irreducible polynomial $p \in k[\alpha_1]$ such that $p \nmid u$ (there are infinitely many irreducible polynomials in the unique factorization domain $k[\alpha_1]$). Now for p^{-1} there exists an integrality equation

$$p^{-m} + b_1 p^{-(m-1)} + \dots + b_m = 0$$

with $b_i \in k[\alpha_1, 1/u]$. Multiplication by p^m and a sufficiently large power of u yields

$$u^r + a_1p + \cdots + a_m p^m = 0$$

with $a_i \in k[\alpha_1]$. This implies $p \mid u$, a contradiction. \square

Corollary 1.2.10. *Let A be a finitely generated k -algebra and let \mathfrak{m} be a maximal ideal in A . Then A/\mathfrak{m} is a finite field extension of k .*

Proof. A/\mathfrak{m} is still finitely generated as k -algebra and also a field. \square

Corollary 1.2.11 (Hilbert's Nullstellensatz, geometric form). *Let $I \subsetneq k[x]$ be a proper ideal. Then $\mathcal{V}(I) \neq \emptyset$.*

Proof. Choose a maximal ideal \mathfrak{m} of $k[x]$ with $I \subseteq \mathfrak{m}$. By Corollary 1.2.10 $k[x]/\mathfrak{m}$ is a finite field extension of k . Hence there is a k -embedding of $k[x]/\mathfrak{m}$ into K , and we can assume

$$k \subseteq k[x]/\mathfrak{m} \subseteq K.$$

Now let $a_i := \bar{x}_i$, the residue class of x_i in $k[x]/\mathfrak{m} \subseteq K$. For every $p \in k[x]$ we then have

$$p(a) = p(\bar{x}) = \bar{p},$$

and for $p \in I$ (even for $p \in \mathfrak{m}$) this implies $p(a) = 0$. Thus $a \in \mathcal{V}(I)$. \square

Remark 1.2.12. In the last proof, without Corollary 1.2.10 we would only get that $\mathcal{V}(I)$ has an element over *some* extension field of k (namely $k[x]/\mathfrak{m}$). With Corollary 1.2.10 we see that there exists an element over a finite extension field and thus over *any* algebraically closed extension field. Over k itself this does not have to be the case, as we can see for example for $k = \mathbb{Q}$ and $I = (x^2 - 2)$, or $k = \mathbb{R}$ and $I = (x^2 + 1)$. \triangle

Remark 1.2.13. Corollary 1.2.11 says that a system of polynomial equations

$$p_1 = 0, \dots, p_r = 0$$

with $p_i \in k[x]$ does *not* have a solution over K if and only if there is an identity of the form

$$q_1 p_1 + \cdots + q_r p_r = 1$$

with $q_i \in k[x]$. In Chapter 2 we will see how this last condition, and thus the solvability of polynomial equation systems, can be checked algorithmically. \triangle

Theorem 1.2.14 (Hilbert's Nullstellensatz, ideal theoretic form). *For every ideal $I \subseteq k[\underline{x}]$ we have*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Proof. " \supseteq " is clear. For " \subseteq " let $0 \neq p \in \mathcal{I}(\mathcal{V}(I))$. Consider the ideal

$$J = (I, tp - 1) \subseteq k[t, \underline{x}].$$

Since $p \equiv 0$ on $\mathcal{V}(I)$, we have $\mathcal{V}(J) = \emptyset$. By Corollary 1.2.11 there is an identity

$$1 = a(tp - 1) + \sum_i b_i p_i$$

with $a, b_i \in k[t, \underline{x}]$ and $p_i \in I$. Substituting p^{-1} for t and multiplying with a sufficiently large power of p yields

$$p^r = \sum_i \tilde{b}_i p_i,$$

with $\tilde{b}_i \in k[\underline{x}]$, hence $p \in \sqrt{I}$. □

Remark 1.2.15. Note that Theorem 1.2.14 is true for any choice of the algebraically closed field K , and that \sqrt{I} does not depend on this choice. Thus it does not really matter over which algebraically closed field K we consider the variety. △

Example 1.2.16. For a hypersurface $V \subseteq \mathbb{A}^n$, say $V = \mathcal{V}(p)$, such that

$$p = p_1^{e_1} \cdots p_r^{e_r}$$

is the decomposition into irreducible polynomials, we have

$$\mathcal{I}(V) = (p_1 \cdots p_r). \quad \triangle$$

Corollary 1.2.17. *Let $I, J \subseteq k[\underline{x}]$ be ideals. Then we have*

$$\mathcal{V}(I) \subseteq \mathcal{V}(J) \Leftrightarrow \sqrt{J} \subseteq \sqrt{I}.$$

Proof. From $\mathcal{V}(I) \subseteq \mathcal{V}(J)$ it obviously follows that $\mathcal{I}(\mathcal{V}(J)) \subseteq \mathcal{I}(\mathcal{V}(I))$, and by Theorem 1.2.14 therefore $\sqrt{J} \subseteq \sqrt{I}$. The other implication follows from Lemma 1.2.3 since $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$ and $\mathcal{V}(J) = \mathcal{V}(\sqrt{J})$. □

Corollary 1.2.18. *The mapping $V \mapsto \mathcal{I}(V)$ is an inclusion-reversing bijection between the set of all affine k -varieties in \mathbb{A}^n and the set of all radical ideals in $k[\underline{x}]$. The inverse mapping is given by $I \mapsto \mathcal{V}(I)$.*

Proof. Lemma 1.2.8 (iv), Theorem 1.2.14, and Corollary 1.2.17. \square

Remark 1.2.19. (i) In particular, maximal ideals in $k[\underline{x}]$ correspond to minimal affine k -varieties $\neq \emptyset$.

(ii) Let $a \in k^n \subseteq \mathbb{A}^n$. Then we have $\{a\} = \mathcal{V}(\mathfrak{m}_a)$, where

$$\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n).$$

It is easy to see that $\mathcal{I}(\{a\}) = \mathfrak{m}_a$ holds. In particular \mathfrak{m}_a is maximal.

(iii) Not every maximal ideal $\mathfrak{m} \subseteq k[\underline{x}]$ has to be of the form \mathfrak{m}_a , and not every minimal variety $\neq \emptyset$ has to be of the form $\{a\}$. For $k = \mathbb{R}$ and $K = \mathbb{C}$ for instance, $\mathfrak{m} = (x_1^2 + 1)$ is maximal in $\mathbb{R}[x_1]$ and corresponds to the minimal \mathbb{R} -variety $\{i, -i\} \subseteq \mathbb{A}^1$. \triangle

Corollary 1.2.20. *The mapping $a \mapsto \mathfrak{m}_a$ is a bijection between k^n and the maximal ideals in $k[\underline{x}]$ with residue field k . In case $k = \bar{k}$, these are all maximal ideals in $k[\underline{x}]$.*

Proof. The injectivity is clear, and we have $k[\underline{x}]/\mathfrak{m}_a = k$. For the surjectivity let $\mathfrak{m} \subseteq k[\underline{x}]$ be a maximal ideal with $k[\underline{x}]/\mathfrak{m} = k$. Let $a_i := \bar{x}_i \in k$. Then we have $x_i - a_i = 0$, and thus $\mathfrak{m}_a \subseteq \mathfrak{m}$. From this we get equality. Since $k[\underline{x}]/\mathfrak{m}$ is a finite field extension of k for every maximal ideal by Corollary 1.2.10, we always have $k[\underline{x}]/\mathfrak{m} = k$ in case $k = \bar{k}$. \square

1.3 The Zariski Topology

Definition 1.3.1. The k -Zariski topology on \mathbb{A}^n has the affine k -varieties as its closed sets. The k -Zariski topology on a subset $X \subseteq \mathbb{A}^n$ is the induced subspace topology. \triangle

As long as we speak about k -varieties, all subsequent topological notions always refer to the k -Zariski topology.

Remark/Example 1.3.2. (i) The k -Zariski topology is indeed a topology. This is a consequence of Lemma 1.2.5.

(ii) The Zariski topology is the most natural topology on \mathbb{A}^n , since the definition only needs polynomial equations, i.e. just the already given field structure is used.

(iii) In the case $k = \bar{k} = K$ the closed subsets of \mathbb{A}^1 are exactly the finite subsets, as well as \mathbb{A}^1 (cf. Exercise 9).

(iv) The Zariski topology on $\mathbb{A}^{n+m} = \mathbb{A}^n \times \mathbb{A}^m$ does *not* coincide with the product topology (Exercise 17).

(v) For $k \subseteq k' \subseteq K$ the k' -Zariski topology is in general finer than the k -Zariski topology. For instance $\{i\} \subseteq \mathbb{A}^1 = \mathbb{C}^1$ is closed in the \mathbb{C} -Zariski topology, but not in the \mathbb{R} -Zariski topology.

(vi) For every $p \in k[x]$ the set

$$\mathcal{D}(p) := \{a \in \mathbb{A}^n \mid p(a) \neq 0\}$$

is open. Every open set is of the form

$$\mathcal{D}(p_1) \cup \cdots \cup \mathcal{D}(p_r),$$

where $p_1, \dots, p_r \in k[x]$ (cf. Corollary 1.2.4).

(vii) The map

$$a \mapsto (1/p(a), a)$$

defines a canonical bijection between the open set $\mathcal{D}(p) \subseteq \mathbb{A}^n$ and the variety $\mathcal{V}(tp - 1) \subseteq \mathbb{A}^{n+1}$. \triangle

Lemma 1.3.3. (i) For $X \subseteq \mathbb{A}^n$ we have $\overline{X} = \mathcal{V}(\mathcal{I}(X))$.

(ii) $U_1 \cap U_2 \neq \emptyset$ holds for any two nonempty open sets $U_1, U_2 \subseteq \mathbb{A}^n$. In particular, every nonempty open set is dense in \mathbb{A}^n .

(iii) The Zariski topology is not Hausdorff.

Proof. (i) $\mathcal{V}(\mathcal{I}(X))$ is closed and contains X , which shows " \subseteq ". For " \supseteq " note that $\mathcal{I}(X) \supseteq \mathcal{I}(\overline{X})$ and thus $\mathcal{V}(\mathcal{I}(X)) \subseteq \mathcal{V}(\mathcal{I}(\overline{X})) = \overline{X}$ (Lemma 1.2.8 (iv)).

(ii) $\mathcal{D}(p) \cap \mathcal{D}(q) = \emptyset$ implies $pq \equiv 0$ on \mathbb{A}^n and therefore $pq = 0$, since K is infinite. It follows that $p = 0$ or $q = 0$, or equivalently $\mathcal{D}(p) = \emptyset$ or $\mathcal{D}(q) = \emptyset$.

(iii) Follows immediately from (ii). \square

Definition 1.3.4. Let X be a topological space.

(i) X is called **irreducible** if $X \neq \emptyset$ and for all closed subsets $A, B \subseteq X$ we have

$$X = A \cup B \Rightarrow A = X \text{ or } B = X.$$

Otherwise X is called **reducible**.

(ii) $Y \subseteq X$ is called **irreducible component of X** if Y is a maximal irreducible subset of X (w.r.t. set inclusion and the subspace topology). \triangle

Remark 1.3.5. For a subset $Y \subseteq X$ the irreducibility (w.r.t. the subspace topology) can be stated as follows:

$$A, B \text{ closed in } X, Y \subseteq A \cup B \Rightarrow Y \subseteq A \text{ or } Y \subseteq B.$$

Note that irreducibility is a property of a topological space itself, and is not defined relative to an ambient space (in contrast to e.g. closedness). \triangle

Lemma 1.3.6. Let $X \neq \emptyset$ be a topological space.

(i) The following are equivalent:

(a) X is irreducible.

(b) Every nonempty open subset of X is dense.

(c) Any two nonempty open subsets of X have a nonempty intersection.

(ii) For $Y \subseteq X$ we have: Y irreducible $\Leftrightarrow \overline{Y}$ irreducible.

(iii) Every irreducible component of X is closed.

Proof. Exercise 21. \square

Remark/Example 1.3.7. (i) For Hausdorff spaces the notion of irreducibility is not quite meaningful. A Hausdorff space X is irreducible if and only if $|X| = 1$. In particular, the irreducible components of a Hausdorff space are just the singleton subsets.

(ii) \mathbb{A}^n is irreducible in the k -Zariski topology. This follows from Lemma 1.3.3 and Lemma 1.3.6. \triangle

Lemma 1.3.8. Every irreducible subset of a topological space X is contained in an irreducible component of X . In particular, X is the union of its irreducible components.

Proof. Let $Y \subseteq X$ be irreducible. Consider

$$\mathcal{M} = \{Z \subseteq X \mid Z \text{ irreducible}, Y \subseteq Z\}.$$

\mathcal{M} is not empty since $Y \in \mathcal{M}$. Let $(Z_\lambda)_{\lambda \in \Lambda}$ be a chain in \mathcal{M} . Then $\bigcup_\lambda Z_\lambda$ is still irreducible: From $\bigcup_\lambda Z_\lambda \subseteq A \cup B$ with A, B closed it follows for all λ that $Z_\lambda \subseteq A$ or $Z_\lambda \subseteq B$. Because the Z_λ form a chain, it must be the same case for all λ .

Therefore $\bigcup_\lambda Z_\lambda \in \mathcal{M}$, and by Zorn's Lemma \mathcal{M} has a maximal element. This is obviously an irreducible component of X containing Y .

The second assertion follows from the fact that $\{x\} \subseteq X$ is irreducible for all $x \in X$. \square

Remark 1.3.9. Every irreducible topological space is connected. In particular, every connected component of X is the union of irreducible components of X . \triangle

Definition 1.3.10. A topological space X is called **noetherian** if every descending chain $A_1 \supseteq A_2 \supseteq \cdots$ of closed subsets of X becomes stationary. \triangle

Lemma 1.3.11. For a topological space X the following are equivalent:

- (i) X is noetherian.
- (ii) Every nonempty system of closed subsets of X contains a minimal element.
- (iii) Every open subset of X is quasi-compact, i.e. every open cover has a finite subcover.

Proof. Exercise 22. \square

Example 1.3.12. (i) Every subspace of a noetherian topological space is itself noetherian.

(ii) \mathbb{A}^n is noetherian in the k -Zariski topology (Lemma 1.2.8 (v)).

(iii) Every affine k -variety is noetherian in the k -Zariski topology. \triangle

Theorem 1.3.13. Let X be a noetherian topological space. Then

(i) X has only finitely many irreducible components.

(ii) If X_1, \dots, X_r are the pairwise distinct irreducible components of X , then

$$X_i \not\subseteq \bigcup_{j \neq i} X_j$$

for all $i = 1, \dots, r$.

(iii) If $X = Y_1 \cup \cdots \cup Y_s$ is a covering with closed irreducible subsets Y_i which satisfy

$$Y_i \not\subseteq \bigcup_{j \neq i} Y_j$$

for all $i = 1, \dots, s$, then Y_1, \dots, Y_s are the irreducible components of X .

Proof. (i) Let \mathcal{M} be the set of all closed subsets of X that are *not* the union of finitely many irreducible subsets. We show that $\mathcal{M} = \emptyset$. Assume $\mathcal{M} \neq \emptyset$. Then there exists a minimal element Y in \mathcal{M} , since X is noetherian. But then Y must be reducible, that is $Y = Y_1 \cup Y_2$ with $Y_i \subseteq X$ closed and $Y_1, Y_2 \subsetneq Y$. This implies $Y_1, Y_2 \notin \mathcal{M}$, and hence Y_1, Y_2 are both a union of finitely many irreducible subsets and so is Y , a contradiction.

In particular X is the union of finitely many irreducible subsets and by Lemma 1.3.8 of finitely many irreducible components X_1, \dots, X_r . Now let Z be another irreducible component of X . Then

$$Z = \bigcup_{i=1}^r Z \cap X_i,$$

and since the X_i are closed and Z is irreducible, it follows $Z \subseteq X_i$ for some i . Because Z is even an irreducible component, we have equality. This proves (i).

(ii) From $X_i \subseteq \bigcup_{j \neq i} X_j$ we obtain $X_i \subseteq \bigcup_{j \neq i} X_i \cap X_j$, and as before $X_i \subseteq X_j$ for some $j \neq i$, a contradiction.

(iii) Again let X_1, \dots, X_r be the irreducible components of X . As before $X_i \subseteq \bigcup_j Y_j \cap X_i$ implies $X_i \subseteq Y_j$ for some j , and thus equality. So the irreducible components of X are among the Y_j . From $Y_i \not\subseteq \bigcup_{j \neq i} Y_j$ it follows that there are no further sets among the Y_j . \square

Corollary 1.3.14. *Let V be an affine k -variety. Then there are irreducible affine k -varieties V_1, \dots, V_r with*

$$V = V_1 \cup \dots \cup V_r$$

and

$$V_i \not\subseteq \bigcup_{j \neq i} V_j$$

for $i = 1, \dots, r$. These constraints uniquely determine the V_i as the irreducible components of V .

Theorem 1.3.15. *Let V be an affine k -variety. Then*

$$V \text{ irreducible} \Leftrightarrow \mathcal{I}(V) \subseteq k[\underline{x}] \text{ prime ideal.}$$

Proof. Let $I := \mathcal{I}(V)$ and thus $V = \mathcal{V}(I)$.

First assume V is irreducible. $V \neq \emptyset$ implies $I \neq (1)$. Let $p, q \in k[\underline{x}]$ such that $pq \in I$. Then we have $V \subseteq \mathcal{V}(p) \cup \mathcal{V}(q)$, and from the irreducibility of V we deduce w.l.o.g. $V \subseteq \mathcal{V}(p)$. This means $p \in I$. Therefore I is a prime ideal.

Now assume I is a prime ideal and

$$V \subseteq \mathcal{V}(I_1) \cup \mathcal{V}(I_2) = \mathcal{V}(I_1 I_2)$$

for two ideals $I_1, I_2 \subseteq k[\underline{x}]$. This implies $I_1 I_2 \subseteq I$ and from I being a prime ideal we obtain w.l.o.g. $I_1 \subseteq I$. This in turn yields $V \subseteq \mathcal{V}(I_1)$, and thus V is irreducible. \square

Corollary 1.3.16. *The mapping $\mathfrak{p} \mapsto \mathcal{V}(\mathfrak{p})$ defines an inclusion-reversing bijection between $\text{Spec}(k[\underline{x}])$ and the set of all irreducible affine k -varieties in \mathbb{A}^n .*

Corollary 1.3.17. *Every ideal $I \subseteq k[\underline{x}]$ is contained in only finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. We have*

$$\sqrt{I} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r.$$

$\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are called the **minimal prime divisors** of I .

Proof. The minimal prime ideals over I are the minimal prime ideals over \sqrt{I} , and by Corollary 1.2.18 and Corollary 1.3.16 these exactly correspond to the maximal irreducible subsets of $V = \mathcal{V}(I)$, of which there are only finitely many by Corollary 1.3.14. According to Theorem 1.1.4 the intersection of the minimal prime ideals containing I is the radical of I . \square

Example 1.3.18. (i) Let $V = \mathcal{V}(p)$ be a hypersurface, and let p_1, \dots, p_r the irreducible factors of p . Then the varieties $\mathcal{V}(p_i)$ are the irreducible components of V , and the ideals (p_i) are the minimal prime divisors of (p) .

(ii) The variety $\mathcal{V}(x_1x_2)$ (cf. Example 1.2.2 (ii)) has the two irreducible components $\mathcal{V}(x_1)$ and $\mathcal{V}(x_2)$. Here V is connected.

(iii) The variety $\mathcal{V}(x_1(x_1 - 1), x_2(x_1 - 1))$ has the two irreducible components $\mathcal{V}(x_1 - 1)$ and $\mathcal{V}(x_1, x_2)$. The easiest way to see this is Corollary 1.3.14. Here the irreducible components are also the connected components.

(iv) The variety $V = \mathcal{V}(x_1^2 + x_2^2) \subseteq \mathbb{A}^2 = \mathbb{C}^2$ is irreducible in the \mathbb{R} -Zariski topology, because $x_1^2 + x_2^2$ is irreducible in $\mathbb{R}[x_1, x_2]$. However, in the \mathbb{C} -Zariski topology V is reducible:

$$V = \mathcal{V}(x_1 + ix_2) \cup \mathcal{V}(x_1 - ix_2).$$

In both topologies V is connected. \triangle

1.4 Regular Functions and Morphisms

Let $V \subseteq \mathbb{A}^n$ be an affine k -variety. Every polynomial $p \in k[\underline{x}]$ defines a polynomial map $p: \mathbb{A}^n \rightarrow K = \mathbb{A}^1$, and by restricting the domain to V we obtain a map

$$p: V \rightarrow \mathbb{A}^1.$$

The restrictions of two polynomials $p, q \in k[\underline{x}]$ define the same map on V if and only if $p - q \equiv 0$ on V and therefore $p - q \in \mathcal{I}(V)$ holds. Thus the polynomial maps $V \rightarrow \mathbb{A}^1$ can be identified with the elements of $k[\underline{x}]/\mathcal{I}(V)$.

Remainder: A k -**algebra** is a ring extension of k .

Definition 1.4.1. Let $V \subseteq \mathbb{A}^n$ be an affine k -variety. Then

$$k[V] := k[\underline{x}]/\mathcal{I}(V)$$

is called the **affine coordinate ring** or the **affine coordinate algebra of V** . We emphasize again that $\mathcal{I}(V)$ and therefore $k[V]$ does not depend on the choice of the algebraically closed field K !

Elements of $k[V]$ are called **regular functions on V** . Regular functions can be interpreted as polynomial functions on V . \triangle

Example 1.4.2. (i) $k[\mathbb{A}^n] = k[\underline{x}]$ and $k[\emptyset] = \{0\}$.

(ii) If $V = \mathcal{V}(p)$ is a hypersurface and p is square free, then $k[V] = k[\underline{x}]/(p)$. \triangle

Lemma 1.4.3. (i) For every affine variety V the coordinate algebra $k[V]$ is a finitely generated reduced k -algebra.

(ii) If $V, W \subseteq \mathbb{A}^n$ are affine varieties with $V \cap W = \emptyset$, then

$$k[V \cup W] \cong k[V] \times k[W].$$

In particular, for every finite variety $V = \{a_1, \dots, a_r\} \subseteq k^n$ we have

$$k[V] \cong \underbrace{k \times \dots \times k}_r.$$

Proof. (i) Follows from the fact that $k[\underline{x}]$ is finitely generated and $\mathcal{I}(V)$ is a radical ideal. For (ii) observe that

$$(1) = \mathcal{I}(V \cap W) = \sqrt{\mathcal{I}(V) + \mathcal{I}(W)} = \mathcal{I}(V) + \mathcal{I}(W),$$

where we have used Lemma 1.1.3 (ii) for the last equation. From the Chinese Remainder Theorem it follows that

$$k[V \cup W] = k[\underline{x}]/(\mathcal{I}(V) \cap \mathcal{I}(W)) \cong k[\underline{x}]/\mathcal{I}(V) \times k[\underline{x}]/\mathcal{I}(W) = k[V] \times k[W].$$

The second claim follows from $k[\{a\}] = k[\underline{x}]/\mathfrak{m}_a \cong k$ for $a \in k^n$. \square

Remark 1.4.4. The ideals of $k[V]$ are in bijection with the ideals of $k[\underline{x}]$ that contain $\mathcal{I}(V)$. The same is true for radical and prime ideals. These ideals in turn correspond to subvarieties of V . Thus we obtain the following relative version of Corollary 1.2.18:

The mapping

$$I \mapsto \mathcal{V}_V(I) := \{a \in V \mid p(a) = 0 \forall p \in I\}$$

yields a bijection between the radical ideals of $k[V]$ and the subvarieties of V . Under this bijection prime ideals exactly correspond to irreducible subvarieties. The inverse mapping is given by

$$W \mapsto \mathcal{I}_V(W) := \{p \in k[V] \mid p(a) = 0 \forall a \in W\}.$$

For $p \in k[V]$ let

$$\mathcal{D}_V(p) := \{a \in V \mid p(a) \neq 0\}.$$

Every open set in the k -Zariski topology of V is a finite union of such $\mathcal{D}(p_i)$. \triangle

Remark 1.4.5. In the transition from V to $k[V]$ no information is lost: From $k[V]$ together with the generators $\bar{x}_1, \dots, \bar{x}_n$ we can reconstruct V in several ways.

(i) Let $\pi: k[\underline{x}] \twoheadrightarrow k[V]$ be the canonical projection $x_i \mapsto \bar{x}_i$. Then we obviously have $\ker(\pi) = \mathcal{I}(V)$, and hence

$$V = \mathcal{V}(\ker(\pi)).$$

(ii) The following maps are mutually inverse bijections

$$\begin{aligned} K^n &\longleftrightarrow \text{Hom}_{k\text{-alg}}(k[\underline{x}], K) \\ a &\mapsto e_a \\ (\alpha(x_1), \dots, \alpha(x_n)) &\longleftarrow \alpha. \end{aligned}$$

Here e_a denotes the evaluation of a polynomial in a . Under this bijection the points $a \in V$ exactly correspond to the homomorphisms $\alpha \in \text{Hom}(k[\underline{x}], K)$ with $\alpha \equiv 0$ on $\mathcal{I}(V)$, and thus to the elements in $\text{Hom}(k[V], K)$. Therefore we obtain the bijection

$$\begin{aligned} \text{Hom}(k[V], K) &\rightarrow V \\ \alpha &\mapsto (\alpha_1(\bar{x}_1), \dots, \alpha(\bar{x}_n)). \end{aligned} \quad \triangle$$

Corollary 1.4.6. *Every finitely generated reduced k -algebra A is isomorphic to the coordinate algebra of an affine k -variety.*

Proof. Follows immediately from the construction in Remark 1.4.5 (i): Choose generators a_1, \dots, a_n of A and consider the surjection

$$\begin{aligned} \pi: k[\underline{x}] &\twoheadrightarrow A \\ x_i &\mapsto a_i. \end{aligned}$$

For $V = \mathcal{V}(\ker(\pi))$ we then have

$$k[V] = k[\underline{x}]/\mathcal{I}(V) = k[\underline{x}]/\ker(\pi) \cong A.$$

Here we use that $\ker(\pi)$ is a radical ideal, which follows from the fact that A is reduced. \square

Definition 1.4.7. Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine k -varieties.

(i) A (k -) **morphism from V to W** is a map

$$p: V \rightarrow W$$

for which there are $p_1, \dots, p_m \in k[V]$ such that

$$p(a) = (p_1(a), \dots, p_m(a)) \quad \forall a \in V.$$

Shorthand we write $p = (p_1, \dots, p_m)$.

(ii) A k -morphism $p: V \rightarrow W$ is called **k -isomorphism** if there is a k -morphism $q: W \rightarrow V$ such that

$$q \circ p = \text{id}_V \quad \text{and} \quad p \circ q = \text{id}_W.$$

(iii) Two affine k -varieties V, W are called **k -isomorphic** if there exists a k -isomorphism $p: V \rightarrow W$. In this situation we write $V \cong_k W$ (or $V \cong W$ in case k is clear from the context).

(iv) With $\text{Hom}_k(V, W)$ we denote the set of all k -morphisms from V to W . \triangle

Theorem 1.4.8. *Let $p: V \rightarrow W$ be a k -morphism. For every $q \in k[W]$ the composition*

$$p^*(q) := q \circ p \in k[V]$$

is a regular function on V . The map p^ defined as*

$$\begin{aligned} p^*: k[W] &\rightarrow k[V] \\ q &\mapsto p^*(q) \end{aligned}$$

is a k -algebra homomorphism. The map

$$\begin{aligned} *: \text{Hom}(V, W) &\rightarrow \text{Hom}(k[W], k[V]) \\ p &\mapsto p^* \end{aligned}$$

is bijective.

Proof. Since the composition of two polynomial maps is obviously again polynomial, the composition of two k -morphisms is again a k -morphism. Thus we have $p^*(q) = q \circ p \in \text{Hom}(V, \mathbb{A}^1) = k[V]$. The map p^* is obviously a k -algebra homomorphism. It remains to show that $*$ is bijective. In order to do this we explicitly define the inverse map. We have

$$k[W] = k[y_1, \dots, y_m] / \mathcal{I}(W) = k[\bar{y}_1, \dots, \bar{y}_m].$$

Now let $\varphi: k[W] \rightarrow k[V]$ be a k -algebra homomorphism. Let

$$p_\varphi := (\varphi(\bar{y}_1), \dots, \varphi(\bar{y}_m)) \in \text{Hom}(V, \mathbb{A}^m).$$

We show that even $p_\varphi: V \rightarrow W$ holds. For $q \in k[\underline{y}]$ and $a \in V$ we have

$$q(p_\varphi(a)) = q(\varphi(\bar{y}_1)(a), \dots, \varphi(\bar{y}_m)(a)) = \varphi(q(\bar{y}))(a) = \varphi(\bar{q})(a).$$

Now if $q \in \mathcal{I}(W)$, then $\bar{q} = 0$ holds in $k[W]$, and hence $q(p_\varphi(a)) = 0$. This proves $p_\varphi(a) \in W$, and thus $p_\varphi \in \text{Hom}(V, W)$. Now the map defined by

$$\begin{aligned} \text{Hom}(k[W], k[V]) &\rightarrow \text{Hom}(V, W) \\ \varphi &\mapsto p_\varphi \end{aligned}$$

is the inverse map of $*$ (Exercise 35). \square

Remark 1.4.9. We put the following on record once more: For $k[W] = k[\underline{y}]/\mathcal{I}(W)$ the inverse of

$$\begin{aligned} * : \text{Hom}(V, W) &\rightarrow \text{Hom}(k[W], k[V]) \\ p &\mapsto p^* \end{aligned}$$

is the map

$$\begin{aligned} \text{Hom}(k[W], k[V]) &\rightarrow \text{Hom}(V, W) \\ \varphi &\mapsto (\varphi(\bar{y}_1), \dots, \varphi(\bar{y}_m)). \end{aligned} \quad \triangle$$

Remark 1.4.10. The bijections from Remark 1.4.9 yield an *equivalence of categories* between the category of affine k -varieties together with k -morphisms and the category of finitely generated reduced k -algebras together with k -algebra homomorphisms. Without defining these terms precisely we summarize the following: First, every finitely generated reduced k -algebra is the coordinate algebra of some k -variety (Corollary 1.4.6). Further, the map $*$ is a bijection, i.e. the morphisms of the varieties and the morphisms of the associated algebras correspond to each other one to one. Moreover, $*$ has the following *functorial property*: For each $p \in \text{Hom}(V, W)$ and $q \in \text{Hom}(W, X)$ we have

$$(q \circ p)^* = p^* \circ q^*$$

as well as

$$\text{id}_V^* = \text{id}_{k[V]}.$$

It follows that every problem about varieties can be translated into an equivalent problem about finitely generated reduced algebras and vice versa. In particular, we obtain the following result. \triangle

Theorem 1.4.11. *A k -morphism $p: V \rightarrow W$ is an isomorphism of varieties if and only if $p^*: k[W] \rightarrow k[V]$ is an isomorphism of k -algebras. In particular, we have*

$$V \cong W \Leftrightarrow k[V] \cong k[W].$$

Proof. Exercise 36. □

Example 1.4.12. (i) First simple examples for k -morphisms are

- inclusions of subvarieties $\mathcal{V}_V(I) \hookrightarrow V$ for $I \subseteq k[V]$.
- projections $\pi: V_1 \times V_2 \rightarrow V_1; (a, b) \mapsto a$, for affine varieties V_1, V_2 .

(ii) Let $P = \mathcal{V}(x_1^2 - x_2) \subseteq \mathbb{A}^2$ be a parabola, and $\pi: P \rightarrow \mathbb{A}^1; (a_1, a_2) \mapsto a_1$. Then π is a k -isomorphism. One can either state the inverse map $r \mapsto (r, r^2)$ directly. Or one considers

$$k[P] = k[x_1, x_2]/(x_1^2 - x_2) = k[\bar{x}_1, \bar{x}_2] = k[\bar{x}_1]$$

and $k[\mathbb{A}^1] = k[t]$, and the induced map

$$\begin{aligned} \pi^*: k[\mathbb{A}^1] &\rightarrow k[P] \\ t &\mapsto \bar{x}_1. \end{aligned}$$

Then π^* is obviously an isomorphism with inverse $\bar{x}_1 \mapsto t, \bar{x}_2 \mapsto t^2$.

(iii) Let $C = \mathcal{V}(x_1^3 - x_2^2)$ be a cuspidal curve (cf. Example 1.2.2 (iv)). Then there is a morphism

$$\begin{aligned} p: \mathbb{A}^1 &\rightarrow C \\ r &\mapsto (r^2, r^3) \end{aligned}$$

that is even bijective, as is easily verified (Exercise 37). Hence, the curve C admits a bijective *polynomial parametrization* by \mathbb{A}^1 . However, p is *not* a k -isomorphism. This is because for the inverse map we would had to take roots, which cannot be done polynomially. An exact proof makes use of the coordinate algebras. We have $k[C] = k[x_1, x_2]/(x_1^3 - x_2^2) = k[\bar{x}_1, \bar{x}_2], k[\mathbb{A}^1] = k[t]$ and

$$\begin{aligned} p^*: k[C] &\rightarrow k[\mathbb{A}^1] \\ \bar{x}_1 &\mapsto t^2 \\ \bar{x}_2 &\mapsto t^3. \end{aligned}$$

The map p^* is obviously not surjective, and hence neither p^* nor p is an isomorphism (by Theorem 1.4.11). *Thus bijectivity of a k -morphism does not imply that it is an isomorphism!* △

Theorem 1.4.13. Let $p: V \rightarrow W$ be a k -morphism between affine varieties and let $p^*: k[W] \rightarrow k[V]$ be the induced algebra homomorphism of the coordinate algebras.

(i) For every ideal $J \subseteq k[W]$ we have

$$p^{-1}(\mathcal{V}_W(J)) = \mathcal{V}_V(p^*(J)).$$

In particular, the preimage of an affine variety under a morphism is again an affine variety.

(ii) For every ideal $I \subseteq k[V]$ we have

$$\overline{p(\mathcal{V}_V(I))} = \mathcal{V}_W((p^*)^{-1}(I)).$$

In particular, we have $\mathcal{I}_W(p(V)) = \ker(p^*)$.

Proof. For $a \in V$ and $q \in k[W]$ we have $q(p(a)) = p^*(q)(a)$. This implies (i) because of

$$\begin{aligned} p(a) \in \mathcal{V}_W(J) &\Leftrightarrow q(p(a)) = 0 \forall q \in J \\ &\Leftrightarrow p^*(q)(a) = 0 \forall q \in J \\ &\Leftrightarrow a \in \mathcal{V}_V(p^*(J)). \end{aligned}$$

(ii) For $q \in k[W]$ we have

$$\begin{aligned} q \equiv 0 \text{ on } \overline{p(\mathcal{V}_V(I))} &\Leftrightarrow q \equiv 0 \text{ on } p(\mathcal{V}_V(I)) \\ &\Leftrightarrow q \circ p \equiv 0 \text{ on } \mathcal{V}_V(I) \\ &\Leftrightarrow p^*(q) \in \mathcal{I}_V(\mathcal{V}_V(I)) = \sqrt{I} \\ &\Leftrightarrow q \in (p^*)^{-1}(\sqrt{I}) = \sqrt{(p^*)^{-1}(I)} \\ &\Leftrightarrow q \equiv 0 \text{ on } \mathcal{V}_W((p^*)^{-1}(I)). \end{aligned}$$

But when two varieties have the same vanishing ideal, they are equal by Lemma 1.2.8 (iv). In particular, we get for $I = (0)$ that

$$\mathcal{I}_W(p(V)) = \mathcal{I}_W(\overline{p(V)}) = \mathcal{I}_W(\mathcal{V}_W(\ker(p^*))) = \sqrt{\ker(p^*)} = \ker(p^*),$$

because $\ker(p^*)$ is a radical ideal, since $k[V]$ is reduced. \square

Remark/Example 1.4.14. (i) Every k -morphism $p: V \rightarrow W$ between affine k -varieties is continuous with respect to the k -Zariski topology. This follows from Theorem 1.4.13 (i). In particular, an isomorphism is always a homeomorphism of the topological spaces.

(ii) A k -morphism between affine varieties can be a homeomorphism with respect to the Zariski topology *without* being an isomorphism. Example 1.4.12 (iii) demonstrates this (Exercise 37).

(iii) The image $p(V)$ of a variety under a morphism is in general neither closed nor open. For example for

$$\begin{aligned} p: \mathbb{A}^2 &\rightarrow \mathbb{A}^2 \\ (a_1, a_2) &\mapsto (a_1, a_1 a_2) \end{aligned}$$

we have

$$p(\mathbb{A}^2) = \mathbb{A}^2 \setminus \{(0, b) \mid b \neq 0\}.$$

(iv) If V is irreducible, then so is $\overline{p(V)}$. This follows either by a direct topological argument using continuity of p , or algebraically from the fact that $k[V]$ is a domain and thus $\mathcal{I}_W(\overline{p(V)}) = \ker(p^*)$ is a prime ideal. \triangle

To conclude this chapter we want to address finite varieties.

Definition 1.4.15. An ideal $I \subseteq k[\underline{x}]$ is called **0-dimensional** if

$$\dim_k k[\underline{x}]/I < \infty$$

holds. \triangle

Theorem 1.4.16. For an ideal $I \subseteq k[\underline{x}]$ the following are equivalent:

- (i) I is 0-dimensional
- (ii) $|\mathcal{V}(I)| < \infty$
- (iii) $I \cap k[x_i] \neq \{0\}$ for all $i = 1, \dots, n$.

If this is fulfilled, then $\mathcal{V}(I) \subseteq \overline{k}^n$ and $|\mathcal{V}(I)| \leq \dim_k k[\mathcal{V}(I)] \leq \dim_k k[\underline{x}]/I$.

Proof. First let $V := \mathcal{V}(I) \subseteq K^n$ be a finite set. By Hilbert's Nullstellensatz we have

$$\sqrt{I} = \bigcap_{a \in V \cap \overline{k}^n} \mathcal{I}(\{a\}).$$

Applying the fundamental theorem on homomorphisms to the evaluation map e_a yields

$$k[\underline{x}]/\mathcal{I}(\{a\}) \cong k[a_1, \dots, a_n],$$

and for $a \in \bar{k}^n$ the right-hand side is a field. Therefore $\mathcal{I}(\{a\})$ is a maximal ideal of $k[x]$ for all $a \in V \cap \bar{k}^n$. Hence

$$\sqrt{I} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$$

is a finite intersection of maximal ideals. Since the \mathfrak{m}_i are pairwise distinct and maximal, it obviously follows that $\mathfrak{m}_i + \mathfrak{m}_j = (1)$ holds for $i \neq j$. The Chinese Remainder Theorem now yields

$$k[V] = k[x]/\sqrt{I} \cong L_1 \times \cdots \times L_r,$$

where $L_i = k[x]/\mathfrak{m}_i$ is a finite algebraic field extension of k by Corollary 1.2.10. Together with Remark 1.4.5 (ii) we obtain

$$V = \text{Hom}_k(k[V], K) = \text{Hom}_k(L_1 \times \cdots \times L_r, K) = \text{Hom}_k(L_1 \times \cdots \times L_r, \bar{k}).$$

For the last equality we used that the elements of all L_i fulfill polynomial equations over k , and are thus always mapped to \bar{k} by k -algebra homomorphisms. This already shows $V \subseteq \bar{k}^n$. Because of

$$\text{Hom}_k(L_1 \times \cdots \times L_r, \bar{k}) = \bigcup_{i=1}^r \text{Hom}_k(L_i, \bar{k})$$

(Exercise 40), we now have

$$\begin{aligned} |V| &= \sum_{i=1}^r |\text{Hom}_k(L_i, \bar{k})| \\ &\leq \sum_{i=1}^r [L_i : k] \\ &= \dim_k k[V] \\ &\leq \dim_k k[x]/I. \end{aligned}$$

This proves the last statement. Now for the equivalence of (i)–(iii) let

$$\pi_i : \mathbb{A}^n \rightarrow \mathbb{A}^1$$

be the projection onto the i -th component. We have

$$\begin{aligned} V \text{ finite} &\Leftrightarrow \pi_i(V) \text{ finite for all } i = 1, \dots, n \\ &\Leftrightarrow \overline{\pi_i(V)} \text{ finite for all } i = 1, \dots, n, \end{aligned}$$

where we have used $\pi_i(V) \subseteq \bar{k}$ for the last equivalence. By Theorem 1.4.13 the variety $\overline{\pi_i(V)}$ is defined by the ideal $I \cap k[x_i]$, and thus (ii) is equivalent to (iii). For (iii) \Rightarrow (i) we use that for every i some $x_i^{d_i}$ can modulo I be replaced by lower powers of x_i . Therefore $k[\underline{x}]/I$ is finite dimensional. For (i) \Rightarrow (iii) we use that $k[x_i]/(I \cap k[x_i])$ can be embedded into $k[\underline{x}]/I$. Therefore $k[x_i]/(I \cap k[x_i])$ is finite dimensional, which can only be true if $I \cap k[x_i] \neq \{0\}$. \square

Example 1.4.17. Again note that $\mathcal{V}(I)$ must be defined in K^n and not in k^n . For example, $\mathcal{V}(x_1^2 + x_2^2)$ is not finite, since $k[x_1, x_2]/(x_1^2 + x_2^2)$ is not a finite dimensional k -vector space. In \mathbb{R}^2 however, we just see a single point of the variety. \triangle

Chapter 2

Algorithmic Aspects

The previous chapter addressed questions that we would also like to solve algorithmically. For example:

- Given $p, p_1, \dots, p_r \in k[\underline{x}]$, is it true that $p \in (p_1, \dots, p_r)$?
- Given ideals $I = (p_1, \dots, p_r), J = (q_1, \dots, q_s) \subseteq k[\underline{x}]$, find generators for the ideals $I \cap J, (I : J), \sqrt{I}$.
- Given a homomorphism $\varphi: k[\underline{x}] \rightarrow k[\underline{y}]$ and an ideal $J = (q_1, \dots, q_r) \subseteq k[\underline{y}]$, find generators for $\varphi^{-1}(J)$.

The theory of *Gröbner bases* will allow us to settle such questions algorithmically, in fact with *symbolic computations*, i.e. computations that are exact and not just numerical approximations (at least if all the input data is exact, for example polynomials over \mathbb{Q}). These algorithms are also implemented in most computer algebra systems that we use today.

2.1 Monomial Ideals

Notation 2.1.1. Let again k be a field and set $\underline{x} = (x_1, \dots, x_n)$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $c \in k$ we call

$$\underline{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

a **monomial**, and

$$c\underline{x}^\alpha$$

a **term**. We further set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

The monomials form a k -vector space basis of $k[\underline{x}]$. For

$$p = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \underline{x}^\alpha \in k[\underline{x}]$$

we define

$$\text{supp}(p) = \{\alpha \in \mathbb{N}^n \mid p_\alpha \neq 0\}.$$

Note that $\text{supp}(p)$ is a finite set for all polynomials p . The **degree** $\deg(p)$ of p is defined as

$$\deg(p) = \max \{|\alpha| \mid \alpha \in \text{supp}(p)\}.$$

The **natural partial ordering** on \mathbb{N}^n is defined by

$$\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \forall i = 1, \dots, n. \quad \triangle$$

Definition 2.1.2. An ideal $I \subseteq k[\underline{x}]$ is called **monomial**, if it is generated by monomials. △

Lemma 2.1.3. Let $M \subseteq \mathbb{N}^n$ and $I = (\underline{x}^\alpha \mid \alpha \in M)$. Then the elements of I are precisely the k -linear combinations of elements \underline{x}^β , with $\alpha \leq \beta$ for some $\alpha \in M$.

Proof. This is clear. □

Remark 2.1.4. (i) The monomials \underline{x}^β with $\alpha \leq \beta$ for some $\alpha \in M$ in fact form a k -vector space basis of I .

(ii) A general ideal $I \subseteq k[\underline{x}]$ will often not contain even a single monomial, for example $I = (x_1 + 1) \subseteq k[x_1]$. △

Theorem 2.1.5 (Dickson's Lemma). For every subset $M \subseteq \mathbb{N}^n$, the set M_{\min} of minimal elements with respect to \leq is finite.

Proof. The ideal

$$I := (\underline{x}^\alpha \mid \alpha \in M)$$

admits a finite generating set, using Hilbert's Basis Theorem. So there is $S \subseteq M$ finite with

$$\forall \beta \in M \exists \alpha \in S \quad \alpha \leq \beta.$$

Since \mathbb{N}^n does not contain infinite descending chains with respect to \leq , this is precisely the statement of the theorem. □

2.2 Monomial Orderings and Gröbner Bases

Definition 2.2.1. A **monomial ordering** is a total ordering \preceq on \mathbb{N}^n , such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$:

$$(i) \ 0 \preceq \alpha$$

$$(ii) \ \alpha \preceq \beta \Rightarrow \alpha + \gamma \preceq \beta + \gamma. \quad \triangle$$

Remark/Example 2.2.2. (i) A monomial ordering can also be understood as a total ordering of the monomials of $k[\underline{x}]$, via

$$\underline{x}^\alpha \preceq \underline{x}^\beta :\Leftrightarrow \alpha \preceq \beta.$$

Condition (ii) in Definition 2.2.1 then simply means compatibility with multiplication:

$$\underline{x}^\alpha \preceq \underline{x}^\beta \Rightarrow \underline{x}^\alpha \underline{x}^\gamma \preceq \underline{x}^\beta \underline{x}^\gamma.$$

(ii) For $n = 1$ there exists exactly one monomial ordering: $1 \prec x_1 \prec x_1^2 \prec \dots$. For $n \geq 2$ there are more, for example the **lexicographic ordering**

$$\alpha \preceq_{\text{lex}} \beta :\Leftrightarrow \alpha = \beta \text{ or } \alpha_i < \beta_i \text{ for } i = \min\{j \mid \alpha_j \neq \beta_j\}$$

or the **graded-lexicographic ordering**

$$\alpha \preceq_{\text{grlex}} \beta :\Leftrightarrow |\alpha| < |\beta| \text{ or } (|\alpha| = |\beta| \text{ and } \alpha \preceq_{\text{lex}} \beta). \quad \triangle$$

Lemma 2.2.3. Monomial orderings are well-orderings on \mathbb{N}^n , i.e. every nonempty subset has a smallest element.

Proof. Let \preceq be a monomial ordering and $\emptyset \neq M \subseteq \mathbb{N}^n$. By Theorem 2.1.5 the set M_{\min} of minimal elements with respect to \leq is finite. But $\alpha \leq \beta$ implies $\alpha \preceq \beta$, and thus the smallest element of M_{\min} with respect to \preceq is also the smallest element of M . \square

Notation 2.2.4. Let \preceq be a monomial ordering and $0 \neq p = \sum_{\alpha} p_{\alpha} \underline{x}^{\alpha} \in k[\underline{x}]$. Let $\gamma = \max_{\preceq} \text{supp}(p)$. We define

$$\text{LM}_{\preceq}(p) = \underline{x}^{\gamma} \quad \text{LC}_{\preceq}(p) = p_{\gamma} \quad \text{LT}_{\preceq}(p) = p_{\gamma} \underline{x}^{\gamma}$$

and call this the **leading monomial**, **leading coefficient** and **leading term** of p . We call p **monic**, if $\text{LC}_{\preceq}(p) = 1$. We set

$$\text{LM}_{\preceq}(0) = \text{LC}_{\preceq}(0) = \text{LT}_{\preceq}(0) = 0.$$

If the choice of the monomial ordering is clear from the context, we sometimes omit the subscript \preceq , and just write $\text{LM}(p)$ instead of $\text{LM}_{\preceq}(p)$ etc. \triangle

Lemma 2.2.5. For $p, q \in k[\underline{x}]$ we have

$$\text{LM}(pq) = \text{LM}(p) \cdot \text{LM}(q)$$

and

$$\text{LM}(p + q) \preceq \max \{ \text{LM}(p), \text{LM}(q) \},$$

where equality holds in the second equation if $\text{LM}(p) \neq \text{LM}(q)$.

Proof. Clear. □

Definition 2.2.6. Let $I \subseteq k[\underline{x}]$ be an ideal and \preceq a monomial ordering. We call

$$\text{LI}_{\preceq}(I) := (\text{LM}_{\preceq}(p) \mid p \in I)$$

the **leading ideal of I with respect to \preceq** . A monomial *not* contained in $\text{LI}_{\preceq}(I)$ is called a **standard monomial of I with respect to \preceq** . △

Remark 2.2.7. (i) $\text{LI}(I)$ is a monomial ideal. The monomials contained in it are precisely the $\text{LM}(p)$ with $p \in I$, since $\underline{x}^\beta \text{LM}(p) = \text{LM}(\underline{x}^\beta p)$.

(ii) $I = (p_1, \dots, p_r)$ implies

$$(\text{LM}(p_1), \dots, \text{LM}(p_r)) \subseteq \text{LI}(I),$$

but this is in general *not an equality!* In a representation

$$p = \sum_i q_i p_i \in I$$

the leading terms of the sum on the right can cancel. Thus from p alone we don't have control over the complexity of the q_i . For example, for $I = (p_1, p_2)$ with $p_1 = xy + 1, p_2 = y^2 - 1$ we find

$$x + y = yp_1 - xp_2 \in I,$$

and thus (depending on the monomial ordering) either x or y belongs to $\text{LI}(I)$. On the other hand we have $\text{LM}(p_1) = xy$ and $\text{LM}(p_2) = y^2$ (for every monomial ordering), and

$$x, y \notin (xy, y^2).$$

It is precisely this problem that we will solve with the notion of a Gröbner basis below. △

Theorem 2.2.8 (Macaulay). *Let $I \subseteq k[\underline{x}]$ be an ideal and \preccurlyeq a monomial ordering. Then the standard monomials of I with respect to \preccurlyeq form a k -vector space basis of $k[\underline{x}]/I$.*

Proof. Linear independence: Let $p = \sum_i p_i \underline{x}^{\alpha_i} \in I$ with $\underline{x}^{\alpha_i} \notin \text{LI}(I)$ for all i . If $p_i \neq 0$ for some i , then $\text{LM}(p)$ is one of the \underline{x}^{α_i} , and thus belongs to $\text{LI}(I)$, a contradiction.

Generating set: Denote by V the k -vector space spanned by the standard monomials of I . We will show $V + I = k[\underline{x}]$. Assume for contradiction that this fails. Choose $p \in k[\underline{x}] \setminus (V + I)$ with smallest leading monomial with respect to \preccurlyeq (using Lemma 2.2.3). Then $\text{LM}(p)$ is not a standard monomial, for $\text{LT}(p) \in V$ and thus $p - \text{LT}(p) \notin V + I$ otherwise, contradicting minimality. So we must have $\text{LM}(p) \in \text{LI}(I)$, i.e. there exists $q \in I$ with $\text{LM}(q) = \text{LM}(p)$. Now consider

$$h = p - \frac{\text{LC}(p)}{\text{LC}(q)}q.$$

We see that $h \notin V + I$, but $\text{LM}(h) \prec \text{LM}(p)$, again a contradiction. \square

Definition 2.2.9. Let $I \subseteq k[\underline{x}]$ be an ideal, \preccurlyeq a monomial ordering, and V the subspace of $k[\underline{x}]$ spanned by the standard monomials of I with respect to \preccurlyeq . By Theorem 2.2.8 for every $p \in k[\underline{x}]$ there exists a *uniquely determined* $q \in V$ with

$$p \equiv q \pmod{I}.$$

We call this q **the canonical form of p modulo I (with respect to \preccurlyeq)**, and use the notation

$$q = \text{cf}_{I, \preccurlyeq}(p).$$

Note that

$$\text{cf}_{I, \preccurlyeq}: k[\underline{x}] \rightarrow V$$

is a surjective linear map with I as its kernel. \triangle

Definition 2.2.10. Let $I \subseteq k[\underline{x}]$ be an ideal and \preccurlyeq a monomial ordering. A *finite* subset $G \subseteq I$ with $0 \notin G$ is called a **Gröbner basis of I (with respect to \preccurlyeq)**, if

$$\text{LI}_{\preccurlyeq}(I) = (\text{LM}_{\preccurlyeq}(g) \mid g \in G).$$

A finite subset $G \subseteq k[\underline{x}]$ is called a **Gröbner basis (with respect to \preccurlyeq)**, if G is a Gröbner basis of the ideal (G) . \triangle

Lemma 2.2.11. *Let $J \subseteq I$ be ideals with $\text{LI}_{\preccurlyeq}(J) = \text{LI}_{\preccurlyeq}(I)$. Then $J = I$.*

Proof. From $\text{LI}_{\preccurlyeq}(I) = \text{LI}_{\preccurlyeq}(J)$ we see that both ideals possess the same standard monomials. So the space V spanned by those standard monomials coincides for both ideals, and from $k[\underline{x}] = V \oplus I = V \oplus J$ (Theorem 2.2.8) we immediately obtain $I = J$. \square

Theorem 2.2.12. *Every ideal $I \subseteq k[\underline{x}]$ possesses a Gröbner basis (with respect to every monomial ordering), and each such Gröbner basis generates I as an ideal.*

Proof. By Hilbert's Basis Theorem, the monomial ideal $\text{LI}_{\preccurlyeq}(I)$ is generated by finitely many monomials $\text{LM}_{\preccurlyeq}(p_1), \dots, \text{LM}_{\preccurlyeq}(p_r)$ with $p_i \in I$. Then the set $G = \{p_1, \dots, p_r\}$ is clearly a Gröbner basis of I with respect to \preccurlyeq .

Now set $J = (p_1, \dots, p_r)$. Then $J \subseteq I$ is an ideal with $\text{LI}_{\preccurlyeq}(J) = \text{LI}_{\preccurlyeq}(I)$, and by Lemma 2.2.11 we get $I = J$. So G generates I . \square

Example 2.2.13. (i) Consider $I = (p_1, p_2) \subseteq k[x, y]$ with $p_1 = xy + 1, p_2 = y^2 - 1$ as in Remark 2.2.7 (ii). We have seen there that $\{p_1, p_2\}$ is not a Gröbner basis of I for any monomial ordering.

(ii) Consider $I = (p_1, p_2) \subseteq k[x, y, z]$ with $p_1 = x + z, p_2 = y + z$.

- With respect to the lexicographic monomial ordering $z \prec y \prec x$ we have

$$\text{LM}_{\preccurlyeq}(p_1) = x, \quad \text{LM}_{\preccurlyeq}(p_2) = y,$$

and thus

$$\text{LI}_{\preccurlyeq}(I) \supseteq (x, y).$$

On the other hand we have $I \cap k[z] = (0)$, which can for example be seen from $\mathcal{V}(I) = \{(t, t, -t) \mid t \in K\}$. So the leading monomial of an element from I cannot be a power of z , since the monomial ordering would imply that only z appears in the polynomial. So each leading monomial is divisible by either x or y , and this implies that $\text{LI}_{\preccurlyeq}(I) = (x, y)$. So $\{p_1, p_2\}$ is a Gröbner basis of I with respect to \preccurlyeq .

- Now let \preccurlyeq be a monomial ordering with $x \prec z, y \prec z$. Then

$$\text{LM}_{\preccurlyeq}(p_1) = \text{LM}_{\preccurlyeq}(p_2) = z.$$

From

$$x - y = p_1 - p_2 \in I$$

we see that either x or y belongs to $\text{LI}_{\preccurlyeq}(I)$. So $\{p_1, p_2\}$ is not a Gröbner basis of I with respect to \preccurlyeq . \triangle

2.3 The Buchberger Algorithm

Throughout this section let \preccurlyeq be a fixed monomial ordering on \mathbb{N}^n . All notions such as $\text{LM}(p)$, $\text{LI}(I)$, Gröbner basis, . . . refer to this ordering.

Theorem 2.3.1 (Division Algorithm). *Let $0 \neq g_1, \dots, g_s \in k[\underline{x}]$. For every $p \in k[\underline{x}]$ there exist $q_1, \dots, q_s, r \in k[\underline{x}]$ with*

$$p = q_1 g_1 + \dots + q_s g_s + r$$

and:

- (1) No monomial in r is divisible by some $\text{LM}(g_i)$
- (2) $\text{LM}(q_i g_i) \preccurlyeq \text{LM}(p)$ for all $i = 1, \dots, s$.

Proof. Write $p = \sum_{\alpha} p_{\alpha} \underline{x}^{\alpha}$. If no monomial in p is divisible by some $\text{LM}(g_i)$, we just set $q_1 = \dots = q_s = 0$ and $r = p$.

Otherwise let \underline{x}^{α} be the largest monomial in p (with respect to \preccurlyeq) which is divisible by some $\text{LM}(g_i)$, say $\underline{x}^{\alpha} = \underline{x}^{\beta} \cdot \text{LM}(g_i)$. We now set

$$q := \frac{p_{\alpha}}{\text{LC}(g_i)} \cdot \underline{x}^{\beta} \quad \text{and} \quad \tilde{p} = p - q g_i.$$

The coefficient of \underline{x}^{α} in \tilde{p} vanishes, and each monomial of \tilde{p} which is divisible by some $\text{LM}(g_i)$ is thus strictly smaller than \underline{x}^{α} (otherwise it would already appear in p).

We iterate this process, which terminates after finitely many steps (\preccurlyeq is a well ordering!). We finally obtain some polynomial r , in which no monomial is divisible by some $\text{LM}(g_i)$. Backwards substitution yields the desired identity. Note that we have

$$\text{LM}(q g_i) = \underline{x}^{\beta} \text{LM}(g_i) = \underline{x}^{\alpha} \preccurlyeq \text{LM}(p)$$

and thus

$$\text{LM}(\tilde{p}) \preccurlyeq \text{LM}(p)$$

in the first step. □

Remark 2.3.2. (i) The proof of Theorem 2.3.1 is constructive. Given p and g_1, \dots, g_s , we can find the q_i and r explicitly through the given procedure.

(ii) The polynomials q_i and r are *not* uniquely determined by the conditions (1) and (2) in Theorem 2.3.1. The division algorithm allows for choices, that this can indeed lead to different results.

- Let $n = 1$ and $g_1 = x, g_2 = x + 1$. With $p = x$ we have

$$p = g_1 + 0 = g_2 - 1.$$

Both identities fulfill (1) und (2) and can indeed arise from the division algorithm.

- Let $n = 2$ and $g_1 = xy + 1, g_2 = y^2 - 1$ as in Example 2.2.13 (i). For $p = xy^2 - x$ we have

$$p = yg_1 - (x + y) = xg_2 + 0.$$

Both identities fulfill (1) und (2) (independent of the monomial ordering) and can arise through the division algorithm. \triangle

Definition 2.3.3. A polynomial r fulfilling the conditions from Theorem 2.3.1 is called a **normal form of p modulo g_1, \dots, g_s with respect to \preccurlyeq** . This is not to be confused with the canonical form $\text{cf}_I(p)$ from Definition 2.2.9, which is uniquely determined. \triangle

Corollary 2.3.4. If $\{g_1, \dots, g_s\}$ is a Gröbner basis of I , then every normal form r of p modulo g_1, \dots, g_s coincides with the canonical form:

$$r = \text{cf}_I(p).$$

In particular, the division algorithm always leads to the same normal form r , independent of the choices made.

Proof. Let r be a normal form. Then every monomial in r is a standard monomial of I , due to property (1) in Theorem 2.3.1, and since the $\text{LM}(g_i)$ generate the leading ideal of I . On the other hand we have $p \equiv r$ modulo I . But these two properties precisely characterize $\text{cf}_I(p)$. \square

We will now develop a method to actually compute a Gröbner basis for a given ideal. We will need some preliminaries for this.

Definition 2.3.5. Let $p, q \in k[x] \setminus \{0\}$. Then

$$S(p, q) := \frac{\text{LT}(q) \cdot p - \text{LT}(p) \cdot q}{\text{gcd}(\text{LM}(p), \text{LM}(q))}$$

is called the **S -polynomial of p and q** . \triangle

Remark 2.3.6. (i) The S -polynomial is really a polynomial. The greatest common divisor of $\text{LM}(p)$ and $\text{LM}(q)$ divides the enumerator.

(ii) The leading terms in the enumerator cancel by construction, so one has

$$\text{LM}(S(p, q)) \prec \text{lcm}(\text{LM}(p), \text{LM}(q)).$$

The next lemma states that cancellation of leading monomials in linear combination is basically always due to this phenomenon in S -polynomials. \triangle

Lemma 2.3.7. Let $g_1, \dots, g_s \in k[\underline{x}] \setminus \{0\}$ all have the same leading monomial $\text{LM}(g_i) = \underline{x}^\alpha$. Let $a_1, \dots, a_s \in k$ with

$$\text{LM}\left(\sum_{i=1}^s a_i g_i\right) \prec \underline{x}^\alpha.$$

Then $\sum_i a_i g_i$ is a linear combination of the $S(g_i, g_{i+1})$, for $i = 1, \dots, s-1$.

Proof. Set $b_i := \text{LC}(g_i)$ and $p_i := \frac{1}{b_i} g_i$ for $i = 1, \dots, s$. From

$$\text{LM}\left(\sum_{i=1}^s a_i g_i\right) \prec \underline{x}^\alpha$$

we obtain

$$\sum_{i=1}^s a_i b_i = 0.$$

With $p_{s+1} = 0$ we thus get:

$$\begin{aligned} \sum_{i=1}^s a_i g_i &= \sum_{i=1}^s a_i b_i p_i \\ &= \sum_{i=1}^s \left(\sum_{j=1}^i a_j b_j \right) (p_i - p_{i+1}) \\ &= \sum_{i=1}^{s-1} \left(\sum_{j=1}^i a_j b_j \right) (p_i - p_{i+1}). \end{aligned}$$

The last equality uses $\sum_{i=1}^s a_i b_i = 0$. The claim now follows from

$$S(g_i, g_j) = \frac{b_j \underline{x}^\alpha g_i - b_i \underline{x}^\alpha g_j}{\underline{x}^\alpha} = b_j g_i - b_i g_j = b_i b_j (p_i - p_j). \quad \square$$

Theorem 2.3.8 (Buchberger Criterion for Gröbner bases). *Let $g_1, \dots, g_s \in k[\underline{x}] \setminus \{0\}$, and let h_{ij} be a normal form of $S(g_i, g_j)$ with respect to g_1, \dots, g_s , for all $i, j \in \{1, \dots, s\}$. Then $\{g_1, \dots, g_s\}$ is a Gröbner basis if and only if $h_{ij} = 0$ for all $i < j$.*

Proof. We set $I = (g_1, \dots, g_s)$. First assume that $\{g_1, \dots, g_s\}$ is a Gröbner basis. From Corollary 2.3.4 we obtain that $h_{ij} = \text{cf}_I(S(g_i, g_j))$ is the canonical form with respect to I . Since $S(g_i, g_j) \in I$ we thus have $h_{ij} = 0$ for all $i < j$.

For the other direction assume $h_{ij} = 0$ for all $i < j$. We have to show that for all $0 \neq p \in I$ there exists some $i \in \{1, \dots, s\}$ with

$$\text{LM}(g_i) \mid \text{LM}(p).$$

By assumption there exists an identity

$$p = \sum_{i=1}^s q_i g_i \quad (*)$$

with $q_i \in k[\underline{x}]$. Let

$$\underline{x}^\gamma := \max\{\text{LM}(q_i g_i) \mid i = 1, \dots, s\}.$$

We now show that if $\text{LM}(p) \prec \underline{x}^\gamma$ holds, we can find a new identity as in (*), in which γ is strictly smaller. By iteration we then end up with $\text{LM}(p) = \underline{x}^\gamma$, proving that $\text{LM}(p)$ is divisible by some $\text{LM}(g_i)$.

So assume $\text{LM}(p) \prec \underline{x}^\gamma$. After relabelling we can assume

$$\gamma_1 = \dots = \gamma_t = \gamma \succ \gamma_{t+1}, \dots, \gamma_s$$

where $\underline{x}^{\gamma_i} = \text{LM}(q_i g_i)$. We rewrite (*) as

$$p = \underbrace{\sum_{i=1}^t \text{LT}(q_i) g_i}_{\tilde{p}} + \sum_{i=1}^t (q_i - \text{LT}(q_i)) g_i + \sum_{i=t+1}^s q_i g_i.$$

Every term in \tilde{p} has \underline{x}^γ as its leading monomial, every other term has a strictly smaller leading monomial. We further have $\text{LM}(\tilde{p}) \prec \underline{x}^\gamma$, since this is true for p . So it suffices to find a representation

$$\tilde{p} = \sum_{j=1}^s \tilde{q}_j g_j \quad (**)$$

with $\text{LM}(\tilde{q}_j g_j) \prec \underline{x}^\gamma$ for all j .

To the polynomials $\text{LM}(q_i)g_i$ ($i = 1, \dots, t$) we can apply Lemma 2.3.7. So \tilde{p} is a linear combination of the

$$s_{ij} := S(\text{LM}(q_i)g_i, \text{LM}(q_j)g_j) \quad \text{for } i < j = 1, \dots, t.$$

Setting $\underline{x}^{\alpha_i} := \text{LM}(g_i)$ we get $\text{LM}(q_i) = \underline{x}^{\gamma - \alpha_i}$ and thus

$$\begin{aligned} s_{ij} &= \frac{1}{\underline{x}^\gamma} \left(\underline{x}^{\gamma - \alpha_j} \text{LT}(g_j) \underbrace{\text{LM}(q_i)g_i}_{\underline{x}^{\gamma - \alpha_i}} - \underline{x}^{\gamma - \alpha_i} \text{LT}(g_i) \underbrace{\text{LM}(q_j)g_j}_{\underline{x}^{\gamma - \alpha_j}} \right) \\ &= \underline{x}^{\gamma - \alpha_i - \alpha_j} \underbrace{(\text{LT}(g_j)g_i - \text{LT}(g_i)g_j)}_{S(g_i, g_j) \cdot \text{gcd}(\underline{x}^{\alpha_i}, \underline{x}^{\alpha_j})} \\ &= \underline{x}^{\beta_{ij}} \cdot S(g_i, g_j). \end{aligned}$$

Here

$$\underline{x}^{\beta_{ij}} = \underline{x}^{\gamma - \alpha_i - \alpha_j} \cdot \text{gcd}(\underline{x}^{\alpha_i}, \underline{x}^{\alpha_j}) = \frac{\underline{x}^\gamma}{\text{lcm}(\underline{x}^{\alpha_i}, \underline{x}^{\alpha_j})}$$

holds. Our conditions $h_{ij} = 0$ now provide identities

$$S(g_i, g_j) = \sum_{k=1}^s p_{ijk} g_k$$

with $\text{LM}(p_{ijk}g_k) \preceq \text{LM}(S(g_i, g_j))$. In total we see that \tilde{p} is a linear combination of the elements

$$\sum_{k=1}^s p_{ijk} \underline{x}^{\beta_{ij}} g_k,$$

and from $\text{LM}(S(g_i, g_j)) \prec \text{lcm}(\underline{x}^{\alpha_i}, \underline{x}^{\alpha_j})$ (by Remark 2.3.6 (ii)) we get

$$\text{LM}(p_{ijk} \underline{x}^{\beta_{ij}} g_k) = \underline{x}^{\beta_{ij}} \text{LM}(p_{ijk} g_k) \preceq \underline{x}^{\beta_{ij}} \text{LM}(S(g_i, g_j)) \prec \underline{x}^\gamma.$$

That is precisely the desired identity (**). \square

Theorem 2.3.9 (Buchberger Algorithm). *Let $g_1, \dots, g_s \in k[\underline{x}] \setminus \{0\}$. The following algorithm terminates after finitely many steps, and outputs a Gröbner basis of $I = (g_1, \dots, g_s)$:*

- Compute a normal form h_{ij} of $S(g_i, g_j)$ with respect to g_1, \dots, g_s , for all $1 \leq i < j \leq s$.

- If $h_{ij} = 0$ for all i, j , terminate and output $\{g_1, \dots, g_s\}$.
- If $h_{ij} \neq 0$ for some i, j , add it to the g_i and start again.

Proof. We have $S(g_i, g_j) \in I$ for all i, j . This implies $h_{ij} \in I$ for all i, j , and thus I is not enlarged by adding the h_{ij} . Upon termination of the algorithm we obtain a Gröbner basis of I , by Theorem 2.3.8. What remains to show is that the algorithm does indeed terminate after finitely many steps.

If $h_{ij} \neq 0$ then

$$(\text{LM}(g_1), \dots, \text{LM}(g_s)) \subsetneq (\text{LM}(g_1), \dots, \text{LM}(g_s), \text{LM}(h_{ij})),$$

since none of the monomials in h_{ij} is divisible by some $\text{LM}(g_i)$. By Hilbert's Basis Theorem this can happen only finitely many times. \square

Remark 2.3.10. (i) All steps in the Buchberger Algorithm are constructive. The $S(g_i, g_j)$ are explicitly given, and a normal form h_{ij} can be computed with the division algorithm (Theorem 2.3.1). So if the input data is accurately representable in a computer (for example if all polynomials have rational coefficients), a computer can compute a Gröbner basis accurately.

(ii) The above described algorithm will in general produce a very redundant Gröbner basis.

(iii) If G is a Gröbner basis of I and $p, q \in G$ with $p \neq q$ and $\text{LM}(p) \mid \text{LM}(q)$, then $G \setminus \{q\}$ is also a Gröbner basis of I . \triangle

Definition 2.3.11. A Gröbner basis G is called **minimal**, if

$$\text{LM}(p) \nmid \text{LM}(q)$$

for all $p, q \in G, p \neq q$. \triangle

Lemma 2.3.12. Let G be a minimal Gröbner basis and $I = (G)$. Then the $\text{LM}(g)$ ($g \in G$) are exactly the different minimal monomials (with respect to \leq) in $\text{LI}(I)$.

Proof. Every minimal monomial (with respect to \leq) in $\text{LI}(I)$ must be of the form $\text{LM}(g)$ for some $g \in G$. On the other hand, a minimal Gröbner basis can only provide these monomials. \square

Remark 2.3.13. Any two minimal Gröbner bases (generating the same ideal I) have the same cardinality. They are minimal with respect to inclusion, and also of smallest cardinality. \triangle

Definition 2.3.14. A Gröbner basis G is called **reduced**, if all $g \in G$ are monic, and for $p \neq q \in G$ we have :

$$\text{LM}(p) \text{ divides no monomial from } q.$$

Obviously, every reduced Gröbner basis is also minimal. △

Theorem 2.3.15. Every ideal $I \subseteq k[x]$ admits a unique reduced Gröbner basis (with respect to the fixed monomial ordering \preceq).

Proof. Existence: Let G be a minimal Gröbner basis of I . We call an element $g \in G$ *reduced with respect to G* , if no monomial of g is divisible by some $\text{LM}(q)$ for $q \in G \setminus \{g\}$. Now choose and fix some $g \in G$ and compute a normal form g' of g with respect to $G \setminus \{g\}$. Then set

$$G' := (G \setminus \{g\}) \cup \{g'\}.$$

From $0 \neq g' \in I$ and $\text{LM}(q) \nmid \text{LM}(g')$ for all $q \in G \setminus \{g\}$ we get $\text{LM}(g) = \text{LM}(g')$. So G' is again a minimal Gröbner basis of I . Now g' is reduced with respect to G' , since it is a normal form with respect to $G \setminus \{g\}$. By iteration we obtain a reduced Gröbner basis.

Uniqueness: Let G, G' be two reduced Gröbner bases of I . By minimality the sets $\text{LM}(G)$ and $\text{LM}(G')$ coincide, using Lemma 2.3.12. For $g \in G$ let $g' \in G'$ be the unique element with $\text{LM}(g) = \text{LM}(g')$. Then $g - g' \in I$, and if $g - g' \neq 0$ we find $q \in G, q' \in G'$ with

$$\text{LM}(q), \text{LM}(q') \mid \text{LM}(g - g').$$

On the other hand we have $\text{LM}(g - g') \prec \text{LM}(g) = \text{LM}(g')$, which implies $q \neq g, q' \neq g'$. Since $\text{LM}(g - g')$ is a monomial of either g or g' , this contradicts reducedness of either G or G' . Thus we have $g - g' = 0$, which implies $G = G'$. □

Remark/Example 2.3.16. (i) The computation of a reduced Gröbner basis (from a given Gröbner basis) of I in the last proof is constructive.

(ii) Let $g_1, \dots, g_s \in k[x]$ be linear forms, $g_i = \sum_{j=1}^n c_{ij}x_j$ say. Let

$$A = (c_{ij})_{i,j} \in M_{s \times n}(k)$$

be the matrix of coefficients and

$$B = (b_{ij})_{i,j} \in M_{s \times n}(k)$$

its reduced row-echelon form (pivots are 1, pivotal columns otherwise 0). Let $q_i = \sum_{j=1}^n b_{ij}x_j$ for $i = 1, \dots, r (= \text{rank}(A))$. If \preccurlyeq is a monomial ordering with $x_1 \succ x_2 \succ \dots \succ x_n$, then $\{q_1, \dots, q_r\}$ is the reduced Gröbner basis of (g_1, \dots, g_s) (Exercise 56). The Buchberger Algorithm (+ the construction of reduced Gröbner bases) thus generalizes the Gaussian Algorithm from linear algebra.

(iii) Let $n = 1$ and $p, q \in k[x] \setminus \{0\}$. The only minimal Gröbner basis of

$$(p, q) = (\text{gcd}(p, q))$$

is $\{\text{gcd}(p, q)\}$, up to scaling. The Buchberger Algorithm thus generalizes the Euclidean Algorithm for $k[x]$. \triangle

2.4 Applications

In this section we see how the theory of Gröbner bases gives constructive answers to the questions from the beginning of this chapter. We emphasize once more that computation of (reduced) Gröbner bases and the division algorithm are constructive.

Application 2.4.1 (Membership in an ideal). Let $p, p_1, \dots, p_s \in k[x]$ be given. The question whether p belongs to $I = (p_1, \dots, p_s)$ can be solved as follows. Choose an arbitrary monomial ordering \preccurlyeq . Then compute a Gröbner basis $\{g_1, \dots, g_t\}$ of I with respect to \preccurlyeq and a normal form r of p modulo g_1, \dots, g_t . We know that $r = \text{cf}_{I, \preccurlyeq}(p)$ holds, and thus

$$p \in I \Leftrightarrow r = 0.$$

This approach also provides a representation $p = \sum_{i=1}^s q_i p_i$, in case $p \in I$. First we obtain a representation

$$p = \sum_{i=1}^t \tilde{q}_i g_i$$

if $r = 0$ in the division algorithm. However, in the construction of the Gröbner basis $\{g_1, \dots, g_t\}$ we might have added some h_{ij} to the p_i (see Theorem 2.3.9). But since the $S(p_i, p_j)$ are explicit combinations of the p_i , so are the h_{ij} . After iterative substitutions we thus obtain an explicit representation

$$p = \sum_{i=1}^s q_i p_i. \quad \triangle$$

Application 2.4.2 (Solvability of a system of polynomial equations). In Corollary 1.2.13 we have seen that a system of equations

$$p_1(x) = 0, \dots, p_s(x) = 0$$

with $p_i \in k[x]$ has a common solution in an algebraically closed field extension of k , if and only if

$$1 \notin (p_1, \dots, p_s) \subseteq k[x].$$

We can thus use Application 2.4.1 to decide solvability. In fact 1 belongs to the ideal if and only if the Gröbner basis contains 1. This follows directly from the definition of a Gröbner basis. \triangle

Application 2.4.3 (Membership in the radical). Checking whether

$$p \in \sqrt{(p_1, \dots, p_s)}$$

holds is also possible with Gröbner bases (see Exercise 59). \triangle

Application 2.4.4 (Containment and equality of ideals). One has

$$I = (p_1, \dots, p_s) \subseteq (q_1, \dots, q_t) = J$$

if and only if $p_i \in J$ for all i . From Application 2.4.1 we know how to check this. In particular we can also check $I = J$. \triangle

Application 2.4.5 (Elimination). Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. The ideal

$$I_j := I \cap k[x_{j+1}, \dots, x_n]$$

is called the j -**th elimination ideal** of I . Its geometric meaning has been studied in Theorem 1.4.13 (ii): It defines the closure of the projection of $\mathcal{V}(I)$ onto the coordinates x_{j+1}, \dots, x_n . The following theorem says how to compute generators of I_j . \triangle

Theorem 2.4.6. *Let G be a Gröbner basis of I with respect to the lexicographic monomial ordering with $x_1 \succ \dots \succ x_n$. Then $G \cap k[x_{j+1}, \dots, x_n]$ is a Gröbner basis of I_j with respect to this monomial ordering, and in particular it generates I_j .*

Proof. Let $p \in I_j$ be fixed. From $p \in I$ we know that there is some $g \in G$ with $\text{LM}(g) \mid \text{LM}(p)$. Since x_1, \dots, x_j do not appear in p , they also do not appear in $\text{LM}(g)$ and thus not in g , by the choice of the monomial ordering. So $g \in G \cap k[x_{j+1}, \dots, x_n]$. \square

Example 2.4.7. Consider

$$I = (x^2 + y + z - 1, y^2 + x + z - 1, z^2 + x + y - 1) \subseteq k[x, y, z].$$

The reduced Gröbner basis of I with respect to the lexicographic monomial ordering with $x \succ y \succ z$ consists of the following 4 polynomials:

$$\begin{aligned} g_1 &= x + y + z^2 - 1 \\ g_2 &= y^2 - y - z^2 + z \\ g_3 &= 2yz^2 + z^4 - z^2 \\ g_4 &= z^6 - 4z^4 + 4z^3 - z^2 = z^2(z - 1)^2(z^2 + 2z - 1). \end{aligned}$$

Thus

$$I \cap k[y, z] = (g_2, g_3, g_4)$$

and

$$I \cap k[z] = (g_4).$$

In particular, the projection of $\mathcal{V}(I)$ onto the third coordinate consists of 0, 1 and $-1 \pm \sqrt{2}$. \triangle

Application 2.4.8 (Finite varieties). Let $I \subseteq k[x]$ be an ideal. We can check whether $\mathcal{V}(I)$ is finite, i.e. whether I is a 0-dimensional ideal (see Theorem 1.4.16). To this end we compute generators for the ideals $I \cap k[x_i]$, as demonstrated in Theorem 2.4.6. We then just check whether $I \cap k[x_i] \neq \{0\}$ holds for all i . In case this is true, consider $M_i := \mathcal{V}(I \cap k[x_i]) \subseteq \mathbb{A}^1$. Then

$$\mathcal{V}(I) \subseteq M_1 \times \cdots \times M_n,$$

and we can directly check the finitely many elements from $M_1 \times \cdots \times M_n$ for membership in $\mathcal{V}(I)$. \triangle

Example 2.4.9. With the equations from Example 2.4.7 we obtain

$$I \cap k[x] = (g_4(x)), \quad I \cap k[y] = (g_4(y)) \quad \text{and} \quad I \cap k[z] = (g_4(z)).$$

So $\mathcal{V}(I)$ is indeed finite, and contained in

$$\{0, 1, -1 \pm \sqrt{2}\}^3 \subseteq K^3.$$

We then find

$$\mathcal{V}(I) = \left\{ (-1 \pm \sqrt{2}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \quad \triangle$$

Application 2.4.10 (Inverse images of ideals). Let $\varphi: k[\underline{x}] \rightarrow k[\underline{y}]$ be a homomorphism of k -algebras, given by $\varphi(x_i) = p_i \in k[\underline{y}]$ for $i = 1, \dots, n$. Let $J = (g_1, \dots, g_s) \subseteq k[\underline{y}]$ be an ideal. We want to compute generators for $\varphi^{-1}(J)$, see Theorem 1.4.13 for the geometric interpretation. The following theorem shows how to reduce this to elimination. \triangle

Theorem 2.4.11. Let $\tilde{J} := (x_1 - p_1, \dots, x_n - p_n, g_1, \dots, g_s) \subseteq k[\underline{x}, \underline{y}]$. Then

$$\varphi^{-1}(J) = \tilde{J} \cap k[\underline{x}].$$

Proof. For every (commutative) ring R and $a \in R^m$ the evaluation map

$$\begin{aligned} R[z_1, \dots, z_m] &\rightarrow R \\ p &\mapsto p(a_1, \dots, a_m) \end{aligned}$$

has kernel $(z_1 - a_1, \dots, z_m - a_m)$. This can be seen easiest in case $a = 0$, the general case follows with a linear transformation. So the homomorphism

$$\begin{aligned} \psi: k[\underline{x}, \underline{y}] &\rightarrow k[\underline{y}] \\ x_i &\mapsto p_i \\ y_i &\mapsto y_i \end{aligned}$$

has kernel $(x_1 - p_1, \dots, x_n - p_n)$. We also have $\psi = \varphi$ on $k[\underline{x}]$. This implies

$$\varphi^{-1}(J) = \psi^{-1}(J) \cap k[\underline{x}]$$

and the claim now follows from the following identity in $k[\underline{x}, \underline{y}]$:

$$\psi^{-1}(J) = (g_1, \dots, g_s) + \ker(\psi).$$

Here " \supseteq " is obvious. For " \subseteq " we use that for all $p \in k[\underline{x}, \underline{y}]$ we have

$$\psi(p - \psi(p)) = \psi(p) - \psi(p) = 0.$$

So $p \in \psi^{-1}(J)$ implies

$$p = \underbrace{\psi(p)}_{\in J} + \underbrace{(p - \psi(p))}_{\in \ker(\psi)} \in (g_1, \dots, g_s) + \ker(\psi). \quad \square$$

Example 2.4.12. Consider the following morphism of varieties:

$$\begin{aligned} p: \mathbb{A}^2 &\rightarrow \mathbb{A}^3 \\ (a, b) &\mapsto (ab, ab^2, a^2). \end{aligned}$$

On the level of coordinate rings it corresponds to the following homomorphism:

$$\begin{aligned} p^*: k[x, y, z] &\rightarrow k[u, v] \\ x &\mapsto uv \\ y &\mapsto uv^2 \\ z &\mapsto u^2. \end{aligned}$$

With the above described method we can compute

$$\ker(p^*) = (x^4 - y^2z).$$

By Theorem 1.4.13 we thus have

$$\overline{p(\mathbb{A}^2)} = \mathcal{V}(x^4 - y^2z).$$

We can see that the full y -axis is contained in $\overline{p(\mathbb{A}^2)}$, the image of p however only contains its point $(0, 0, 0)$. \triangle

Application 2.4.13 (Intersection of ideals). Let $I = (p_1, \dots, p_s)$ and $J = (q_1, \dots, q_r)$ be ideals in $k[\underline{x}]$. Generators for the product IJ are easy to find: just take the pairwise products p_iq_j . It is harder to compute generators for the intersection $I \cap J$. The following Theorem shows how to do that with elimination (already settled in Application 2.4.5). \triangle

Theorem 2.4.14. Let t be a new variable and Q the ideal generated by

$$tp_1, \dots, tp_s, (1-t)q_1, \dots, (1-t)q_r$$

in $k[\underline{x}, t]$. Then

$$I \cap J = Q \cap k[\underline{x}].$$

Proof. For " \subseteq " let $p \in I \cap J$. From $p = tp + (1-t)p$ we get $p \in Q$.

For " \supseteq " let

$$k[\underline{x}] \ni p = t \sum_i h_i p_i + (1-t) \sum_j g_j q_j \in Q,$$

where $h_i, g_j \in k[\underline{x}, t]$. Substituting $t = 0$ we obtain $p \in J$, substituting $t = 1$ yields $p \in I$. \square

Application 2.4.15 (Homogenization of an ideal). For $I \subseteq k[x_1, \dots, x_n]$ we let

$$I^h = (p^h \mid p \in I) \subseteq k[x_0, \dots, x_n]$$

be its homogenization (see Theorem 3.3.18 for the geometric interpretation). If $I = (p_1, \dots, p_s)$, then in general we will have

$$(p_1^h, \dots, p_s^h) \subsetneq I^h,$$

see Remark 3.3.19. We now want to compute generators for I^h . To this end, we call a monomial ordering \preceq **degree compatible**, if

$$|\alpha| < |\beta| \Rightarrow \alpha \prec \beta$$

for all $\alpha, \beta \in \mathbb{N}^n$. For example, the graded-lexicographic monomial ordering is degree compatible. The following Theorem shows how to compute generators for I^h . \triangle

Theorem 2.4.16. *Let \preceq be a degree compatible monomial ordering on \mathbb{N}^n and G a Gröbner basis of the ideal $I \subseteq k[x_1, \dots, x_n]$ with respect to \preceq . Then in $k[x_0, \dots, x_n]$ we have*

$$I^h = (g^h \mid g \in G).$$

Proof. On $k[x_0, \dots, x_n]$ we define a monomial ordering by

$$x_0^r \cdot \underline{x}^\alpha \preceq' x_0^s \underline{x}^\beta \Leftrightarrow \underline{x}^\alpha \prec \underline{x}^\beta \text{ or } (\alpha = \beta \text{ and } r \leq s).$$

Let $G^h = \{g^h \mid g \in G\}$. We show that G^h is even a Gröbner basis of I^h with respect to \preceq' , this will imply the statement.

For $0 \neq p \in k[x_1, \dots, x_n]$ we have

$$\deg(\text{LM}_{\preceq}(p)) = \deg(p),$$

due to degree compatibility of \preceq . This immediately implies

$$\text{LM}_{\preceq'}(p^h) = \text{LM}_{\preceq}(p)$$

for all $p \in k[x_1, \dots, x_n]$. By Lemma 3.3.16 (i) every homogeneous $q \in I^h$ is of the form

$$q = x_0^r \cdot p^h$$

for some $p \in I$. Thus

$$\text{LM}_{\preceq'}(q) = x_0^r \cdot \text{LM}_{\preceq'}(p^h).$$

Now there exists some $g \in G$ with

$$\text{LM}_{\preceq'}(g^h) = \text{LM}_{\preceq}(g) \mid \text{LM}_{\preceq}(p) = \text{LM}_{\preceq'}(p^h),$$

since G is a Gröbner basis of I . This implies $\text{LM}_{\preceq'}(g^h) \mid \text{LM}_{\preceq'}(q)$, what was to be shown. \square

Chapter 3

Projective Varieties

3.1 Projective Spaces

Definition 3.1.1. Let K be a field and V a K -vector space.

(i) The **projective space** $\mathbb{P}(V)$ is the set of all one-dimensional K -subspaces of V .

(ii) The **dimension** of a projective space is defined as

$$\dim \mathbb{P}(V) := \dim_K(V) - 1.$$

(iii) If $W \subseteq V$ is a K -subspace, then $\mathbb{P}(W)$ is called **linear or projective subspace** of $\mathbb{P}(V)$. Linear subspaces of $\mathbb{P}(V)$ of dimension $0, 1, 2, \dim \mathbb{P}(V) - 1$ are called **(projective) points, lines, planes, and hyperplanes**, respectively.

(iv) We write

$$\mathbb{P}^n(K) := \mathbb{P}(K^{n+1})$$

and call $\mathbb{P}^n(K)$ the n -**dimensional projective space over K** . \triangle

The elements of $\mathbb{P}(V)$ are the sets $[v] = K \cdot v$ for $0 \neq v \in V$. Here we have that

$$[v] = [w] \Leftrightarrow \exists c \in K^* \quad v = cw.$$

Definition 3.1.2. Elements of $\mathbb{P}^n(K)$ are often denoted in **homogeneous coordinates**. For $0 \neq (a_0, \dots, a_n) \in K^{n+1}$ we write

$$(a_0 : \dots : a_n) := [(a_0, \dots, a_n)] = K \cdot (a_0, \dots, a_n) \in \mathbb{P}^n(K).$$

The colon notation indicates that not the vector but rather the line spanned by the vector is meant to be the *element* of the projective space. Note that the point $(0 : \dots : 0)$ is *not defined*. We have

$$(a_0 : \dots : a_n) = (b_0 : \dots : b_n) \Leftrightarrow \exists c \in K^* \quad a_i = cb_i \text{ for } i = 0, \dots, n.$$

One can also use the following characterization:

$$(a_0 : \dots : a_n) = (b_0 : \dots : b_n) \Leftrightarrow a_i b_j = a_j b_i \text{ for all } i, j = 0, \dots, n.$$

This is because two nontrivial vectors $(a_0, \dots, a_n), (b_0, \dots, b_n)$ are collinear if and only if the matrix

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{pmatrix}$$

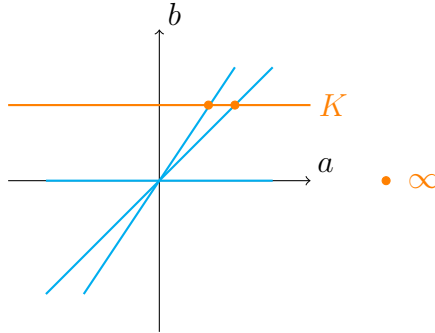
has rank 1, i.e. if all 2×2 -minors vanish. △

Example 3.1.3. (i) $\mathbb{P}^0(K)$ consists of exactly one point.

(ii) $\mathbb{P}^1(K)$ can be identified with $K \cup \{\infty\}$ via

$$\begin{aligned} \mathbb{P}^1(K) &\rightarrow K \cup \{\infty\} \\ (a : b) &\mapsto a/b \text{ if } b \neq 0 \\ (a : 0) &\mapsto \infty. \end{aligned}$$

Geometrically this corresponds to intersecting lines through the origin in K^2 with the line $b = 1$. Every line through the origin corresponds to exactly one point on the line $b = 1$, except for the line $b = 0$, which corresponds to ∞ .



(iii) Regarded as a set we have the identification $\mathbb{P}^n(\mathbb{R}) = S^n / \sim$, where S^n is the unit sphere in \mathbb{R}^{n+1} and $a \sim -a$ holds for all $a \in S^n$. In terms of differential geometry this is a compact, smooth, and for $n \geq 2$ non-orientable manifold of dimension n .

(iv) $\emptyset, \mathbb{P}(V)$ are linear subspaces of $\mathbb{P}(V)$. We have $\dim \emptyset = -1$ because $\emptyset = \mathbb{P}(\{0\})$. Every intersection of linear subspaces is again a linear subspace:

$$\bigcap_i \mathbb{P}(W_i) = \mathbb{P}\left(\bigcap_i W_i\right).$$

A line in $\mathbb{P}(V)$ is of the form $\mathbb{P}(W)$ for a 2-dimensional subspace $W \subseteq V$. △

Theorem 3.1.4. Let $U_1 = \mathbb{P}(W_1)$ and $U_2 = \mathbb{P}(W_2)$ be linear subspaces of $\mathbb{P}(V)$. Then we have

$$\dim U_1 + \dim U_2 = \dim(U_1 \cap U_2) + \dim \mathbb{P}(W_1 + W_2).$$

In particular, from $\dim U_1 + \dim U_2 \geq \dim \mathbb{P}(V)$ it already follows that $U_1 \cap U_2 \neq \emptyset$. Hence any two lines in $\mathbb{P}^2(K)$ intersect.

Proof. Follows immediately from the dimension formula for vector spaces

$$\dim_K W_1 + \dim_K W_2 = \dim_K(W_1 \cap W_2) + \dim_K(W_1 + W_2)$$

when we subtract 2 on both sides. \square

Definition 3.1.5. Let $f: V \hookrightarrow W$ be an injective linear map. Then

$$\begin{aligned} \mathbb{P}(f): \mathbb{P}(V) &\rightarrow \mathbb{P}(W) \\ [v] &\mapsto [f(v)] \end{aligned}$$

is a well-defined map. If f is bijective, then $\mathbb{P}(f)^{-1} = \mathbb{P}(f^{-1})$ and $\mathbb{P}(f)$ is called a **projectivity from $\mathbb{P}(V)$ to $\mathbb{P}(W)$** . \triangle

Theorem 3.1.6. The projectivities from $\mathbb{P}(V)$ to itself form a group, which is isomorphic to

$$\mathrm{PGL}(V) := \mathrm{GL}(V) / (K^* \cdot \mathrm{id}_V).$$

Proof. Let P be the group of projectivities. Consider the group homomorphism

$$\begin{aligned} \mathrm{GL}(V) &\twoheadrightarrow P \\ f &\mapsto \mathbb{P}(f). \end{aligned}$$

The kernel consists of those $f \in \mathrm{GL}(V)$ for which every vector is an eigenvector. It is well-known that these are precisely the multiples of the identity (see Exercise 64). \square

Example 3.1.7. For $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$ we have

$$\mathbb{P}(f): (x : y) \mapsto (ax + by : cx + dy).$$

Under the identification $\mathbb{P}^1(K) = K \cup \{\infty\}$ from Example 3.1.3 (ii) this translates to the map

$$r \mapsto \frac{ar + b}{cr + d}, \quad \infty \mapsto \frac{a}{c},$$

a so called **Möbius transformation** of the line. \triangle

Construction 3.1.8. In $\mathbb{P}^n(K)$ the set

$$H := \{(a_0 : \dots : a_n) \mid a_0 = 0\}$$

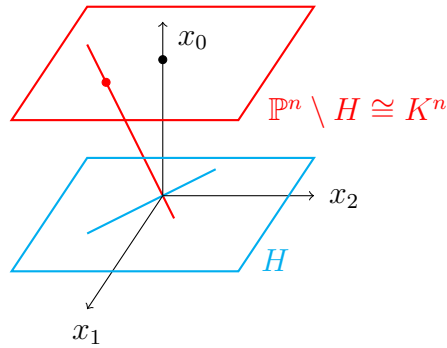
is a hyperplane, and in particular we have $H \cong \mathbb{P}^{n-1}(K)$. There is a bijection from the complement

$$\begin{aligned} \mathbb{P}^n(K) \setminus H &\leftrightarrow K^n \\ (a_0 : \dots : a_n) &\mapsto \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0} \right) \\ (1 : b_1 : \dots : b_n) &\leftarrow (b_1, \dots, b_n). \end{aligned}$$

More generally, for every hyperplane $W \subseteq V$ there is a bijection

$$\begin{aligned} W &\rightarrow \mathbb{P}(V) \setminus \mathbb{P}(W) \\ w &\mapsto [v + w], \end{aligned}$$

where $v \in V \setminus W$ is arbitrary but fixed (see Exercise 65). From this point of view $\mathbb{P}(V)$ is a disjoint union of W and $\mathbb{P}(W)$. The points in $\mathbb{P}(W)$ are the **points at infinity** with respect to W .



△

3.2 Graded Rings

In this chapter we are going to discuss the underlying algebra of projective spaces. For this purpose always let $(G, +, 0)$ be an abelian group.

Definition 3.2.1. (i) A **G -graded ring** is a ring R together with additive subgroups $R_g \subseteq R$ for every $g \in G$, such that

$$R = \bigoplus_{g \in G} R_g$$

$$\text{and } R_g \cdot R_h \subseteq R_{g+h}$$

for all $g, h \in G$.

(ii) An element $a \in R$ is called **homogeneous** if $a \in R_g$ for some $g \in G$. In case $a \neq 0$ we call $\deg(a) := g$ the **degree** of a . We set $\deg(0) := -\infty$.

(iii) Every element $a \in R$ has a unique decomposition

$$a = \sum_{g \in G} a_g,$$

where a_g is homogeneous of degree g , and just finitely many a_g are $\neq 0$. With this notation we call a_g the **homogeneous component of degree g of a** .

(iv) If R, S are two G -graded rings, then a ring homomorphism $\varphi: R \rightarrow S$ is called **graded** if

$$\varphi(R_g) \subseteq S_g$$

holds for all $g \in G$. △

Example 3.2.2. (i) Every ring R can be trivially G -graded by $R_0 := R$ and $R_g := \{0\}$ for $g \neq 0$.

(ii) On $R = k[\underline{x}]$ a \mathbb{Z} -grading is uniquely defined by

$$k \subseteq R_0 \text{ und } \deg(x_i) = 1.$$

Here R_d consists precisely of the polynomials

$$p = \sum_{|\alpha|=d} p_\alpha \underline{x}^\alpha,$$

and $R_d = \{0\}$ for $d < 0$. This grading is also called the **standard grading** of $k[\underline{x}]$.

(iii) More generally we can prescribe $k \subseteq R_0$ and $\deg(x_i) = d_i \in \mathbb{Z}$, and obtain a unique such grading on $R = k[\underline{x}]$, called a **weighted degree-grading**.

(iv) For $R = k[\underline{x}]$ there is for instance also the \mathbb{Z}^n -grading with $R_\alpha = k \cdot \underline{x}^\alpha$ for $\alpha \in \mathbb{N}^n$ and $R_\beta = \{0\}$ for $\beta \in \mathbb{Z}^n \setminus \mathbb{N}^n$. △

Lemma 3.2.3. For every G -graded ring R we have $1 \in R_0$, and R_0 is a subring of R .

Proof. Write

$$1 = \sum_{g \in G} e_g$$

with $e_g \in R_g$. For homogeneous $a \in R_h$ we then have

$$a = 1 \cdot a = \sum_g e_g a$$

and from $e_g a \in R_{g+h}$ it follows that $e_0 a = a$ holds. Here we used the uniqueness of the decomposition into homogeneous summands. This implies $e_0 a = a$ for all $a \in R$, and hence $e_0 = 1$. \square

Remark 3.2.4. In graded rings computation are sometimes easier than in ungraded rings. For example, below we are often going to use the following fact: If

$$a = \sum_i b_i c_i$$

and both a and all c_i are homogeneous, then the b_i can be assumed as homogeneous as well, with

$$\deg(b_i) = \deg(a) - \deg(c_i).$$

This is because we can replace every b_i by its homogeneous component of degree $\deg(a) - \deg(c_i)$ and the equation will still hold. \triangle

Lemma 3.2.5. *Let R be a G -graded ring and $I \subseteq R$ an ideal. Then the following conditions on I are equivalent:*

- (i) $a \in I \Rightarrow a_g \in I$ for all $g \in G$
- (ii) $I = \bigoplus_{g \in G} (I \cap R_g)$
- (iii) I is generated by homogeneous elements.

Proof. Exercise 67. \square

Definition 3.2.6. An ideal I which satisfies the conditions in Lemma 3.2.5 is called **homogeneous ideal**. \triangle

Example 3.2.7. (i) Let $\varphi: R \rightarrow S$ be a graded homomorphism. Then $\ker(\varphi)$ is a homogeneous ideal. More generally, $\varphi^{-1}(J)$ is a homogeneous ideal for every homogeneous ideal $J \subseteq S$.

(ii) If R is \mathbb{Z} -graded with $R_d = \{0\}$ for $d < 0$, then

$$R_+ = \bigoplus_{d \geq 1} R_d$$

is a homogeneous ideal. R_+ is called the **irrelevant ideal**. If the ring R_0 is a field, then every proper homogeneous ideal is contained in R_+ . In order to give the set of proper homogeneous ideals multiple maximal elements, R_+ is often disregarded.

(iii) With respect to the grading in Example 3.2.2 (iv) the homogeneous ideals are precisely the ideals generated by monomials (cf. Definition 2.1.2). \triangle

Lemma 3.2.8. *Let R be a G -graded ring.*

(i) *Sums, products, and intersection of homogeneous ideals in R are again homogeneous.*

(ii) *If I is a homogeneous ideal, then R/I becomes G -graded via the grading*

$$(R/I)_g := R_g/I$$

for $g \in G$.

Proof. (i) is clear, for example with the property (i) in Lemma 3.2.5. For (ii) first note that the R_g/I are additive subgroups of R/I . It is also clear that $R_g/I \cdot R_h/I \subseteq R_{g+h}/I$ as well as $R/I = \sum_g R_g/I$. It remains to show that the sum is direct. So let $a_g \in R_g$ such that

$$\sum_g \bar{a}_g = 0 \text{ in } R/I.$$

This means $\sum_g a_g \in I$, and from the homogeneity of I it follows $a_g \in I$ for all g . This implies $\bar{a}_g = 0$ for all g . \square

Remark 3.2.9. Let I be a homogeneous ideal of the graded ring R , and let R/I be equipped with the grading just introduced. Then one can easily show that the homogeneous ideals of R/I exactly correspond to the homogeneous ideals J of R with $I \subseteq J$. \triangle

From now on we assume that G is an **ordered abelian group**, i.e. there is a total ordering \leq on G such that

$$a \leq b \Rightarrow a + c \leq b + c$$

for all $a, b, c \in G$. Almost always we will have $G = \mathbb{Z}$ anyway, and in this case such an ordering exists. Our ring R from now on shall always be G -graded.

Lemma 3.2.10. *Let I be a proper homogeneous ideal in R . Then I is a prime ideal if and only if for all homogeneous $a, b \in R$ we have*

$$ab \in I \Rightarrow a \in I \text{ or } b \in I.$$

Proof. One implication is trivial. Now let $a, b \in R$, not necessarily homogeneous, with $a, b \notin I$. Further let $g, h \in G$ be the maximal indices with respect to \leq with $a_g \notin I, b_h \notin I$. Then we have

$$(ab)_{g+h} = a_g b_h + \underbrace{\cdots}_{\in I}$$

because for $g' \neq g, h' \neq h$ with $g' + h' = g + h$ either $g < g'$ or $h < h'$ must hold. From the assumption it follows that $a_g b_h \notin I$, and hence also $(ab)_{g+h} \notin I$. Since I is homogeneous, we obtain $ab \notin I$. \square

Lemma 3.2.11. *All minimal prime ideals over the homogeneous ideal I are also homogeneous.*

Proof. Exercise 69. \square

Corollary 3.2.12. *For every homogeneous ideal I of R , the radical \sqrt{I} is the intersection of all homogeneous prime ideals over I . In particular, \sqrt{I} is itself homogeneous.*

Lemma 3.2.13. *Let R be a G -graded ring and $M \subseteq R$ a multiplicative subset consisting of homogeneous elements. Then the localization $M^{-1}R$ becomes a G -graded ring via the grading*

$$(M^{-1}R)_g := \left\{ \frac{a}{m} \mid m \in M, a \in R_{\deg(m)+g} \right\}.$$

Proof. Obviously $(M^{-1}R)_g$ is an additive subgroup of $M^{-1}R$, and we have $(M^{-1}R)_g \cdot (M^{-1}R)_h \subseteq (M^{-1}R)_{g+h}$ as well as

$$M^{-1}R = \sum_g (M^{-1}R)_g.$$

It remains to show that the sum is direct. So let

$$0 = \sum_g \frac{a_g}{m_g} = \frac{\sum_g a_g \prod_{h \neq g} m_h}{\prod_h m_h} \text{ in } M^{-1}R,$$

be a finite sum with $a_g \in R$ homogeneous, $m_g \in M$, and

$$\deg(a_g) = \deg(m_g) + g.$$

Then there is an $m \in M$ such that

$$m \cdot \left(\sum_g a_g \prod_{h \neq g} m_h \right) = 0 \text{ in } R.$$

Here the summand with index g is homogeneous of degree

$$\deg(m) + g + \deg\left(\prod_h m_h\right).$$

Since these are different degrees for different g , it follows that

$$m \cdot a_g \cdot \prod_{h \neq g} m_h = 0$$

for all g , and this implies $\frac{a_g}{m_g} = 0$ in $M^{-1}R$ for all g . \square

Definition 3.2.14. Let M be a multiplicative set consisting of homogeneous elements in R . The subring

$$R_{(M)} := (M^{-1}R)_0 = \left\{ \frac{a}{m} \mid a \in R_{\deg(m)} \right\}$$

of the localization $M^{-1}R$ is called **homogeneous localization with respect to M** . We will learn about its geometric interpretation for example in Theorem 4.1.20 below. \triangle

Example 3.2.15. (i) Let $m \in R$ be homogeneous of degree $g \in G$. Set $M = \{1, m, m^2, \dots\}$ and write

$$R_{(m)} = R_{(M)} = \left\{ \frac{a}{m^r} \mid a \in R_{r \cdot g} \right\}.$$

(ii) Let $\mathfrak{p} \subseteq R$ be a homogeneous prime ideal, and M the set of all *homogeneous* elements in $R \setminus \mathfrak{p}$. Then M is a multiplicative set and we call

$$R_{(\mathfrak{p})} := R_{(M)} = \left\{ \frac{a}{m} \mid a, m \in R \text{ homogeneous, } m \notin \mathfrak{p}, \deg(a) = \deg(m) \right\}$$

the **homogeneous localization of R in \mathfrak{p}** .

(iii) Note that there is a natural homomorphism $R \rightarrow M^{-1}R$, but in general there is no such homomorphism from R to $R_{(M)}$. \triangle

3.3 Projective Algebraic Varieties

Again let k be an arbitrary field and K an algebraically closed extension field. We write

$$\mathbb{P}^n := \mathbb{P}^n(K) = \mathbb{P}(K^{n+1}).$$

Note that with regard to Definition 3.1.1 we consider K^{n+1} as K -vector space, and not as k -vector space. Thus elements of \mathbb{P}^n are one-dimensional K -subspaces of K^{n+1} . From now on we set $\underline{x} = (x_0, \dots, x_n)$ and always equip $k[\underline{x}]$ with the standard grading. For *homogeneous* $p \in k[\underline{x}]$ and $v \in K^{n+1}$ as well as $\lambda \in K$ we have

$$p(\lambda \cdot v) = \lambda^{\deg(p)} \cdot p(v).$$

So if a homogeneous polynomial vanishes on a point, it vanishes on the entire line spanned by that point, and hence for $a = [v] \in \mathbb{P}^n$ the equation

$$p(a) = 0 \quad :\Leftrightarrow \quad p(v) = 0$$

is well-defined. Similarly we write $p(a) \neq 0$ if $p(v) \neq 0$. For an arbitrary $p \in k[\underline{x}]$ write

$$p = p_0 + p_1 + \dots + p_d$$

with homogeneous $p_i \in k[\underline{x}]_i$ and define

$$p(a) = 0 \quad :\Leftrightarrow \quad p_0(a) = p_1(a) = \dots = p_d(a) = 0.$$

Because of

$$p(\lambda \cdot v) = p_0 + \lambda \cdot p_1(v) + \lambda^2 \cdot p_2(v) + \dots + \lambda^d \cdot p_d(v),$$

this is equivalent to

$$p(\lambda v) = 0 \text{ for all } \lambda \in K,$$

i.e. p vanishes on the entire line spanned by v .

Definition 3.3.1. Let $P \subseteq k[\underline{x}]$ and $\emptyset \neq V \subseteq \mathbb{P}^n$.

(i) We define

$$\mathcal{V}_+(P) = \{a \in \mathbb{P}^n \mid p(a) = 0 \text{ for all } p \in P\}$$

and

$$\mathcal{I}_+(V) := \{p \in k[\underline{x}] \mid p(a) = 0 \text{ for all } a \in V\}.$$

We call $\mathcal{V}_+(P)$ the **projective variety** defined by P , and $\mathcal{I}_+(V)$ the **vanishing ideal** of V .

(ii) A set of the form $\mathcal{V}_+(P)$ with $P \subseteq k[\underline{x}]$ is called a **projective k -variety**.

(iii) We set

$$\widehat{V} := \bigcup_{a \in V} a = \{v \in K^{n+1} \mid v \neq 0, [v] \in V\} \cup \{0\}$$

and call \widehat{V} the **affine cone over** V .

(iv) We set

$$\mathcal{I}_+(\emptyset) := (x_0, \dots, x_n)$$

(the irrelevant ideal) and

$$\widehat{\emptyset} := \{0\}.$$

(v) For homogeneous $p \in k[\underline{x}]$ let

$$\mathcal{D}_+(p) := \{a \in \mathbb{P}^n \mid p(a) \neq 0\} = \mathbb{P}^n \setminus \mathcal{V}_+(p). \quad \triangle$$

Remark 3.3.2. (i) For arbitrary $V \subseteq \mathbb{P}^n$ we have $k[\underline{x}]$

$$\mathcal{I}_+(V) = \mathcal{I}(\widehat{V}).$$

In particular, $\mathcal{I}_+(V)$ is a radical ideal. On the other hand $\mathcal{I}_+(V)$ is by the definition of " $p(a) = 0$ " also a homogeneous ideal.

(ii) For $M \subseteq k[\underline{x}]$ let M_h be the set of all homogeneous components of elements in M , and $I = (M_h)$ the ideal generated by this set. Then we have

$$\mathcal{V}_+(M) = \mathcal{V}_+(M_h) = \mathcal{V}_+(I) = \mathcal{V}_+(\sqrt{I})$$

(iii) We have

$$\mathcal{V}_+(k[\underline{x}]) = \mathcal{V}_+((x_0, \dots, x_n)) = \emptyset$$

and

$$\mathcal{V}_+(0) = \mathbb{P}^n.$$

For homogeneous ideals I, J, I_λ ($\lambda \in \Lambda$) we have

$$\mathcal{V}_+(I) \cup \mathcal{V}_+(J) = \mathcal{V}_+(I \cap J) = \mathcal{V}_+(IJ)$$

and

$$\bigcap_{\lambda \in \Lambda} \mathcal{V}_+(I_\lambda) = \mathcal{V}_+\left(\sum_{\lambda \in \Lambda} I_\lambda\right).$$

(iv) For arbitrary subsets $V_\lambda \subseteq \mathbb{P}^n$ ($\lambda \in \Lambda$) we have

$$\widehat{\bigcup_{\lambda} V_\lambda} = \bigcup_{\lambda} \widehat{V_\lambda} \quad \text{and} \quad \widehat{\bigcap_{\lambda} V_\lambda} = \bigcap_{\lambda} \widehat{V_\lambda}.$$

(v) If $I \subsetneq k[x]$ is a homogeneous ideal, then we have

$$\widehat{\mathcal{V}_+(I)} = \mathcal{V}(I) \subseteq \mathbb{A}^{n+1}.$$

This is because for every point $v \in \mathcal{V}(I)$ we have $[v] \subseteq \mathcal{V}(I)$, since I is generated by homogeneous elements. Note that this is not true for $I = k[x]$, because $\widehat{\emptyset} = \{0\} \neq \emptyset$. For $I = (x_0, \dots, x_n)$ however it is true. Further note that the statement is also not true for non-homogeneous ideals. For instance, by definition we have

$$\mathcal{V}_+(x_0 - 1) = \emptyset \subseteq \mathbb{P}^1, \quad \mathcal{V}_+(\widehat{x_0 - 1}) = \{0\},$$

but

$$\mathcal{V}(x_0 - 1) = \{(1, r) \mid r \in K\} \subseteq \mathbb{A}^2. \quad \triangle$$

Definition 3.3.3. The k -Zariski topology on \mathbb{P}^n is the topology with the projective k -varieties as its closed sets. By Remark 3.3.2 (iii) this indeed is a topology. \triangle

Lemma 3.3.4. (i) A subset $V \subseteq \mathbb{P}^n$ is closed if and only if $\widehat{V} \subseteq \mathbb{A}^{n+1}$ is closed.
(ii) The k -Zariski topology on \mathbb{P}^n is noetherian.

Proof. For (i) first let $V = \mathcal{V}_+(I)$ for a homogeneous ideal $I \neq 1$ (cf. Remark 3.3.2 (ii)). Then by Remark 3.3.2 (v) we have $\widehat{V} = \mathcal{V}(I)$, which is closed in \mathbb{A}^{n+1} . Conversely let $\widehat{V} \subseteq \mathbb{A}^{n+1}$ be closed. Then $I := \mathcal{I}(\widehat{V})$ is a homogeneous ideal $\neq 1$, because \widehat{V} is nonempty and a union of lines through the origin. For $W := \mathcal{V}_+(I)$ we then have

$$\widehat{W} = \mathcal{V}(I) = \widehat{V},$$

where we have used that \widehat{V} is closed for the last equality. But then $V = W = \mathcal{V}_+(I)$ is closed.

For (ii) let

$$V_0 \supseteq V_1 \supseteq \dots$$

be a descending chain of closed sets in \mathbb{P}^n . Then

$$\widehat{V}_0 \supseteq \widehat{V}_1 \supseteq \dots$$

is a descending chain of closed sets in \mathbb{A}^{n+1} . This chain becomes stationary, and thus this is also true for the initial chain. \square

Theorem 3.3.5. Let $I \neq 1$ be a homogeneous ideal in $k[x]$, and let $V \subseteq \mathbb{P}^n$ be a subset. Then we have

- (i) $\mathcal{I}_+(\mathcal{V}_+(I)) = \sqrt{I}$.
- (ii) $\mathcal{V}_+(\mathcal{I}_+(V)) = \overline{V}$.
- (iii) $V \mapsto \mathcal{I}_+(V)$ is a bijection between the closed subsets of \mathbb{P}^n and the homogeneous radical ideals $\neq 1$ in $k[\underline{x}]$. The inverse mapping is given by $I \mapsto \mathcal{V}_+(I)$.

Proof. (i) can be seen by

$$\mathcal{I}_+(\mathcal{V}_+(I)) = \mathcal{I}(\widehat{\mathcal{V}_+(I)}) = \mathcal{I}(\mathcal{V}(I)) = \sqrt{I},$$

where we have used Remark 3.3.2 and Theorem 1.2.14. (ii) follows immediately from the definition of the Zariski topology, because $\mathcal{V}_+(\mathcal{I}_+(V))$ is obviously the smallest closed superset of V . (iii) follows immediately from (i) and (ii). \square

Remark 3.3.6. In the affine case, the empty variety is defined precisely by the ideal $I = (1)$ and by no other ideal (cf. Theorem 1.2.11). In the projective space, \emptyset arises either from a homogeneous system of equations that is already unsolvable in \mathbb{A}^{n+1} , i.e. from the trivial homogeneous radical ideal $k[\underline{x}]$, or from a homogeneous system that defines $\{0\}$ in \mathbb{A}^{n+1} , for example from the irrelevant ideal $k[\underline{x}]_+$. Both are homogeneous radical ideals, and in Theorem 3.3.5 we have chosen $k[\underline{x}]_+$.

The (not necessarily radical) homogeneous ideals that define $\emptyset \subseteq \mathbb{P}^n$ can be described even more precisely. In order to do this we consider the **ideal quotient**

$$(I : J) = \{a \in A \mid aJ \subseteq I\},$$

and further define

$$(I : J^\infty) := \bigcup_{d=0}^{\infty} (I : J^d).$$

Note that $(I : J^\infty)$ is the union of the ascending chain

$$I \subseteq (I : J) \subseteq (I : J^2) \subseteq \dots$$

is again an ideal. It is called the **saturation of I with respect to J** . \triangle

Theorem 3.3.7. Let $I \subseteq k[\underline{x}]$ be a homogeneous ideal and $\mathfrak{m} = (x_0, \dots, x_n)$ the irrelevant ideal. Then the following are equivalent:

- (i) $\mathcal{V}_+(I) = \emptyset$

(ii) $\mathfrak{m}^d \subseteq I$ for some $d \geq 0$

(iii) $(I : \mathfrak{m}^\infty) = (1)$

(iv) $k[\underline{x}]_d \subseteq I$ for some $d \geq 0$.

Proof. We have $(I : \mathfrak{m}^\infty) = \bigcup_{d \geq 0} (I : \mathfrak{m}^d)$ and therefore

$$(I : \mathfrak{m}^\infty) = (1) \Leftrightarrow 1 \in (I : \mathfrak{m}^d) \text{ for some } d \geq 0 \Leftrightarrow \mathfrak{m}^d \subseteq I \text{ for some } d \geq 0.$$

This shows the equivalence of (ii) and (iii). For (i) \Rightarrow (ii) let $V = \mathcal{V}_+(I) = \emptyset$ and $I \neq 1$. From Theorem 3.3.5 it follows that

$$\mathfrak{m} = \mathcal{I}_+(V) = \sqrt{I},$$

and in particular we have $x_i^r \in I$ for all i and some r . This implies $\mathfrak{m}^{r(n+1)} \subseteq I$, as can be easily verified. The implication (ii) \Rightarrow (iv) follows immediately from

$$k[\underline{x}]_d \subseteq \mathfrak{m}^d.$$

(iv) \Rightarrow (i) holds because $\mathcal{V}_+(x_0^d, \dots, x_n^d) = \emptyset$. □

Definition 3.3.8. Let $V \subseteq \mathbb{P}^n$ be a projective k -variety. Then the \mathbb{Z} -graded k -algebra

$$k_+[V] := k[\underline{x}]/\mathcal{I}_+(V)$$

is called the **projective coordinate ring of V** (the grading is defined as in Lemma 3.2.8 (ii)). △

Remark 3.3.9. Let $V \subseteq \mathbb{P}^n$ be a projective k -variety.

(i) The affine cone $\widehat{V} \subseteq \mathbb{A}^{n+1}$ over V is an affine k -variety, and we have $\mathcal{I}(\widehat{V}) = \mathcal{I}_+(V)$. Therefore

$$k[\widehat{V}] = k_+[V]$$

holds for the *ungraded* rings.

(ii) The elements of $k_+[V]$ *cannot* be regarded as functions on V as in the affine case. Although the condition $p(v) = 0$ is well defined for $p \in k_+[V]$ and $v \in V$, the *value* $p(v)$ is not. However if $p, q \in k_+[V]$ are *homogeneous of the same degree*, then

$$v \mapsto \frac{p(v)}{q(v)}$$

is a well-defined map $V \setminus \mathcal{V}_+(q) \rightarrow \mathbb{A}^1$.

(iii) For a homogeneous ideal $J \subseteq k_+[V]$ we define

$$\mathcal{V}_{V,+}(J) := \{v \in V \mid p(v) = 0 \forall p \in J\},$$

and for a homogeneous element $p \in k_+[V]$ we define

$$\mathcal{D}_{V,+}(p) := V \setminus \mathcal{V}_{V,+}(p).$$

The homogeneous radical ideals of $k_+[V]$ correspond to the homogeneous radical ideals of $k[\underline{x}]$ that contain $\mathcal{I}_+(V)$, and these in turn correspond to the projective k -subvarieties of V . Thus we obtain the following analogue to Remark 1.4.4: The map $\mathcal{V}_{V,+}(\cdot)$ yields a bijection between homogeneous radical ideals $\neq 1$ in $k_+[V]$ and k -subvarieties of V . \triangle

Construction 3.3.10. For $i \in \{0, \dots, n\}$ consider the well-defined bijection

$$\begin{aligned} \phi_i: \mathcal{D}_+(x_i) &\rightarrow \mathbb{A}^n \\ (a_0 : \dots : a_n) &\mapsto \left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i} \right) \\ (b_1 : \dots : 1 : \dots : b_n) &\leftarrow (b_1, \dots, b_n) \end{aligned}$$

For simplicity we restrict ourselves to the case $i = 0$. For $0 \neq p \in k[x_1, \dots, x_n]$ we define

$$\begin{aligned} p^h &:= x_0^{\deg(p)} \cdot p \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \in k[x_0, \dots, x_n]_{\deg(p)} \\ 0^h &:= 0 \end{aligned}$$

and call p^h the **homogenization of p by x_0** . For $\deg(p) = d$ and $p = \sum_{|\alpha| \leq d} p_\alpha \underline{x}^\alpha$ we have

$$p^h := \sum_{|\alpha| \leq d} p_\alpha \cdot x_0^{d-|\alpha|} \cdot \underline{x}^\alpha.$$

Thus we make p homogeneous by multiplying every monomial in p , which does not have the maximal degree d , with x_0 until it has degree d . For example we have

$$(x_1^2 - x_2 + 1)^h = x_1^2 - x_0 x_2 + x_0^2.$$

By the definition

$$(p_1 \cdot p_2)^h = p_1^h \cdot p_2^h$$

is clear. Conversely let $q \in k[x_0, \dots, x_n]$ be homogeneous. Then we set

$$\tilde{q} := q(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n]$$

and call \tilde{q} the **dehomogenization of q by x_0** . We obviously have

$$\tilde{p}^h = p,$$

and

$$x_0^m \cdot (\tilde{q})^h = q \text{ for some } m \geq 0.$$

In the last equation $m \geq 1$ occurs if and only if the degree of q decreases when dehomogenized, that is when every monomial in q is divisible by x_0 . Hence we have $m = \deg(q) - \deg(\tilde{q})$. In the following lemma we show that the algebraic construction of homogenizing and dehomogenizing precisely corresponds to applying ϕ_0 on the geometric side. \triangle

Lemma 3.3.II. *With the previous constructions we obtain for $p \in k[x_1, \dots, x_n]$ that*

$$\phi_0^{-1}(\mathcal{V}(p)) = \mathcal{V}_+(p^h) \cap \mathcal{D}_+(x_0),$$

and for homogeneous $q \in k[x_0, \dots, x_n]$ that

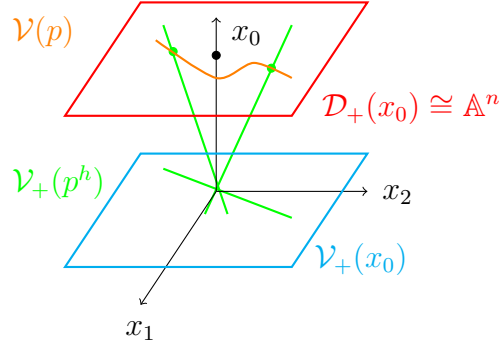
$$\phi_0(\mathcal{V}_+(q) \cap \mathcal{D}_+(x_0)) = \mathcal{V}(\tilde{q}).$$

Proof. We have

$$\begin{aligned} \phi_0^{-1}(\mathcal{V}(p)) &= \left\{ (a_0 : \dots : a_n) \mid a_0 \neq 0, p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) = 0 \right\} \\ &= \left\{ (a_0 : \dots : a_n) \mid a_0 \neq 0, a_0^{\deg(p)} \cdot p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) = 0 \right\} \\ &= \left\{ (a_0 : \dots : a_n) \mid a_0 \neq 0, p^h(a_0, \dots, a_n) = 0 \right\} \\ &= \mathcal{V}_+(p^h) \cap \mathcal{D}_+(x_0) \end{aligned}$$

and

$$\begin{aligned} \phi_0(\mathcal{V}_+(q) \cap \mathcal{D}_+(x_0)) &= \left\{ \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) \mid a_0 \neq 0, q(a_0, \dots, a_n) = 0 \right\} \\ &= \left\{ \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) \mid a_0 \neq 0, a_0^{\deg(q)} q\left(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) = 0 \right\} \\ &= \{(b_1, \dots, b_n) \mid \tilde{q}(b_1, \dots, b_n) = 0\} \\ &= \mathcal{V}(\tilde{q}). \quad \square \end{aligned}$$



Theorem 3.3.12. For all $i \in \{0, \dots, n\}$ the map $\phi_i: \mathcal{D}_+(x_i) \rightarrow \mathbb{A}^n$ is a homeomorphism with respect to the Zariski topologies.

Proof. ϕ_i is bijective and by Lemma 3.3.11 images and preimages of closed sets are closed. \square

Remark 3.3.13. Theorem 3.3.12 implies that

$$\mathbb{P}^n = \mathcal{D}_+(x_0) \cup \dots \cup \mathcal{D}_+(x_n)$$

is an open covering with subsets homeomorphic to \mathbb{A}^n . \triangle

Definition 3.3.14. Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. The **homogenization** I^h of I is the homogeneous ideal

$$I^h := (p^h \mid p \in I) \subseteq k[x_0, \dots, x_n]. \quad \triangle$$

Corollary 3.3.15. We have

$$\phi_0^{-1}(\mathcal{V}(I)) = \mathcal{V}_+(I^h) \cap \mathcal{D}_+(x_0).$$

Proof.

$$\begin{aligned} \phi_0^{-1}(\mathcal{V}(I)) &= \phi_0^{-1}\left(\bigcap_{p \in I} \mathcal{V}(p)\right) = \bigcap_{p \in I} \phi_0^{-1}(\mathcal{V}(p)) \\ &= \mathcal{D}_+(x_0) \cap \bigcap_{p \in I} \mathcal{V}_+(p^h) = \mathcal{D}_+(x_0) \cap \mathcal{V}_+(I^h), \end{aligned}$$

where we have used Lemma 3.3.11. \square

Lemma 3.3.16. *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal.*

(i) *Every homogeneous $q \in I^h$ is of the form $q = x_0^r \cdot p^h$ for some $p \in I$.*

(ii) *We have $\sqrt{I^h} = \sqrt{I}^h$. In particular, the homogenization of a radical ideal is again a radical ideal.*

Proof. For (i) write $q = \sum_i g_i p_i^h$ with $g_i \in k[x_0, \dots, x_n]$ and $p_i \in I$. Since q and all p_i^h are homogeneous, we can assume that the g_i are homogeneous as well. Thus

$$q = \sum_i x_0^{r_i} \tilde{g}_i^h p_i^h$$

with $\deg(g_i) + \deg(p_i) = \deg(q) =: d$ and $r_i = \deg(g_i) - \deg(\tilde{g}_i)$. We have

$$d' := \deg\left(\sum_i \tilde{g}_i p_i\right) \leq d,$$

since every term in the sum has at most degree d . We obtain

$$\begin{aligned} x_0^{d-d'} \left(\sum_i \tilde{g}_i p_i\right)^h &= x_0^{d-d'} \cdot \sum_i x_0^{d'-\deg(\tilde{g}_i)-\deg(p_i)} \tilde{g}_i^h p_i^h \\ &= \sum_i x_0^{d-\deg(\tilde{g}_i)-\deg(p_i)} \tilde{g}_i^h p_i^h \\ &= \sum_i x_0^{r_i} \tilde{g}_i^h p_i^h = q. \end{aligned}$$

This shows (i). (ii) is Exercise 70. □

Definition 3.3.17. For every affine k -variety $V \subseteq \mathbb{A}^n$ the set

$$\bar{V} := \overline{\phi_0^{-1}(V)} \subseteq \mathbb{P}^n$$

is called the **projective closure of V in \mathbb{P}^n** (with respect to ϕ_0^{-1}). Often we will suppress ϕ_0 in the notation. Since \mathbb{P}^n precisely induces the Zariski topology on \mathbb{A}^n , we have $\bar{V} \cap \mathbb{A}^n = V$. △

Theorem 3.3.18. *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. For the projective closure we then have*

$$\overline{\mathcal{V}(I)} = \mathcal{V}_+(I^h) \text{ and } \mathcal{I}_+(\overline{\mathcal{V}(I)}) = \sqrt{I^h} = \sqrt{I}^h = \mathcal{I}(\mathcal{V}(I))^h.$$

Proof. Let $V := \mathcal{V}(I)$. The first assertion follows from the second by applying $\mathcal{V}_+(\cdot)$. So first let $p \in \sqrt{I} = \mathcal{I}(V)$. Then $p^h \equiv 0$ holds on $\phi_0^{-1}(V)$ by Lemma 3.3.11. But then $p^h \equiv 0$ also holds on $\overline{\phi_0^{-1}(V)} = \overline{V}$, i.e. $p^h \in \mathcal{I}_+(\overline{V})$. This shows $\sqrt{I}^h \subseteq \mathcal{I}_+(\overline{V})$. Conversely let $g \in \mathcal{I}_+(\overline{V})$, and w.l.o.g. g homogeneous. Write $g = x_0^m g_1$ with $x_0 \nmid g_1$. Then $g_1 \equiv 0$ holds on $\mathcal{D}_+(x_0) \cap \overline{V}$, and hence $\tilde{g}_1 \equiv 0$ holds on V again by Lemma 3.3.11. This implies $\tilde{g}_1 \in \sqrt{I}$, and $g_1 = \tilde{g}_1^h$ holds because $x_0 \nmid g_1$. Thus g_1 and likewise g are in \sqrt{I}^h . \square

Remark 3.3.19. For $I = (p_1, \dots, p_s)$ we have

$$(p_1^h, \dots, p_s^h) \subseteq I^h$$

but in general this is *not an equality*. For $p_1 = x_1, p_2 = x_1 + 1$ we have

$$I = (p_1, p_2) = (1)$$

and therefore $I^h = (1)$. But

$$(p_1^h, p_2^h) = (x_1, x_1 + x_0) \neq (1). \quad \triangle$$

Definition 3.3.20. (i) Let $V \subseteq \mathbb{A}^n$ be an affine k -variety and $\overline{V} \subseteq \mathbb{P}^n$ its projective closure. Let $H = \mathcal{V}_+(x_0) \subseteq \mathbb{P}^n$. Then the points in

$$H \cap \overline{V} = \overline{V} \setminus V$$

are called the **points at infinity of V** . They form a closed set in $H = \mathbb{P}^{n-1}$.

(ii) For $0 \neq p \in k[x_1, \dots, x_n]$ write $p = p_0 + p_1 + \dots + p_d$ with p_i homogeneous of degree $i, p_d \neq 0$. Then

$$\text{LF}(p) := p_d$$

is called the **leading form** of p . Note that

$$\text{LF}(p) = p^h(0, x_1, \dots, x_n)$$

holds.

(iii) For an ideal $I \subseteq k[x_1, \dots, x_n]$ the ideal

$$\text{LF}(I) := (\text{LF}(p) \mid p \in I)$$

is called the **leading form ideal of I** , a homogeneous ideal in $k[x_1, \dots, x_n]$. \triangle

Corollary 3.3.21. *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and $V = \mathcal{V}(I) \subseteq \mathbb{A}^n$. Let $H = \mathcal{V}_+(x_0)$. Then we have*

$$\overline{V} \cap H = \mathcal{V}_+(\text{LF}(I)) \subseteq H \cong \mathbb{P}^{n-1},$$

where we consider $H \cong \mathbb{P}^{n-1}$ with homogeneous coordinates $(x_1 : \dots : x_n)$. In particular, we have $\mathcal{I}_+(\overline{V} \cap H) = \sqrt{\text{LF}(I)}$ in $k[x_1, \dots, x_n]$.

Proof. Because $\overline{V} = \mathcal{V}_+(I^h)$, we obtain in \mathbb{P}^n that

$$\overline{V} \cap H = \mathcal{V}_+((x_0) + I^h) = \mathcal{V}_+((x_0) + \text{LF}(I)),$$

since $p^h \equiv \text{LF}(p)$ modulo (x_0) . When we restrict ourselves to H , we obtain $\mathcal{V}_+(\text{LF}(I)) \subseteq H = \mathbb{P}^{n-1}$. \square

Example 3.3.22. (i) The projective closure of an affine hypersurface

$$\mathcal{V}(p) \subseteq \mathbb{A}^n$$

(with $0 \neq p \in k[x_1, \dots, x_n]$) is

$$\overline{V} = \mathcal{V}_+(p^h)$$

because $(p)^h = (p^h)$. If p is square-free, then $\mathcal{I}_+(\overline{V}) = \sqrt{(p)^h} = (p)^h = (p^h)$. The points at infinity of V are the roots of $\text{LF}(p)$ in $\mathcal{V}_+(x_0) = \mathbb{P}^{n-1}$.

(ii) For $p \in k[x_1, x_2]$ with $\deg(p) \geq 1$ consider the affine curve $V = \mathcal{V}(p) \subseteq \mathbb{A}^2$. There is a factorization

$$\text{LF}(p) = c \cdot \prod_{i=1}^d (a_i x_1 + b_i x_2)$$

with $(a_i : b_i) \in \mathbb{P}^1(\overline{k})$, $c \in k^*$, where the $(a_i : b_i)$ are uniquely determined. This factorization can be obtained by factorizing $\text{LF}(p)(x_1, 1) \in k[x_1]$ over \overline{k} and re-homogenizing the factors. Hence the points at infinity of V are just the

$$(0 : b_i : -a_i) \in \mathbb{P}^2 \quad i = 1, \dots, d,$$

or alternatively the points

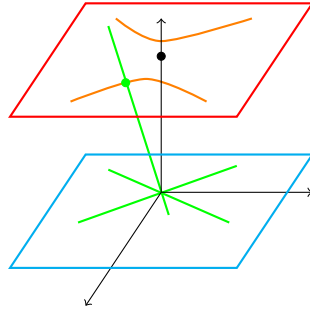
$$(b_i : -a_i) \in \mathbb{P}^1 = \mathcal{V}_+(x_0) \quad i = 1, \dots, d.$$

(iii) For $p = x_1^2 - x_2^2 - 1$ the set $\mathcal{V}(p)$ is a hyperbola. Since

$$\text{LF}(p) = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2),$$

we obtain the two points at infinity

$$(0 : 1 : 1) \quad (0 : 1 : -1).$$



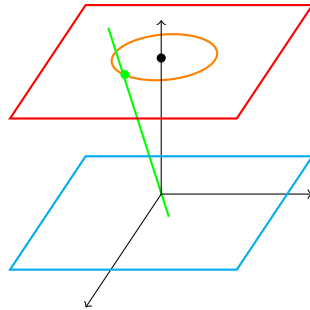
(iv) For $p = (x_1 - a_1)^2 + (x_2 - a_2)^2 - r^2$ the set $\mathcal{V}(p)$ is a circle. Because

$$\text{LF}(p) = x_1^2 + x_2^2,$$

there are the two points at infinity

$$(0 : 1 : \sqrt{-1}) \quad (0 : 1 : -\sqrt{-1}),$$

which cannot be seen in the real image however.

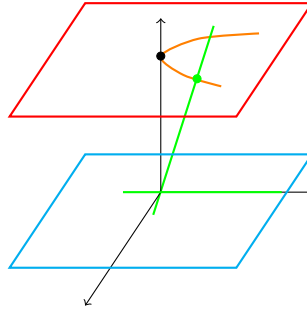


(v) For $p = x_1^2 - x_2$ the set $\mathcal{V}(p)$ is a parabola. Since

$$\text{LF}(p) = x_1^2,$$

we just obtain the single point at infinity

$$(0 : 0 : 1).$$

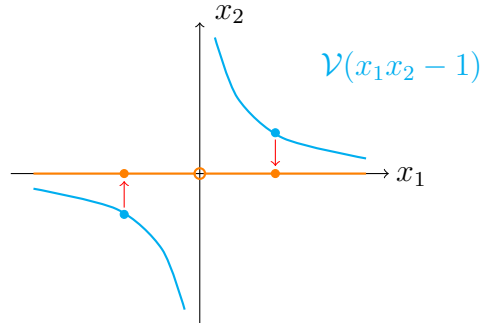


△

Remark 3.3.23. Even if $I \subseteq k[\underline{x}]$ is a radical ideal, this does not have to be the case for $\text{LF}(I)$. This can be seen for instance in the last example $I = (x_1^2 - x_2)$. Here we have $\text{LF}(I) = (x_1^2)$. △

3.4 The Fundamental Theorem of Elimination Theory

In Section 1.4 we have seen that the image of an affine variety under a morphism is not necessarily closed, i.e. is not necessarily again an affine variety. An example is the projection of the variety $\mathcal{V}(x_1x_2 - 1) \subseteq \mathbb{A}^2$ onto the x_1 -component. The image in this example is $\mathbb{A}^1 \setminus \{0\}$.



The reason that the image fails to be closed is that the preimage of a point $a \in \mathbb{A}^1$ "vanishes towards infinity" when a goes to 0. In the projective space we now also have points at infinity available, which is the reason why the situation is different. In this section let

$$\underline{x} = (x_0, \dots, x_m), \quad \underline{y} = (y_1, \dots, y_n)$$

be two tuples of variables. A polynomial $p \in k[\underline{x}, \underline{y}]$ is called **homogeneous in \underline{x}** if it is homogeneous in the polynomial ring $(k[\underline{y}])[\underline{x}]$ with respect to the standard grading. This just means that for every monomial $\underline{x}^\alpha \underline{y}^\beta$ of p we have $|\alpha| = d$ for some fixed $d \in \mathbb{N}$. Now let $p_1, \dots, p_r \in k[\underline{x}, \underline{y}]$ be homogeneous in \underline{x} . Then the following set is well-defined:

$$X := \{(a, b) \in \mathbb{P}^m \times \mathbb{A}^n \mid p_1(a, b) = \dots = p_r(a, b) = 0\}.$$

Let

$$\begin{aligned} \pi: \mathbb{P}^m \times \mathbb{A}^n &\rightarrow \mathbb{A}^n \\ (a, b) &\mapsto b \end{aligned}$$

be the projection onto the second component.

Theorem 3.4.1 (Fundamental Theorem of Elimination Theory). *The set $\pi(X)$ is k -closed in \mathbb{A}^n .*

Proof. For $b \in \mathbb{A}^n$ we have

$$b \in \pi(X) \Leftrightarrow \mathcal{V}_+(p_1(\underline{x}, b), \dots, p_r(\underline{x}, b)) \neq \emptyset.$$

Let $R = K[\underline{x}]$ equipped with the standard grading. From Theorem 3.3.7 it follows

$$b \in \pi(X) \Leftrightarrow R_d \not\subseteq (p_1(\underline{x}, b), \dots, p_r(\underline{x}, b)) \quad \forall d \geq 0.$$

For $d \geq 0$ let

$$Y_d := \{b \in \mathbb{A}^n \mid R_d \not\subseteq (p_1(\underline{x}, b), \dots, p_r(\underline{x}, b))\}.$$

Then we have

$$Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots \quad \text{and} \quad \bigcap_{d \geq 0} Y_d = \pi(X).$$

Hence it suffices to show that Y_d is closed for all large enough d . Let $d_i = \deg_{\underline{x}}(p_i)$ and $d \geq \max\{d_1, \dots, d_r\}$. Then

$$\begin{aligned} b \notin Y_d &\Leftrightarrow R_d \subseteq (p_1(\underline{x}, b), \dots, p_r(\underline{x}, b)) \\ &\Leftrightarrow \text{the } \underline{x}^\alpha \cdot p_i(\underline{x}, b) \text{ with } |\alpha| = d - d_i \text{ span the entire space } R_d \text{ over } K. \end{aligned}$$

For the second equivalence we have used Remark 3.2.4. When we expand the $\underline{x}^\alpha \cdot p_i(\underline{x}, b)$ in a suitable basis of R_d (e.g. the monomial basis) and write the coefficients into a matrix B , then the entries of B are polynomials over k in b and altogether we obtain

$$b \notin Y_d \Leftrightarrow \text{rank}(B) \geq \dim_K R_d.$$

Equivalently we can rewrite this as

$$\begin{aligned} b \in Y_d &\Leftrightarrow \text{rank}(B) < \dim_K R_d \\ &\Leftrightarrow \text{all minors of } B \text{ of size } \dim_K R_d \text{ vanish.} \end{aligned}$$

The last condition defines a k -closed set. □

Remark 3.4.2. The Fundamental Theorem of Elimination Theory states that for all \underline{x} -homogeneous polynomials $p_1, \dots, p_r \in k[\underline{x}, \underline{y}]$ there are $q_1, \dots, q_s \in k[\underline{y}]$ such that for all $b \in K^n$ we have

$$\left(\exists a \in \mathbb{P}^m(K) : \bigwedge_i p_i(a, b) = 0 \right) \Leftrightarrow \bigwedge_j q_j(b) = 0.$$

That is, the existential quantifier can be eliminated. △

Remark 3.4.3. All varieties are noetherian in the Zariski topology and are therefore quasi-compact, i.e. every open cover has a finite subcover. In non-Hausdorff spaces the notion of quasi-compactness is often not quite helpful. The Fundamental Theorem of Elimination Theory is some kind of an alternative compactness property for \mathbb{P}^n . A Hausdorff space X is compact if and only if for all Hausdorff spaces Y the projection

$$\pi: X \times Y \rightarrow Y$$

maps closed sets to closed sets. This can be proven in Exercise 73. △

We also obtain the Fundamental Theorem of Elimination Theory for the case $\mathbb{P}^m \times \mathbb{P}^n$. Let

$$\underline{x} = (x_0, \dots, x_m) \quad \underline{y} = (y_0, \dots, y_n)$$

and let $p_i \in k[\underline{x}, \underline{y}]$ be *bihomogeneous*, i.e. homogeneous in both \underline{x} and \underline{y} . Then

$$X = \{(a, b) \in \mathbb{P}^m \times \mathbb{P}^n \mid p_1(a, b) = \dots = p_r(a, b) = 0\}$$

is well-defined. Let $\pi: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ be the projection.

Corollary 3.4.4. *The set $\pi(X) \subseteq \mathbb{P}^n$ is closed.*

Proof. Let $X' \subseteq \mathbb{P}^m \times \mathbb{A}^{n+1}$ be the set of roots of the p_i and let

$$\pi': \mathbb{P}^m \times \mathbb{A}^{n+1} \rightarrow \mathbb{A}^{n+1}$$

be the projection. By Theorem 3.4.1 the image $\pi'(X')$ is closed in \mathbb{A}^{n+1} . On the other hand we obviously have $\pi'(X') = \widehat{\pi(X)}$, and the assertion follows from Lemma 3.3.4 (i). \square

Corollary 3.4.5. *Let $p_1, \dots, p_r \in k[x_0, \dots, x_m]$ be homogeneous and let $f_0, \dots, f_n \in k[x_0, \dots, x_m]$ homogeneous of the same degree. Further let*

$$V = \mathcal{V}_+(p_1, \dots, p_r) \subseteq \mathbb{P}^m$$

and assume that

$$V \cap \mathcal{V}_+(f_0, \dots, f_n) = \emptyset.$$

Then the image of the (well-defined!) map

$$\begin{aligned} f: V &\rightarrow \mathbb{P}^n \\ a &\mapsto (f_0(a) : \dots : f_n(a)) \end{aligned}$$

is closed in \mathbb{P}^n .

Proof. The graph

$$\Gamma_f := \{(a, b) \in \mathbb{P}^m \times \mathbb{P}^n \mid a \in V, b = f(a)\}$$

can be defined bihomogeneously by the equations $p_i(a) = 0$ for $i = 1, \dots, r$ and

$$b_j f_k(a) - b_k f_j(a) = 0$$

for $j, k = 0, \dots, n$. By Corollary 3.4.4 the set

$$f(V) = \pi(\Gamma_f)$$

is closed in \mathbb{P}^n . \square

In the last corollary we can see how the Fundamental Theorem of Elimination Theory can be applied in situations where the compactness would be invoked in the case of Hausdorff spaces. The set V is quasi-compact in the Zariski topology and the map f is continuous. Thus the image is again quasi-compact, and in Hausdorff spaces this implies that the image is closed.

Chapter 4

Quasi-Projective Varieties

In this chapter we further generalize the notions of varieties and morphisms, to reach a more flexible framework. In particular we obtain a suitable notion of morphism for projective varieties.

4.1 Quasi-Projective Varieties, Regular Functions and Morphisms

Definition 4.1.1. Let X be a topological space. A subset $Y \subseteq X$ is called **locally closed in X** , if it fulfills one of the following equivalent conditions:

- (i) Y is relatively open in \bar{Y}
- (ii) There exists an open subset $O \subseteq X$ and a closed subset $A \subseteq X$ with $Y = O \cap A$. △

Remark 4.1.2. (i) Open and closed subsets of X are locally closed. Finite intersections of locally closed subsets of X are locally closed.

(ii) For a locally closed subset $Y \subseteq X$ and $Z \subseteq Y$ we have

$$Z \text{ locally closed in } Y \Leftrightarrow Z \text{ locally closed in } X.$$

(iii) Inverse images of locally closed sets under continuous maps are locally closed. △

Definition 4.1.3. A **quasi-projective k -variety** V is a locally closed subset of some \mathbb{A}^n oder \mathbb{P}^n , with respect to the k -Zariski topology. If $W \subseteq V$ is a locally closed

(closed, open) subset, we call W a locally closed (closed, open) **subvariety** of V . \triangle

Remark 4.1.4. Closed sets are defined by polynomial equations, open sets by their complements. So a quasi-projective variety can be seen as the set of solutions to a system of polynomial equations and inequations (and unions thereof). \triangle

Definition 4.1.5. Let $V \subseteq \mathbb{A}^n$ ($V \subseteq \mathbb{P}^n$, respectively) be a locally closed subset.

(i) A **(k -)regular function on V** is a function

$$f: V \rightarrow \mathbb{A}^1$$

with the following property: For all $a \in V$ there exists an open neighborhood $U \subseteq V$ of a and polynomials $p, q \in k[x_1, \dots, x_n]$ (homogeneous polynomials $p, q \in k[x_0, \dots, x_n]$ of the same degree, respectively) with $q \neq 0$ on U , and

$$f(b) = \frac{p(b)}{q(b)}$$

for all $b \in U$.

(ii) By $\mathcal{O}(V)$ we denote the set of all k -regular functions on the quasi-projective variety V . \triangle

Lemma 4.1.6. Let V be a quasi-projective variety.

(i) The set $\mathcal{O}(V)$ is a k -algebra with respect to pointwise operations.

(ii) For every $f \in \mathcal{O}(V)$ the set

$$\mathcal{V}_V(f) := \{a \in V \mid f(a) = 0\}$$

is closed in V , and the set

$$\mathcal{D}_V(f) := V \setminus \mathcal{V}_V(f)$$

is open in V .

(iii) If $f \in \mathcal{O}(V)$ fulfills $f(a) \neq 0$ for all $a \in V$, then $\frac{1}{f}$ is a regular function on V .

(iv) If $W \subseteq V$ is locally closed and $f \in \mathcal{O}(V)$, then $f|_W \in \mathcal{O}(W)$. The restriction map $\mathcal{O}(V) \rightarrow \mathcal{O}(W)$ is a homomorphism of algebras.

Proof. (i) and (iv) are obvious. For (ii) it is enough to show that for each $a \in V$ there exists an open neighborhood $U \subseteq V$ with $U \cap \mathcal{V}_V(f)$ closed in U . Now if $f = \frac{p}{q}$ holds on an open neighborhood U of a , then $\mathcal{V}_V(f) \cap U = \mathcal{V}(p) \cap U$ is closed in U . Statement (iii) is easy, since $f^{-1} = \frac{q}{p}$ holds on U , if $f = \frac{p}{q}$ on U . \square

Remark 4.1.7. (i) For each quasi-projective variety V and each locally closed subset $W \subseteq V$ the algebra $\mathcal{O}(W)$ is defined, since W itself is locally closed in \mathbb{A}^n or \mathbb{P}^n . For each inclusion $W' \subseteq W$ we have the restriction map $\mathcal{O}(W) \rightarrow \mathcal{O}(W')$, which is a k -algebra homomorphism.

(ii) The property of being regular (of a function $f: V \rightarrow \mathbb{A}^1$) is *local* with respect to V . That is, whenever $V = \bigcup_{i \in I} U_i$ is an open covering, then

$$f \text{ regular} \Leftrightarrow f|_{U_i} \text{ regular for all } i \in I.$$

(iii) From now on we will say **variety** instead of *quasi-projective k -variety*. \triangle

We will first show that the notion of a regular function coincides with the one from Definition 1.4.1, in case the variety is affine.

Theorem 4.1.8. *Let $V \subseteq \mathbb{A}^n$ be closed and $s \in k[V]$. Then the canonical homomorphism*

$$k[V]_s \rightarrow \mathcal{O}(\mathcal{D}_V(s))$$

is an isomorphism.

Proof. Injectivity: Take $p/s^m \in k[V]_s$ and assume $p/s^m = 0$ in $\mathcal{O}(\mathcal{D}_V(s))$. Then $p|_{\mathcal{D}_V(s)} \equiv 0$, so $ps \equiv 0$ on V , and thus $ps = 0$ in $k[V]$. This implies $\frac{p}{s^m} = 0$ in $k[V]_s$.

For surjectivity let $f \in \mathcal{O}(\mathcal{D}_V(s))$. By quasi-compactness of $\mathcal{D}_V(s)$ there exists a finite open covering

$$\mathcal{D}_V(s) = U_1 \cup \cdots \cup U_r$$

and elements $p_i, q_i \in k[V]$ with

$$f = \frac{p_i}{q_i} \text{ on } U_i.$$

Without loss of generality we can even assume $U_i = \mathcal{D}_V(s_i)$ for certain $s_i \in k[V]$. From $U_i \subseteq \mathcal{D}_V(q_i)$ we conclude

$$U_i = \mathcal{D}_V(q_i^2 s_i),$$

and since

$$\frac{p_i}{q_i} = \frac{p_i q_i s_i}{q_i^2 s_i} \text{ on } U_i$$

we can finally assume

$$U_i = \mathcal{D}_V(q_i)$$

as well as

$$p_i \equiv 0 \text{ on } \mathcal{V}_V(q_i).$$

We now have

$$p_i q_j = p_j q_i$$

as elements of $k[V]$, for all $i, j = 1, \dots, r$. This is because on $\mathcal{V}_V(q_i q_j)$ both sides vanish, and on $\mathcal{D}_V(q_i q_j) = U_i \cap U_j$ we can divide by $q_i q_j$ pointwisely and obtain

$$\frac{p_i}{q_i} = \frac{p_j}{q_j},$$

since both sides represent f on $U_i \cap U_j$. From $\mathcal{D}_V(s) = U_1 \cup \dots \cup U_r$ we conclude

$$\mathcal{V}_V(s) = \mathcal{V}_V(q_1, \dots, q_r),$$

and Hilbert's Nullstellensatz implies

$$s \in \sqrt{(q_1, \dots, q_r)} \subseteq k[V].$$

So let

$$s^m = b_1 q_1 + \dots + b_r q_r$$

for some $m \geq 1$ and $b_1, \dots, b_r \in k[V]$. We set

$$p := p_1 b_1 + \dots + p_r b_r$$

and claim that

$$f = \frac{p}{s^m}$$

holds on the whole of $\mathcal{D}_V(s)$. For $i = 1, \dots, r$ we indeed have

$$q_i p = \sum_{j=1}^r q_i p_j b_j = \sum_{j=1}^r q_j p_i b_j = p_i \sum_{j=1}^r q_j b_j = p_i s^m,$$

and this means

$$\frac{p}{s^m} = \frac{p_i}{q_i} = f \text{ on } U_i = \mathcal{D}_V(q_i). \quad \square$$

Corollary 4.1.9. *For every affine variety $V \subseteq \mathbb{A}^n$ we have $\mathcal{O}(V) = k[V]$, i.e. every regular function on V is globally defined by a polynomial.*

Regular functions on a projective variety will be classified in Theorem 4.3.16 below. We first define and study morphisms between varieties. Also here we will later ensure that the new definition coincides with the old one in the affine case.

Definition 4.1.10. (i) Let V, W be quasi-projective k -varieties. A **k -morphism** from V to W is a continuous map

$$\phi: V \rightarrow W$$

such that for every open subset $W' \subseteq W$ and every $g \in \mathcal{O}(W')$ we have

$$g \circ \phi|_{\phi^{-1}(W')} \in \mathcal{O}(\phi^{-1}(W')).$$

(ii) A morphism $\phi: V \rightarrow W$ is an **isomorphism**, if there exists a morphism $\psi: W \rightarrow V$ with $\phi \circ \psi = \text{id}_W$, $\psi \circ \phi = \text{id}_V$. \triangle

Remark 4.1.11. (i) The map

$$\phi^*(g) := g \circ \phi|_{\phi^{-1}(W')}$$

is called **pullback** of g by ϕ . A continuous map ϕ thus is a morphism, if each pullback of a locally defined regular function on W is a regular function on the respective inverse image.

(ii) For $W' \subseteq W$ open, the induced mapping

$$\phi^*: \mathcal{O}(W') \rightarrow \mathcal{O}(\phi^{-1}(W'))$$

is a k -algebra homomorphism. If ϕ is an isomorphism, then so is ϕ^* .

(iii) If $\phi: V \rightarrow W$ and $\psi: W \rightarrow X$ are morphisms, then so is

$$\psi \circ \phi: V \rightarrow X.$$

For $X' \subseteq X$ open we have

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*: \mathcal{O}(X') \rightarrow \mathcal{O}(\psi^{-1}(X')) \rightarrow \mathcal{O}((\psi \circ \phi)^{-1}(X')).$$

(iv) Being a morphism is a *local property* with respect to the domain variety. That means, if $V = \bigcup_{i \in I} V_i$ is an open covering and $\phi: V \rightarrow W$ is a mapping, then ϕ is a morphism if and only if all restrictions

$$\phi|_{V_i}: V_i \rightarrow W$$

are morphisms. This is immediate from Remark 4.1.7 (ii) and the fact that continuity is a local property.

(v) Being a morphism is also local with respect to the image variety. That means, if $W = \bigcup_{i \in I} W_i$ is an open covering and $\phi: V \rightarrow W$ is a *continuous* mapping, then ϕ is a morphism if and only if

$$\phi|_{\phi^{-1}(W_i)}: \phi^{-1}(W_i) \rightarrow W_i$$

is a morphism, for all $i \in I$. △

The following Lemma shows that domains and codomains of morphisms can be restricted.

Lemma 4.1.12. *Let V be a k -variety and $V' \subseteq V$ a locally closed subset.*

(i) *The inclusion $V' \hookrightarrow V$ is a morphism.*

(ii) *If $\phi: W \rightarrow V$ is a morphism with $\phi(W) \subseteq V'$, so the induced mapping $\phi': W \rightarrow V'$ is again a morphism.*

Proof. (i) is obvious, since the restriction of a regular function onto a locally closed subset is again a regular function. For (ii) let $U \subseteq V'$ be open in V' , and take $g \in \mathcal{O}(U)$. Note that U need not be open in V . But without loss of generality we can assume $g = \frac{p}{q}$ on the whole of U , for some polynomials p, q (by making U smaller, if necessary). Then $\frac{p}{q}$ is regular on the open subset $\mathcal{D}_V(q)$ of V , and thus $f := \phi^* \left(\frac{p}{q} \right)$ is regular on $\phi^{-1}(\mathcal{D}_V(q))$. But then also $f|_{\phi^{-1}(U)}$ is regular, and it coincides with $\phi'^*(g)$. □

Theorem 4.1.13. *Let V, W be varieties, and assume $W \subseteq \mathbb{A}^m$ is closed. Let*

$$\phi = (\phi_1, \dots, \phi_m): V \rightarrow W$$

be a mapping. Then ϕ is a morphism if and only if

$$\phi_i \in \mathcal{O}(V) \text{ for } i = 1, \dots, m.$$

In particular, the morphisms $V \rightarrow \mathbb{A}^1$ are precisely the regular functions on V .

Proof. First assume $\phi_1, \dots, \phi_m \in \mathcal{O}(V)$. For each $p \in k[W]$ we then have $\phi^*(p) = p(\phi_1, \dots, \phi_m) \in \mathcal{O}(V)$, since $\mathcal{O}(V)$ is a k -algebra. For $q \in k[W]$ we know that

$$\phi^{-1}(\mathcal{D}_W(q)) = \mathcal{D}_V(\phi^*(q))$$

is open in V (by Lemma 4.1.6 (ii)), and the pullback of the regular function

$$\frac{p}{q} \in \mathcal{O}(\mathcal{D}_W(q))$$

is

$$\phi^* \left(\frac{p}{q} \right) = \frac{\phi^*(p)}{\phi^*(q)},$$

which is regular on $\phi^{-1}(\mathcal{D}_W(q))$ by Lemma 4.1.6 (iii). So ϕ is a morphism, since every regular function is locally of the form $\frac{p}{q}$.

Conversely, if ϕ is a morphism, then

$$\phi_i = \phi^*(x_i) \in \mathcal{O}(V),$$

since $x_i \in \mathcal{O}(W)$. □

Corollary 4.1.14. *Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be closed. Then a mapping*

$$\phi: V \rightarrow W$$

is a morphism (in the sense of the new Definition 4.1.10), if there are regular functions $p_1, \dots, p_m \in k[V]$ with

$$\phi = (p_1, \dots, p_m).$$

For affine varieties, the new Definition thus coincides with the old Definition 1.4.7.

Proof. By Theorem 4.1.13, ϕ is a morphism if and only if $\phi_i \in \mathcal{O}(V)$ holds for all i . Furthermore we have $\mathcal{O}(V) = k[V]$ by Corollary 4.1.9. □

Remark 4.1.15. A morphism of varieties can be a homeomorphism of topological spaces, without being an isomorphism of varieties. This can be seen in Example 1.4.12 (iii). △

Theorem 4.1.16. *For all $i = 0, \dots, n$ the mapping*

$$\begin{aligned} \phi_i: \mathcal{D}_+(x_i) &\rightarrow \mathbb{A}^n \\ (a_0 : \dots : a_n) &\mapsto \left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a}_i}{a_i}, \dots, \frac{a_n}{a_i} \right) \end{aligned}$$

is an isomorphism of varieties.

Proof. We know from Theorem 3.3.12 that ϕ_i is a homeomorphism. By Theorem 4.1.13 ϕ_i is also a morphism of varieties, since its components x_j/x_i are regular functions on $\mathcal{D}_+(x_i)$. For the inverse functions let us only consider $i = 0$, and let $p, q \in k[x_0, \dots, x_n]$ be homogeneous functions of the same degree. If $\tilde{p} = p(1, x_1, \dots, x_n)$ and $\tilde{q} = q(1, x_1, \dots, x_n)$ are their dehomogenizations, then

$$\phi_0(\mathcal{D}_+(x_0) \cap \mathcal{D}_+(q)) = \mathcal{D}(\tilde{q})$$

and

$$(\phi_0^{-1})^* \left(\frac{p}{q} \right) = \frac{\tilde{p}}{\tilde{q}}$$

is a regular function on $\mathcal{D}(\tilde{q})$. Again, every regular function is locally of the form p/q , and thus ϕ_0^{-1} is a morphism of varieties. \square

Theorem 4.1.17. *Let $V \subseteq \mathbb{A}^n$ be closed and let $s \in k[V]$. Then the open subvariety $\mathcal{D}_V(s)$ of V is isomorphic to the closed subvariety*

$$V' := \{(a, t) \mid a \in V, t \in \mathbb{A}^1, t \cdot s(a) - 1 = 0\} \subseteq \mathbb{A}^{n+1}.$$

In particular we have $k[V'] \cong \mathcal{O}(\mathcal{D}_V(s)) = k[V]_s$.

Proof. The mapping

$$\begin{aligned} \phi: V' &\rightarrow V \\ (a, t) &\mapsto a \end{aligned}$$

is a morphism with domain $\text{im}(\phi) = \mathcal{D}_V(s)$. Thus also $\phi: V' \rightarrow \mathcal{D}_V(s)$ is a morphism. The inverse mapping

$$\begin{aligned} \psi: \mathcal{D}_V(s) &\rightarrow V' \\ a &\mapsto \left(a, \frac{1}{s(a)} \right) \end{aligned}$$

is also a morphism, since its components are regular functions on $\mathcal{D}_V(s)$. \square

We now extend the notions of an affine and a projective variety.

Definition 4.1.18. A quasi-projective k -variety V is called **affine (projective, respectively)**, if V is isomorphic to a closed subvariety of some \mathbb{A}^n (\mathbb{P}^n respectively). If V is affine, we also write $k[V]$ instead of $\mathcal{O}(V)$. \triangle

Remark/Example 4.1.19. (i) If V is an affine variety and $s \in k[V]$, then the open subvariety $\mathcal{D}_V(s)$ is also affine, and we have $k[\mathcal{D}_V(s)] = k[V]_s$. Indeed, an isomorphism $\phi: V \rightarrow W$ to a closed subset $W \subseteq \mathbb{A}^n$ restricts to an isomorphism

$$\mathcal{D}_V(s) \rightarrow \mathcal{D}_W((\phi^{-1})^*(s))$$

by Lemma 4.1.12. We can then apply Theorem 4.1.17 for W .

(ii) The open subvariety $\mathcal{D}_+(x_i) \subseteq \mathbb{P}^n$ is affine, by Theorem 4.1.16.

(iii) Every open subvariety of \mathbb{A}^1 is affine. Since $k[x_1]$ is a principal ideal domain, every open subset is in fact of the form $\mathcal{D}(s)$ for some $s \in k[x_1]$.

(iv) In general, open subvarieties of affine varieties are not affine. For example,

$$\mathbb{A}^2 \setminus \{(0, 0)\}$$

is not affine (Exercise 74).

(v) Every closed subvariety of an affine (projective) variety is again affine (projective, respectively).

(vi) Every quasi-projective variety is an open subvariety of a projective variety. For locally closed $V \subseteq \mathbb{P}^n$ this is clear by definition, and a closed $V \subseteq \mathbb{A}^n$ is open in the projective closure \bar{V} . \triangle

Theorem 4.1.20. Let $V \subseteq \mathbb{P}^n$ be closed and $i \in \{0, \dots, n\}$. Then the open subset

$$V_i := V \cap \mathcal{D}_+(x_i)$$

of V is affine, and the canonical homomorphism

$$k_+[V]_{(\bar{x}_i)} \rightarrow k[V_i]$$

is an isomorphism.

Proof. As a closed subset of the affine variety $\mathcal{D}_+(x_i)$, V_i is affine. Elements of $k_+[V]_{(\bar{x}_i)}$ are of the form $\frac{p}{\bar{x}_i^r}$ with $p \in k_+[V]$ homogeneous of degree r . Thus they define regular functions on V_i . This defines a canonical homomorphism $\varphi: k_+[V]_{(\bar{x}_i)} \rightarrow k[V_i]$ which is injective, since $\varphi\left(\frac{p}{\bar{x}_i^r}\right) = 0$ implies $p \equiv 0$ on V_i , and thus $\bar{x}_i p \equiv 0$ on V . We therefore obtain $\bar{x}_i p = 0$ in $k_+[V]$ and thus $\frac{p}{\bar{x}_i^r} = 0$ in $k_+[V]_{(\bar{x}_i)}$.

For surjectivity first note that the coordinate algebra of an affine variety is generated by the coordinate functions. With respect to the isomorphism $\mathcal{D}_+(x_i) \cong \mathbb{A}^n$ the coordinate functions are exactly the x_j/x_i on $\mathcal{D}_+(x_i)$, and the coordinate functions of V_i are thus \bar{x}_j/\bar{x}_i . They however all lie in the image of φ . \square

Corollary 4.1.21. *Every variety admits a basis of open sets that consists of affine open sets.*

Proof. If the variety V is affine, this follows from Theorem 4.1.17, since the $\mathcal{D}_V(s)$ for $s \in k[V]$ form a basis of open sets. Every projective variety is covered by open affine subsets, by Theorem 4.1.20, and thus the statement is also true for projective varieties and open subsets of such. \square

Example 4.1.22. (i) Let $p_0, \dots, p_n \in k[x_0, \dots, x_m]$ be homogeneous of the same degree. Then the mapping

$$\begin{aligned} \phi: \mathbb{P}^m \setminus \mathcal{V}_+(p_0, \dots, p_n) &\rightarrow \mathbb{P}^n \\ a &\mapsto (p_0(a) : \dots : p_n(a)) \end{aligned}$$

is a morphism. In fact $\mathbb{P}^m \setminus \mathcal{V}_+(p_0, \dots, p_n)$ is the union of open sets $\mathcal{D}_+(p_i)$ and it suffices to show that

$$\phi: \mathcal{D}_+(p_i) \rightarrow \mathbb{P}^n$$

is a morphism for all $i = 0, \dots, n$. Under the isomorphism

$$\phi(\mathcal{D}_+(p_i)) \subseteq \mathcal{D}_+(x_i) \cong \mathbb{A}^n$$

the mapping ϕ translates to

$$a \mapsto \left(\frac{p_0(a)}{p_i(a)}, \dots, \frac{\widehat{p_i(a)}}{p_i(a)}, \dots, \frac{p_n(a)}{p_i(a)} \right).$$

This however is a morphism by Theorem 4.1.13, since $\frac{p_j}{p_i}$ is regular on $\mathcal{D}_+(p_i)$.

(ii) A special case of (i) are linear morphisms. Let M be a matrix of size $(n+1) \times (m+1)$ over k with $\text{rank}(M) = m+1$. Then

$$\begin{aligned} \phi_M: \mathbb{P}^m &\rightarrow \mathbb{P}^n \\ [v] &\mapsto [Mv] \end{aligned}$$

is a morphism, and we have $\phi_{MN} = \phi_M \circ \phi_N$. In case $m = n$ we obtain a group homomorphism

$$\text{GL}_{n+1}(k) \rightarrow \text{Aut}_k(\mathbb{P}^n)$$

with kernel k^*I . This gives an embedding

$$\text{PGL}_{n+1}(k) \hookrightarrow \text{Aut}_k(\mathbb{P}^n),$$

which can be shown to be surjective.

(iii) Consider the following morphism:

$$\begin{aligned} \phi: \mathbb{P}^1 &\rightarrow \mathbb{P}^2 \\ (a_0 : a_1) &\mapsto (a_0^2 : a_0a_1 : a_1^2). \end{aligned}$$

We have

$$\phi(\mathbb{P}^1) \subseteq \mathcal{V}_+(x_1^2 - x_0x_2) =: C$$

and thus $\phi: \mathbb{P}^1 \rightarrow C$ is also a morphism, even an isomorphism. To see this we define the inverse morphism as

$$\begin{aligned} \psi: C &\rightarrow \mathbb{P}^1 \\ (b_0 : b_1 : b_2) &\mapsto \begin{cases} (b_0 : b_1) & \text{if } b_0 \neq 0 \\ (b_1 : b_2) & \text{if } b_2 \neq 0 \end{cases} \end{aligned}$$

We have $C \subseteq \mathcal{D}_+(x_0) \cup \mathcal{D}_+(x_2)$, and on the intersection within C both definitions coincide, since $b_0b_2 = b_1^2$. Thus ψ is well-defined on C and a morphism, since on both subsets $C \cap \mathcal{D}_+(x_0)$, $C \cap \mathcal{D}_+(x_2)$ it is one, by (i). It is now easily checked that ψ is inverse to ϕ . \triangle

Remark 4.1.23. We have just shown

$$\mathbb{P}^1 \cong C = \mathcal{V}_+(x_1^2 - x_0x_2) \subseteq \mathbb{P}^2.$$

Let us now compare the projective coordinate rings of both varieties. We have

$$k_+[C] = k[x_0, x_1, x_2]/(x_1^2 - x_0x_2)$$

and

$$\dim_k k_+[C]_1 = 3,$$

since $(x_1^2 - x_0x_2)$ does not contain any homogeneous polynomial of degree 1. This implies that $k_+[C]$ is *not* generated by 2 elements as a k -algebra (the homogeneous terms of degree 1 of two generators would otherwise span $k_+[C]_1$ over k). On the other hand, $k_+[\mathbb{P}^1] = k[x_0, x_1]$ is obviously generated by two elements. So we have

$$\mathbb{P}^1 \cong C \text{ and } k_+[\mathbb{P}^1] \not\cong k_+[C].$$

The projective coordinate ring is thus not even invariant under isomorphism! This is a significant difference to the affine case, where it even classifies isomorphism completely. \triangle

4.2 The Veronese Embedding

In this section we introduce the Veronese Embedding, which can be used to linearize equations. We will show that we can reduce every setup to quadratic equations in projective space

Construction 4.2.1. Let $\underline{x} = (x_0, \dots, x_n)$ and $d \geq 1$ be fixed. Let

$$m_0, m_1, \dots, m_N \in k[\underline{x}]$$

be all monomials in \underline{x} of degree (exactly) d . We have $N = \binom{n+d}{n}$ by Exercise 46. As in Example 4.1.22 (i) we obtain a k -morphism

$$\begin{aligned} v_d: \mathbb{P}^n &\rightarrow \mathbb{P}^N \\ a &\mapsto (m_0(a) : \dots : m_N(a)) \end{aligned}$$

which is called the **Veronese embedding of degree d** . Its image

$$V_n^d := v_d(\mathbb{P}^n)$$

is called the **Veronese variety of degree d** . △

Theorem 4.2.2. *The Veronese variety $V_n^d = v_d(\mathbb{P}^n)$ is a closed subvariety of \mathbb{P}^N , and*

$$v_d: \mathbb{P}^n \rightarrow V_n^d$$

is an isomorphism of k -varieties.

Proof. Note that we know closedness of V_n^d already from Corollary 3.4.5. However, we will provide an explicit description by homogeneous equations, since we need it for the later construction of the inverse morphism. Set

$$J := \{\alpha \in \mathbb{N}^n \mid |\alpha| = d\}.$$

We can use homogeneous coordinates $(z_\alpha : \alpha \in J)$ on \mathbb{P}^N and have

$$v_d(a) = (a^\alpha)_{\alpha \in J}$$

for all $a \in \mathbb{P}^n$. Now let $Z \subseteq \mathbb{P}^N$ be the zero set of all quadratic polynomials of the form

$$z_\alpha z_\beta - z_\gamma z_\delta$$

for $\alpha, \beta, \gamma, \delta \in J$ with $\alpha + \beta = \gamma + \delta$. Then clearly $V_n^d \subseteq Z$, since

$$a^\alpha a^\beta = a^{\alpha+\beta} = a^{\gamma+\delta} = a^\gamma a^\delta$$

holds for all such $\alpha, \beta, \gamma, \delta$. Thus

$$v_d: \mathbb{P}^n \rightarrow Z$$

is a morphism of varieties. We will now define an inverse mapping. To this end let $\beta \in J$ and $i \in \{0, \dots, n\}$ with $\beta_i \geq 1$. We define

$$\begin{aligned} \phi_{\beta,i}: Z \cap \mathcal{D}_+(z_\beta) &\rightarrow \mathbb{P}^n \\ (b_\alpha)_{\alpha \in J} &\mapsto (b_{\beta-e_i+e_0} : b_{\beta-e_i+e_1} : \dots : b_{\beta-e_i+e_n}). \end{aligned}$$

Then $\phi_{\beta,i}$ is a well-defined morphism as in Example 4.1.22 (i) (the image of $\phi_{\beta,i}$ lies inside $\mathcal{D}_+(x_i)$). Now let $\beta, \gamma \in J, \beta_i \geq 1, \gamma_j \geq 1$ and $b \in Z \cap \mathcal{D}_+(z_\beta) \cap \mathcal{D}_+(z_\gamma)$. We then have

$$\phi_{\beta,i}(b) = \phi_{\gamma,j}(b).$$

As explained in Definition 3.1.2 it is enough to show

$$b_{\beta-e_i+e_k} b_{\gamma-e_j+e_l} = b_{\beta-e_i+e_l} b_{\gamma-e_j+e_k}$$

for all $k, l = 0, \dots, n$ to obtain this. But the last equation is true by definition of Z , since the indices on both sides sum up to the same tuple.

The $\phi_{\beta,i}$ thus define a global morphism

$$\phi: Z \rightarrow \mathbb{P}^n.$$

We have $\phi \circ v_d = \text{id}_{\mathbb{P}^n}$: for $a \in \mathbb{P}^n$ just choose some i with $a_i \neq 0$ and set $\beta := de_i$. Then $a^\beta = a_i^d \neq 0$ and thus $v_d(a) = (a^\alpha)_{\alpha \in J} \in \mathcal{D}_+(z_\beta)$. We therefore have

$$\begin{aligned} \phi(v_d(a)) &= \phi_{\beta,i}(v_d(a)) \\ &= (a^{\beta-e_i+e_0} : \dots : a^{\beta-e_i+e_n}) \\ &= (a_i^{d-1} a_0 : \dots, a_i^{d-1} a_n) \\ &= a. \end{aligned}$$

Conversely we also have $v_d \circ \phi = \text{id}_Z$: choose $\beta \in J, i \in \{0, \dots, n\}$ with $\beta_i \geq 1$ and $b \in Z \cap \mathcal{D}_+(z_\beta)$. Then

$$v_d(\phi(b)) = (b_{\beta-e_i+e_0}^{\alpha_0} \cdots b_{\beta-e_i+e_n}^{\alpha_n})_{\alpha \in J}$$

and this defines the same point as $b = (b_\alpha)_{\alpha \in J}$ in \mathbb{P}^N . To see this we show the following equality, for all $\gamma, \delta \in J$:

$$b_\gamma \cdot b_{\beta - e_i + e_0}^{\delta_0} \cdots b_{\beta - e_i + e_n}^{\delta_n} = b_\delta \cdot b_{\beta - e_i + e_0}^{\gamma_0} \cdots b_{\beta - e_i + e_n}^{\gamma_n}.$$

Note that the indices on both sides sum up to the same:

$$\gamma + \delta_0 \cdot (\beta - e_i + e_0) + \cdots + \delta_n \cdot (\beta - e_i + e_n) = \gamma + d \cdot (\beta - e_i) + \delta = \cdots$$

Thus the equation above is a consequence of the equations defining Z (Exercise 80). \square

Remark 4.2.3. (i) We have found explicit equations defining the Veronese variety V_n^d of degree d . If \mathbb{P}^N is equipped with homogeneous coordinates $(z_\alpha)_{|\alpha|=d}$, these are

$$z_\alpha z_\beta - z_\gamma z_\delta$$

with $\alpha + \beta = \gamma + \delta$. In particular, Z is an intersection of quadratic hypersurfaces, so-called **quadrics**.

(ii) For $n = 1$ one has

$$\begin{aligned} v_d: \mathbb{P}^1 &\rightarrow \mathbb{P}^d \\ (a_0 : a_1) &\mapsto (a_0^d : a_0^{d-1}a_1 : \cdots : a_0a_1^{d-1} : a_1^d) \end{aligned}$$

The image V_1^d is also called **rational normal curve of degree d** . In case $d = 2$ it is the curve from Example 4.1.22 (iii).

(iii) For every locally closed subvariety V of \mathbb{P}^n , the Veronese embedding v_d defines an isomorphism between V and $v_d(V) \subseteq \mathbb{P}^N$. \triangle

Remark 4.2.4. Let $p \in k[x_0, \dots, x_n]$ be homogeneous of degree d , and write $p = \sum_{|\alpha|=d} p_\alpha \underline{x}^\alpha$. Then for $a \in \mathbb{P}^n$ we have

$$a \in \mathcal{V}_+(p) \Leftrightarrow v_d(a) \in \mathcal{V}_+ \left(\sum_{|\alpha|=d} p_\alpha z_\alpha \right).$$

The isomorphism v_d thus maps the degree d hypersurface $\mathcal{V}_+(p)$ to the intersection of V_n^d with a hyperplane. A slightly more general statement is the following Theorem. \triangle

Theorem 4.2.5. *Every closed k -subvariety of \mathbb{P}^n is isomorphic (via some v_d) to an intersection*

$$V_n^d \cap L$$

where L is a linear subspace of \mathbb{P}^N defined over k .

Proof. For $p \in k[\underline{x}]$ homogeneous and all $r \geq 0$ we have

$$\mathcal{V}_+(p) = \mathcal{V}_+(x_0^r p, \dots, x_n^r p).$$

Thus every closed subset of \mathbb{P}^n is definable with homogeneous equations of the same degree, and the statement follows from the last remark. \square

Corollary 4.2.6. *Every projective k -variety is isomorphic to an intersection of quadratic k -hypersurfaces in some \mathbb{P}^n .*

Proof. The linear space L from the last theorem is isomorphic to some \mathbb{P}^n , and V_n^d is defined by quadratic equations. \square

We will now prove a generalization of Theorem 4.1.20 to general hypersurfaces (and a projective version of Theorem 4.1.8).

Theorem 4.2.7. *Let $V \subseteq \mathbb{P}^n$ be closed and $s \in k_+[V]$ homogeneous. Then the open subset $\mathcal{D}_{V,+}(s)$ of V is affine, with*

$$k[\mathcal{D}_{V,+}(s)] = k_+[V]_{(s)}.$$

Proof. It is enough to prove the statement for $V = \mathbb{P}^n$, since for homogeneous $t \in k[\underline{x}]$ we know that $\mathcal{D}_{V,+}(\bar{t}) = V \cap \mathcal{D}_+(t)$ is a closed subvariety of the affine variety $\mathcal{D}_+(t) \subseteq \mathbb{P}^n$, and thus also affine. In addition, construction of quotients and homogeneous localization commute:

$$(R/I)_{(\bar{t})} \cong R_{(t)}/(I).$$

So assume $V = \mathbb{P}^n$ and let $s \in k[\underline{x}]$ be homogeneous of degree d . Consider the Veronese embedding

$$v_d: \mathbb{P}^n \rightarrow \mathbb{P}^N$$

and let $H_s \subseteq \mathbb{P}^N$ be the hypersurface associated to $\mathcal{V}_+(s) \subseteq \mathbb{P}^n$. We have

$$\mathcal{D}_+(s) = v_d^{-1}(\mathbb{P}^N \setminus H_s) \cong V_n^d \cap (\mathbb{P}^N \setminus H_s).$$

From $\mathbb{P}^N \setminus H_s \cong \mathbb{A}^N$ we obtain that $\mathcal{D}_+(s)$ is an affine variety. The canonical homomorphism

$$k[\underline{x}]_{(s)} \rightarrow \mathcal{O}(\mathcal{D}_+(s))$$

is injective, as we have seen in the proof of Theorem 4.1.20. Surjectivity follows exactly in the same way, since the generating coordinate functions on $\mathbb{P}^N \setminus H_s \cong \mathbb{A}^N$ are precisely the z_α/s , which correspond to the $\underline{x}^\alpha/s \in k[\underline{x}]_{(s)}$ under the isomorphism v_d . \square

4.3 Direct Products

In the affine setting there is a canonical identification $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$, and for two closed subsets $V \subseteq \mathbb{A}^m, W \subseteq \mathbb{A}^n$ we know that $V \times W$ is closed in \mathbb{A}^{m+n} . We can thus easily form direct products of affine varieties in the sense of Chapter 1. This is harder for projective varieties. For example, there is *no* canonical identification between $\mathbb{P}^m \times \mathbb{P}^n$ and \mathbb{P}^{m+n} . But for vector spaces V, W over the same field, there is a well-defined embedding

$$\begin{aligned} \mathbb{P}(V) \times \mathbb{P}(W) &\rightarrow \mathbb{P}(V \otimes W) \\ ([v], [w]) &\mapsto [v \otimes w] \end{aligned}$$

the so-called **Segre-Embedding**. We now examine it in the case of \mathbb{P}^m and \mathbb{P}^n in more detail.

Construction 4.3.1. For $m, n \geq 1$ we identify \mathbb{P}^{mn+m+n} with the projective space

$$\mathbb{P}(\text{Mat}_{(m+1) \times (n+1)}(K)).$$

For a matrix $A \in \text{Mat}_{(m+1) \times (n+1)}(K)$ one has

$$\text{rank}(A) = 1 \Leftrightarrow A = uv^t$$

for certain column vectors $0 \neq u \in K^{m+1}, 0 \neq v \in K^{n+1}$. Here u, v are uniquely determined up to scaling. The mapping

$$\begin{aligned} K^{m+1} \times K^{n+1} &\rightarrow \text{Mat}_{(m+1) \times (n+1)} \\ (u, v) &\mapsto uv^t \end{aligned}$$

thus induces a well-defined injective mapping

$$\sigma: \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}(\text{Mat}_{(m+1) \times (n+1)}(K)) = \mathbb{P}^{mn+m+n},$$

the so-called **Segre-embedding**. Its image

$$S_{m,n} := \{[A] \mid \text{rank}(A) = 1\}$$

is called **Segre variety**. $S_{m,n}$ is really a k -closed subset of \mathbb{P}^{mn+m+n} , since it is defined by the vanishing of all 2×2 -minors of A . \triangle

Definition 4.3.2. If we talk about $\mathbb{P}^m \times \mathbb{P}^n$ as a variety from now on, we always mean the projective k -variety $S_{m,n}$. Even if we work with the Cartesian product $\mathbb{P}^m \times \mathbb{P}^n$ set-theoretically, all statements that refer to the structure of a variety are to be understood in $S_{m,n}$ via σ . \triangle

Theorem 4.3.3. (i) Both projections

$$\text{pr}_1: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^m, \quad \text{pr}_2: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^n$$

are morphisms of varieties.

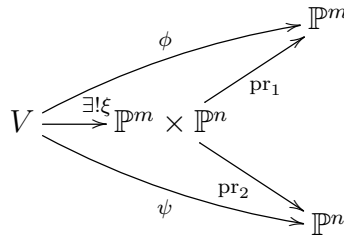
(ii) If $\phi: V \rightarrow \mathbb{P}^m$ and $\psi: V \rightarrow \mathbb{P}^n$ are morphisms of varieties, then

$$\begin{aligned} (\phi, \psi): V &\rightarrow \mathbb{P}^m \times \mathbb{P}^n \\ a &\mapsto (\phi(a), \psi(a)) \end{aligned}$$

is also a morphism.

(iii) The projective variety $\mathbb{P}^m \times \mathbb{P}^n$, together with the morphisms pr_1 and pr_2 , fulfills the universal property of a direct product:

For any two morphisms $\phi: V \rightarrow \mathbb{P}^m$, $\psi: V \rightarrow \mathbb{P}^n$ there is exactly one morphism $\xi: V \rightarrow \mathbb{P}^m \times \mathbb{P}^n$ such that $\text{pr}_1 \circ \xi = \phi$, $\text{pr}_2 \circ \xi = \psi$.



Proof. (i) If we translate to $S_{m,n}$ via σ , we have for $[A] \in S_{m,n}$

$$\text{pr}_1([A]) = [u],$$

where u is an arbitrary non-zero column of A . This yields a local definition of pr_1 as a morphism in the sense of Example 4.1.22 (i), on open subsets of $S_{m,n}$ defined

by the non-vanishing of a certain column. For pr_2 we use the same argument with rows instead.

(ii) Choose a local definition

$$\phi = (1 : \phi_1 : \dots : \phi_m) : \phi^{-1}(\mathcal{D}_+(x_0)) \rightarrow \mathcal{D}_+(x_0) \subseteq \mathbb{P}^m$$

$$\psi = (1 : \psi_1 : \dots : \psi_n) : \psi^{-1}(\mathcal{D}_+(y_0)) \rightarrow \mathcal{D}_+(y_0) \subseteq \mathbb{P}^n$$

with regular functions ϕ_i, ψ_i . Via the identification

$$\mathcal{D}_+(x_0) \times \mathcal{D}_+(y_0) \xleftarrow{\sigma} \mathcal{D}_+(z_{00}) \subseteq \mathbb{P}(\text{Mat}_{(m+1) \times (n+1)}(K))$$

the mapping (ϕ, ψ) then translates to

$$\begin{aligned} & \phi^{-1}(\mathcal{D}_+(x_0)) \cap \psi^{-1}(\mathcal{D}_+(y_0)) \rightarrow \mathcal{D}_+(z_{00}) \\ & a \mapsto \begin{pmatrix} 1 & \psi_1(a) & \cdots & \psi_n(a) \\ \phi_1(a) & \phi_1(a)\psi_1(a) & & \\ \vdots & & \ddots & \\ \phi_m(a) & & & \phi_m(a)\psi_n(a) \end{pmatrix} \end{aligned}$$

Since $\mathcal{D}_+(z_{00})$ is affine and all components are regular functions, (ϕ, ψ) is a morphism. (iii) is then clear, since ξ is uniquely determined by ϕ and ψ . \square

Remark 4.3.4. (i) If $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ are locally closed subsets, then

$$\text{pr}_1^{-1}(V) \cap \text{pr}_2^{-1}(W) \subseteq S_{m,n}$$

is also locally closed, as the inverse image with respect to morphisms. In the direct product $\mathbb{P}^m \times \mathbb{P}^n$, via σ , this corresponds to the product $V \times W$. In this way we can understand the direct product of any two k -varieties as a k -variety, and $V \times W$ fulfills the universal property of direct products. This is immediate from Theorem 4.3.3 and Lemma 4.1.12. The triple

$$(V \times W, \text{pr}_1, \text{pr}_2)$$

is uniquely determined up to isomorphism, by the universal property.

(ii) For closed subsets $V \subseteq \mathbb{A}^m$, $W \subseteq \mathbb{A}^n$ we have already constructed the direct product $V \times W \subseteq \mathbb{A}^{m+n}$ explicitly, in the first chapter. It obviously has the universal property, and thus coincides with the new construction. In particular, the direct product of affine varieties is again affine.

(iii) The direct product of projective varieties is again projective, by construction.
 (iv) In Section 3.4 we have considered $\mathbb{P}^m \times \mathbb{P}^n$ just as a set, and we have used bihomogeneous polynomials to construct subsets X . We now observe that these sets X are precisely the closed subsets with respect to our structure as a variety. Homogeneous polynomials on $S_{m,n}$ correspond, via σ , directly to bihomogeneous polynomials on $\mathbb{P}^m \times \mathbb{P}^n$, which are homogeneous of the same degree in both types of variables \underline{x} and \underline{y} . With the same argument as in the proof of Theorem 4.2.5 we see that this is not a restriction. \triangle

Example 4.3.5. $\mathbb{P}^1 \times \mathbb{P}^1$ is the variety of singular 2×2 -matrices in $\mathbb{P}(\text{Mat}_{2 \times 2}(K))$, and thus a quadric in \mathbb{P}^3 :

$$\mathbb{P}^1 \times \mathbb{P}^1 = \mathcal{V}_+ \left(\det \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \right) = \mathcal{V}_+(x_0x_3 - x_1x_2) \subseteq \mathbb{P}^3.$$

On $\mathbb{P}^1 \times \mathbb{P}^1$ there are two families of lines :

$$\{a\} \times \mathbb{P}^1 \text{ as well as } \mathbb{P}^1 \times \{a\},$$

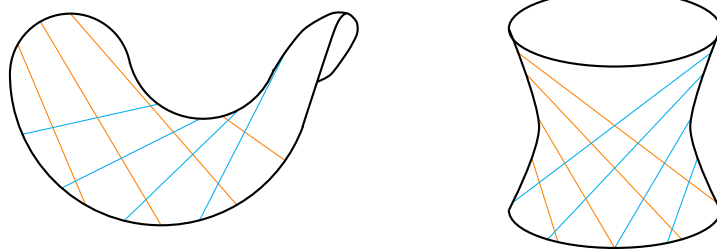
for $a \in \mathbb{P}^1$. If $a = (a_0 : a_1)$ for example, then

$$\{a\} \times \mathbb{P}^1 = \left\{ \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} \mid \text{rank} \begin{pmatrix} a_0 & b_0 & b_1 \\ a_1 & b_2 & b_3 \end{pmatrix} = 1 \right\},$$

and this is defined by the two linear equations

$$a_0x_2 - a_1x_0 \text{ und } a_0x_3 - a_1x_1$$

within $S_{1,1}$. The same is true for $\mathbb{P}^1 \times \{a\}$. Any two different lines from the same family are disjoint, any two lines from different families intersect in precisely one point. The following picture shows two affine sections of $\mathbb{P}^1 \times \mathbb{P}^1$, on the left with $\mathcal{D}_+(x_0)$, on the right with $\mathcal{D}_+(x_0 + x_3)$:



One can show that any two infinite closed subsets of \mathbb{P}^2 have nonempty intersection (compare Corollaries 4.3.18 and 4.3.19). In particular we obtain $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$. \triangle

Theorem 4.3.6. *Let V, W be k -varieties and V projective. Then for any closed $X \subseteq V \times W$ the image $\text{pr}_2(X)$ is closed in W .*

Proof. For $\tilde{X} \subseteq S_{m,n}$ closed we know that $\tilde{X} \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is definable by bihomogeneous polynomials, and $\tilde{X} \cap (\mathbb{P}^m \times \mathbb{A}^n)$ is definable by \underline{x} -homogeneous polynomials. Thus the statement follows for affine W directly from Theorem 3.4.1, since $\tilde{X} \cap (V \times W) \subseteq \mathbb{P}^m \times \mathbb{A}^n$ is again definable like that. For general W we choose an open covering $W = \bigcup_{i \in I} W_i$ by affine subsets W_i and consider $X_i := X \cap (V \times W_i)$ in $V \times W_i$. Then $\text{pr}_2(X_i) = \text{pr}_2(X) \cap W_i$ is closed in W_i , and this implies the statement. \square

The following Lemma proves that quasi-projective varieties are **separated**. Often, nice arguments about topological spaces use the Hausdorff property. But varieties are not Hausdorff, unfortunately. However, separateness can often be used instead. Note that a topological space is Hausdorff if and only if the diagonal $\{(x, x) \mid x \in X\}$ is closed in $X \times X$ (with respect to the product topology).

Lemma 4.3.7 (Separateness). *For every k -variety V , the diagonal*

$$\Delta_V := \{(a, a) \mid a \in V\}$$

is a closed subset of $V \times V$.

Proof. Let $\iota: V \hookrightarrow \mathbb{P}^n$ be a locally closed embedding. By the universal property, the morphisms

$$V \times V \xrightarrow{\text{pr}_1} V \xrightarrow{\iota} \mathbb{P}^n$$

$$V \times V \xrightarrow{\text{pr}_2} V \xrightarrow{\iota} \mathbb{P}^n$$

induce a morphism

$$\iota \times \iota: V \times V \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

and Δ_V is the inverse image of $\Delta_{\mathbb{P}^n}$. So it suffices to prove the statement for $V = \mathbb{P}^n$. With respect to the identification

$$\mathbb{P}^n \times \mathbb{P}^n \xleftarrow{\sigma} S_{n,n}$$

the diagonal corresponds to the symmetric matrices in $S_{n,n}$. So within $S_{n,n}$, $\Delta_{\mathbb{P}^n}$ is defined by the equations

$$z_{ij} - z_{ji} \quad \text{for } i, j = 0, \dots, n.$$

This proves closedness. \square

Remark 4.3.8. Alternatively, one can define $\Delta_{\mathbb{P}^n}$ within $\mathbb{P}^n \times \mathbb{P}^n$ by the bihomogeneous equations

$$x_i y_j - x_j y_i = 0$$

for $i, j = 0, \dots, n$. \triangle

The following statement is obvious for Hausdorff spaces. We will see how we can prove it with separateness instead.

Lemma 4.3.9 (Identity Theorem). *Let V, W be varieties and $\phi, \psi: V \rightarrow W$ morphisms with $\phi \equiv \psi$ on a Zariski dense subset $U \subseteq V$. Then $\phi = \psi$.*

Proof. The set

$$V' := \{a \in V \mid \phi(a) = \psi(a)\}$$

is dense in V , and also the inverse image of Δ_W under the morphism

$$(f, g): V \rightarrow W \times W.$$

So V' is closed, and this implies $V' = V$. \square

Theorem 4.3.10. *Let $\phi: V \rightarrow W$ be a morphism of varieties. Then its graph*

$$\Gamma_\phi := \{(a, \phi(a)) \mid a \in V\}$$

is a closed subset of $V \times W$. The graph morphism

$$\begin{aligned} \gamma_\phi: V &\rightarrow \Gamma_\phi \\ a &\mapsto (a, \phi(a)) \end{aligned}$$

is an isomorphism.

Proof. Γ_ϕ is the inverse image of Δ_W under the morphism

$$\begin{aligned} \phi \times \text{id}_W: V \times W &\rightarrow W \times W \\ (a, b) &\mapsto (\phi(a), b). \end{aligned}$$

Thus Γ_ϕ is closed. The universal property of the product implies that γ_ϕ is a morphism, and the inverse mapping

$$\begin{aligned}\Gamma_\phi &\rightarrow V \\ (a, \phi(a)) &\mapsto a\end{aligned}$$

is the restriction of the projection, and thus also a morphism. \square

Corollary 4.3.11. *For every variety V we have $V \cong \Delta_V$, via $a \mapsto (a, a)$.*

Proof. This follows from Theorem 4.3.10, applied to $\phi = \text{id}_V : V \rightarrow V$. \square

Corollary 4.3.12. *Let V be a variety and let U_1, \dots, U_r be open affine subsets of V . Then $U_1 \cap \dots \cap U_r$ is also affine.*

Proof. It is clearly enough to consider the case $r = 2$. Then

$$U_1 \cap U_2 \cong \Delta_{U_1 \cap U_2}$$

and

$$\Delta_{U_1 \cap U_2} = (U_1 \times U_2) \cap \Delta_V.$$

Thus $\Delta_{U_1 \cap U_2}$, as a closed subset of the affine variety $U_1 \times U_2$, is affine. \square

Theorem 4.3.13. *Let V be projective and $\phi : V \rightarrow W$ a morphism. Then $\phi(V)$ is closed in W .*

Proof. We know by Theorem 4.3.10 that Γ_ϕ is closed in $V \times W$, and thus

$$\phi(V) = \text{pr}_2(\Gamma_\phi)$$

is closed in W by Theorem 4.3.6. \square

Definition 4.3.14. Let V be a locally closed subset of \mathbb{A}^n (or \mathbb{P}^n). Let $k \subseteq L \subseteq K$ be an intermediate field. Then

$$V(L) := V \cap L^n,$$

or

$$V(L) := V \cap \{[v] \in \mathbb{P}^n \mid v \in L^{n+1}\},$$

respectively, is called the set of **L -rational points of V** . \triangle

Remark 4.3.15. (i) Let V be a k -variety and $k \subseteq L \subseteq K$ an intermediate field. For every $f \in \mathcal{O}(V)$ we then have

$$f(V(L)) \subseteq L,$$

since f is defined locally by fractions of polynomials over k . For every k -morphism $\phi: V \rightarrow W$ we thus have

$$\phi(V(L)) \subseteq W(L),$$

since ϕ is defined locally by regular functions. In particular, every isomorphism yields a bijection between the set of L -rational points.

(ii) Finding L -rational points on a variety can be very hard. For example, the fact that

$$\mathcal{V}_+(z^n - x^n - y^n) \subseteq \mathbb{P}^2(\mathbb{C})$$

does not have (nontrivial) \mathbb{Q} -rational points for any n is exactly the famous Fermat's last theorem. \triangle

Theorem 4.3.16. *Let V be an irreducible projective k -variety. Then $\mathcal{O}(V)$ is a finite field extension of k . If $V(k) \neq \emptyset$, then $\mathcal{O}(V) = k$, i.e. every regular function is constant.*

Proof. Let $a \in V$ and $f \in \mathcal{O}(V)$ with $f(a) = 0$. Consider f as a morphism

$$f: V \rightarrow \mathbb{A}^1.$$

By Theorem 4.3.13 f has a closed image, and thus either $f(V) = \mathbb{A}^1$ or $f(V)$ is finite. Now

$$f: V \rightarrow \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$$

is also a morphism with closed image, and thus $f(V) = \mathbb{A}^1$ is impossible. So

$$f(V) = \{0, r_1, \dots, r_d\} = \mathcal{V}(p) \subseteq \mathbb{A}^1$$

for some $p \in k[t]$. Write $p = t^s q$ with $q(0) \neq 0$. Then $f(V) = \{0\} \dot{\cup} \mathcal{V}(q)$. Now $f(V)$ is irreducible (since V is irreducible), and this implies $f(V) = \{0\}$. We have thus shown that a regular function with a zero must be constant. So $\mathcal{O}(V)$ is a field, by Lemma 4.1.6 (iii).

By Hilbert's Nullstellensatz we have $V(\bar{k}) \neq \emptyset$, and thus there exists a finite field extension L/k and some $a \in V(L)$. The k -algebra homomorphism

$$\begin{aligned} \mathcal{O}(V) &\rightarrow L \\ f &\mapsto f(a) \end{aligned}$$

is injective, as we have just shown. Thus $\mathcal{O}(V)$ is a finite field extension of k . In case $V(k) \neq \emptyset$, the same argument shows $\mathcal{O}(V) = k$. \square

Corollary 4.3.17. *If V is both affine and projective, then $|V| < \infty$.*

Proof. V has only finitely many irreducible components, each one of which is again closed and thus both affine and projective. So we can assume that V is irreducible. By Theorem 4.3.16 we have

$$\dim_k k[V] = \dim_k \mathcal{O}(V) < \infty,$$

and thus V is finite by Theorem 1.4.16. \square

Corollary 4.3.18. *Let $V \subseteq \mathbb{P}^n$ be a hypersurface and $Z \subseteq \mathbb{P}^n$ an infinite closed set. Then $V \cap Z \neq \emptyset$.*

Proof. Let $V = \mathcal{V}_+(p)$ for some homogeneous $p \in k[x]$. Now $V \cap Z = \emptyset$ would imply that $Z \subseteq \mathcal{D}_+(p)$ is a closed subset of an affine variety, which is itself affine. This implies $|Z| < \infty$, a contradiction. \square

Corollary 4.3.19. *For $n \geq 2$, any two hypersurfaces in \mathbb{P}^n have nonempty intersection.*

Proof. By Corollary 4.3.18 it is enough to show that hypersurfaces are infinite. It is enough to consider affine hypersurfaces and the case $K = \bar{k}$. So let $V = \mathcal{V}(p) \neq \emptyset$ be a hypersurface in \mathbb{A}^n . Choose some i with $p \notin k[x_i]$. Then $(p) \cap k[x_i] = \{0\}$, and from Theorem 1.4.13 we see that

$$\text{pr}_i(V) \subseteq \mathbb{A}^1 = \bar{k}$$

is k -Zariski dense, and thus infinite. \square

4.4 Rational Functions and Maps

We will now generalize the notion of a morphism, to obtain enough of such mappings also for projective varieties.

Definition 4.4.1. Let V, W be k -varieties.

(i) A **rational map** $f: V \dashrightarrow W$ is an equivalence class of morphisms

$$\phi: U \rightarrow W$$

on open and dense subsets U of V , under the following equivalence relation:

$$(\phi: U \rightarrow W) \sim (\psi: U' \rightarrow W)$$

$$\Downarrow$$

$$\exists U'' \subseteq U \cap U' \text{ open and dense in } V \text{ with } \phi \equiv \psi \text{ on } U''.$$

We then also write $f = [\phi]$ for the equivalence class of ϕ .

(ii) The set of all rational maps from V to W is denoted by

$$\text{Rat}_k(V, W).$$

(iii) A rational map $f: V \dashrightarrow \mathbb{A}^1$ is called a **rational function** on V . △

Definition 4.4.2. Let $f: V \dashrightarrow W$ be a rational map. Then

$$\text{dom}(f) := \bigcup \{U \mid U \subseteq V \text{ open and dense, } \exists \phi: U \rightarrow W \text{ with } f = [\phi]\}$$

is called the **domain** of f . △

Theorem 4.4.3. Let $f: V \dashrightarrow W$ be a rational map. Then there exists a morphism $f_0: \text{dom}(f) \rightarrow W$, such that every representative of f is a restriction of f_0 .

Proof. Let $\phi: U \rightarrow W$ and $\psi: U' \rightarrow W$ be two representatives of f . By the Identity Theorem 4.3.9 we have $\phi \equiv \psi$ on $U \cap U'$. Thus f_0 can be defined locally by the representatives of f . □

Remark/Example 4.4.4. (i) For every open and dense subset $U \subseteq V$ we have $\text{Rat}_k(V, W) = \text{Rat}_k(U, W)$.

(ii) Let V be an irreducible affine k -variety and $F = \text{Quot}(k[V])$. Then every element of F defines a rational function on V . If $\frac{a}{b}$ with $a, b \in k[V], b \neq 0$, then $\frac{a}{b}: \mathcal{D}_V(b) \rightarrow \mathbb{A}^1$ is a morphism. Since $\mathcal{D}_V(b)$ is open and nonempty, it is automatically dense in V , and we obtain the rational function $f = [\frac{a}{b}]$. If

$$\frac{a}{b} = \frac{c}{d}$$

is another representation of the same fraction, the morphisms $\frac{a}{b}$ and $\frac{c}{d}$ coincide on the open and dense subset $\mathcal{D}_V(bd) = \mathcal{D}_V(b) \cap \mathcal{D}_V(d)$, and thus define the same rational function.

(iii) In general, $\text{dom}(f)$ can be larger than visible at first sight. For example, consider the projective curve

$$C = \mathcal{V}_+(x_0^3 - x_0x_1^2 - x_1x_2^2) \subseteq \mathbb{P}^2.$$

The rule

$$(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$$

defines a morphism

$$C \setminus \{(0 : 0 : 1)\} \rightarrow \mathbb{P}^1,$$

and thus a rational map $f: C \rightarrow \mathbb{P}^1$. On the other hand, there exists the morphism

$$(a_0 : a_1 : a_2) \mapsto (a_2^2 : a_0^2 - a_1^2),$$

defined on $C \setminus \{(1 : \pm 1 : 0)\}$. On all points of C where both morphisms are defined, they are easily checked to coincide. So f is defined on the whole of C , i.e. it is even a global morphism.

(iv) Let $V \subseteq \mathbb{P}^m$ be closed and irreducible. Every tuple $p_0, \dots, p_n \in k_+[V]$ of homogeneous elements $\neq 0$ of the same degree then defines the rational map

$$f = (p_0 : \dots : p_n): V \dashrightarrow \mathbb{P}^n$$

with $\text{dom}(f) \supseteq V \setminus \mathcal{V}_+(p_1, \dots, p_n)$. △

Theorem 4.4.5. *Let V be a k -variety. Then the rational functions on V form a k -algebra $k(V)$, with respect to pointwise operations. If V is irreducible, $k(V)$ is a field, called the **function field of V** .*

Proof. For two rational functions $f_1, f_2: V \dashrightarrow \mathbb{A}^1$, $f_1 \pm f_2, f_1 \cdot f_2$ are regular functions (and thus morphisms) on $\text{dom}(f_1) \cap \text{dom}(f_2)$, and thus rational functions on V . If V is irreducible and $f: V \dashrightarrow \mathbb{A}^1$ not identically zero, then $\{a \in \text{dom}(f) \mid f(a) \neq 0\}$ is open, nonempty and thus dense in V . On this set one can define $\frac{1}{f}$ as a regular function, and thus $k(V)$ is a field. □

From now on we will restrict ourselves to irreducible varieties. Then every nonempty open subset is automatically dense.

Theorem 4.4.6. *Let V be an irreducible variety.*

- (i) *One has $k(V) = k(U)$ for every open subset $U \neq \emptyset$ in V .*
- (ii) *One has $k(V) = \bigcup_U \mathcal{O}_V(U)$, where the union runs over all nonempty open subsets of V .*
- (iii) *The field extension $k \subseteq k(V)$ is finitely generated.*
- (iv) *If V is affine, then $k(V) = \text{Quot}(k[V])$.*

Proof. (i) is clear from Remark 4.4.4 (i). (ii) is also clear, since the elements of $\mathcal{O}_V(U)$ are morphisms from U to \mathbb{A}^1 . (iv) follows from

$$k(V) = \bigcup_{\emptyset \neq U \text{ open}} \mathcal{O}(U) = \bigcup_{s \in k[V], s \neq 0} \mathcal{O}(\mathcal{D}_V(s)) = \bigcup_s k[V]_s = \text{Quot}(k[V]),$$

where we have used Theorem 4.1.8. For (iii) we can assume that V is affine, using (i), since by Corollary 4.1.21 V admits a nonempty affine open subset. For affine varieties the statement follows from (iv), since $k[V]$ is finitely generated as a k -algebra, and thus $\text{Quot}(k[V])$ is finitely generated as a field. \square

Remark 4.4.7. Let $V \subseteq \mathbb{P}^n$ be closed and irreducible. Then $k_+[V]$ is a \mathbb{Z} -graded ring. Let T be the set of all homogeneous elements $\neq 0$, and consider the field

$$k_+[V]_{(T)} = (k_+[V]_T)_0 = \left\{ \frac{p}{q} \mid p, 0 \neq q \in k_+[V] \text{ homogeneous of same degree} \right\}.$$

Then in complete analogy to the affine setting we have $k(V) = k_+[V]_{(T)}$, this time using Theorem 4.2.7. \triangle

Example 4.4.8. (i) $k(\mathbb{A}^n) = k(\mathbb{P}^n) = k(x_1, \dots, x_n)$. This field is called the **rational function field in n variables**.

(ii) If $p \in k[x_1, \dots, x_n]$ is irreducible, for $V = \mathcal{V}(p) \subseteq \mathbb{A}^n$ we have

$$k(V) = \text{Quot}(k[x_1, \dots, x_n]/p). \quad \triangle$$

Remark 4.4.9. Rational maps can in general *not* be composed! For example, consider

$$f: \mathbb{A}^1 \rightarrow \mathbb{A}^2, a \mapsto (a, 0)$$

and

$$g: \mathbb{A}^2 \dashrightarrow \mathbb{A}^1; (a, b) \mapsto a/b,$$

where $\text{im}(f) \cap \text{dom}(g) = \emptyset$ and $g \circ f$ is thus not defined. \triangle

Definition 4.4.10. Let V, W be varieties with V irreducible.

(i) A morphism $\phi: V \rightarrow W$ is called **dominant**, if $\phi(V)$ is dense in W .

(ii) A rational map $f: V \dashrightarrow W$ is called **dominant**, if one (equivalently every) of its representing morphisms is dominant.

(iii) The set of all dominant rational maps is denoted by

$$\text{Rat}_k^{\text{dom}}(V, W). \quad \triangle$$

Remark 4.4.11. (i) If $\phi: V \rightarrow W$ and $\psi: W \rightarrow X$ are dominant morphisms, then $\psi \circ \phi$ is also dominant.

(ii) A morphism $\phi: V \rightarrow W$ of affine varieties is dominant if and only if the associated ring homomorphism $\phi^*: k[W] \rightarrow k[V]$ is injective. This is immediate from Theorem 1.4.13. \triangle

Note again that we now assume all varieties to be irreducible!

Theorem 4.4.12. Let $f: V \dashrightarrow W, g: W \dashrightarrow X$ be rational maps, and assume f is dominant. Then $g \circ f: V \dashrightarrow X$ is a well-defined rational map. If also g is dominant, then so is $g \circ f$.

Proof. Let $\phi: V' \rightarrow W$ and $\psi: W' \rightarrow X$ be representatives of f and g . Then $\phi^{-1}(W') \neq \emptyset$, since ϕ has dense image. Thus $\phi^{-1}(W')$ is dense in V , and the morphism $\psi \circ \phi$ is defined here. So we obtain the rational map $g \circ f: V \dashrightarrow X$, and the definition does obviously not depend on the choice of the representatives ϕ and ψ . The rest of the statement is clear. \square

Definition 4.4.13. (i) A dominant rational map $f: V \dashrightarrow W$ is called **birational equivalence** (or just **birational**), if there exists a dominant rational map $g: W \dashrightarrow V$ with $g \circ f = \text{id}_V$ and $f \circ g = \text{id}_W$.

(ii) Two varieties V, W are called **(k -)birationally equivalent**, if there exists a birational equivalence $f: V \dashrightarrow W$.

(iii) A variety is called **(k -)rational**, if it is birationally equivalent to some \mathbb{P}^n . \triangle

Example 4.4.14. (i) Let V be irreducible and $\emptyset \neq U \subseteq V$ open. Then the inclusion $U \hookrightarrow V$ is a birational equivalence. In particular, \mathbb{A}^n is birationally equivalent to \mathbb{P}^n , and every irreducible variety is birationally equivalent to some affine variety.

(ii) For every irreducible projective variety $V \subseteq \mathbb{P}^n$, the affine cone $\widehat{V} \subseteq \mathbb{A}^{n+1}$ is irreducible and birationally equivalent to $V \times \mathbb{A}^1$. In fact, there are mutually inverse mappings

$$V \times \mathbb{A}^1 \dashrightarrow \widehat{V}$$

$$((a_0 : \dots : a_n), t) \mapsto t \cdot \left(1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0} \right)$$

(without loss of generality assume $V \cap \mathcal{D}_+(x_0) \neq \emptyset$), as well as

$$\widehat{V} \dashrightarrow V \times \mathbb{A}^1$$

$$(a_0, \dots, a_n) \mapsto ((a_0 : \dots : a_n), a_0),$$

defined on $\widehat{V} \setminus (0, \dots, 0)$.

(iii) If $f: V \dashrightarrow W$ is dominant and $V(k)$ is Zariski dense, then $W(k)$ is Zariski dense in W . In particular, if V, W are birationally equivalent varieties, $V(k)$ is dense in V if and only if $W(k)$ is dense in W . For a rational variety V , $V(k)$ is always dense in V (at least if k is infinite). \triangle

Theorem 1.4.8 and Theorem 1.4.11 can now easily be proven for rational maps as well. For field extensions $k \subseteq E, k \subseteq F$ we denote by $\text{Hom}_k(E, F)$ the set of k -embeddings of E to F , as usual.

Theorem 4.4.15. *Let V, W be irreducible k -varieties.*

(i) *Every dominant rational map $f: V \dashrightarrow W$ induces a k -embedding $f^*: k(W) \hookrightarrow k(V)$ of function fields. This assignment is functorial, i.e. we have $\text{id}^* = \text{id}$ and $(g \circ f)^* = f^* \circ g^*$ for dominant $g: W \dashrightarrow X$.*

(ii) *The mapping $*$: $\text{Rat}_k^{\text{dom}}(V, W) \rightarrow \text{Hom}_k(k(W), k(V))$ is bijective.*

(iii) *A dominant rational map $f: V \dashrightarrow W$ is a birational equivalence if and only if $f^*: k(W) \rightarrow k(V)$ is an isomorphism of fields.*

Proof. (i) is clear. For (ii) we can assume V, W to be affine. We now construct the inverse mapping to $*$. To this end, let $\varphi: k(W) \rightarrow k(V)$ be a k -embedding. Since $k[W]$ is a finitely generated k -algebra, there exists some $0 \neq s \in k[V]$ with

$$\varphi|_{k[W]}: k[W] \rightarrow k[V]_s = k[\mathcal{D}_V(s)].$$

By Theorem 1.4.8 this $\varphi|_{k[W]}$ corresponds to a morphism

$$f_0: D_V(s) \rightarrow W$$

of affine varieties. Now let $f = [f_0]: V \dashrightarrow W$ be the rational map represented by f_0 . It does not depend on the choice of s , as can be seen in the following diagram, using the equivalence of categories (Remark 1.4.10):

$$\begin{array}{ccc} & k[V]_s & \\ \varphi \nearrow & & \searrow \text{id} \\ k[W] & & k[V]_{st} \\ \varphi \searrow & & \nearrow \text{id} \\ & k[V]_t & \end{array}$$

The assignment $\varphi \mapsto f$ defines the inverse mapping to $*$, as is easily checked (φ is already uniquely determined by $\varphi|_{k[W]}$, and $\varphi|_{k[W]}$ corresponds to f_0).

(iii) If f is a birational equivalence, f^* is an isomorphism, by functoriality from (i). Conversely, if $\varphi: k(V) \rightarrow k(W)$ is an inverse embedding to f^* , and if $\varphi = g^*$ for some $g \in \text{Rat}_k^{\text{dom}}(W, V)$, then

$$(f \circ g)^* = g^* \circ f^* = \varphi \circ f^* = \text{id}_{k(W)}.$$

From bijectivity of $*$ we obtain $f \circ g = \text{id}_W$ (analogously for $g \circ f$). So f is a birational equivalence. \square

Corollary 4.4.16. *For two irreducible k -varieties V and W , the following are equivalent:*

(i) V and W are birationally equivalent.

(ii) $k(V) \cong_k k(W)$

(iii) There are open subsets $\emptyset \neq V' \subseteq V, \emptyset \neq W' \subseteq W$ with $V' \cong W'$.

Proof. "(i) \Leftrightarrow (ii)" is clear from Theorem 4.4.15. Also clear is "(iii) \Rightarrow (i)". For "(i) \Rightarrow (iii)" let $f: V_0 \rightarrow W$ and $g: W_0 \rightarrow V$ be morphisms on open and dense subsets, with $g \circ f = \text{id}_{f^{-1}(W_0)}$ and $f \circ g = \text{id}_{g^{-1}(V_0)}$. This immediately implies

$$f: f^{-1}(W_0) \xrightarrow{\cong} g^{-1}(V_0). \quad \square$$

Corollary 4.4.17. *An irreducible variety V is rational if and only if $k(V)$ is a rational function field $k(x_1, \dots, x_n)$.*

Example 4.4.18. Let $C = \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$ be the curve from Example 1.2.2 (iii). The rational function

$$\begin{aligned} f: C &\dashrightarrow \mathbb{A}^1 \\ (a, b) &\mapsto \frac{b}{a} \end{aligned}$$

is a birational equivalence. The inverse mapping is

$$\begin{aligned} g: \mathbb{A}^1 &\rightarrow C \\ r &\mapsto (r^2 - 1, r(r^2 - 1)). \end{aligned}$$

So C is a rational curve and $k(C) = k(x)$. Geometrically, f maps each point of $C \setminus \{(0, 0)\}$ to the slope of the line through 0 and the point. We obtain an isomorphism between the open subsets $C \setminus \{(0, 0)\}$ and $\mathbb{A}^1 \setminus \{\pm 1\}$. Thus we have solved the equation $y^2 - x^2 - x^3 = 0$ almost completely. \triangle

Example 4.4.19. (i) Let $H \subseteq \mathbb{P}^n$ be a k -hyperplane, and $z \in \mathbb{P}^n(k) \setminus H$. For every point $a \in \mathbb{P}^n$ we denote by $\pi(a) \in H$ the intersection point of the line spanned by z and a with H . We obtain a rational map

$$\pi: \mathbb{P}^n \dashrightarrow H = \mathbb{P}^{n-1},$$

called **projection from z** . We can choose coordinates such that $z = (1 : 0 : \dots : 0)$ and $H = \mathcal{V}_+(x_0)$. Then

$$\pi(a_0 : \dots : a_n) = (0 : a_1 : \dots : a_n),$$

and $\pi: \mathbb{P}^n \setminus \{z\} \rightarrow H$ is a morphism.

(ii) More generally, let Z, L be linear subspaces of \mathbb{P}^n , with $Z \cap L = \emptyset$ and $\dim(Z) + \dim(L) = n - 1$ (i.e. they come from k -subspaces $\tilde{Z}, \tilde{L} \subseteq K^{n+1}$ with $\tilde{Z} \oplus \tilde{L} = K^{n+1}$). Every point $a \in \mathbb{P}^n \setminus Z$ corresponds to a line through the origin $v \subseteq K^{n+1}$, for which $\text{span}(\tilde{Z}, v) \cap \tilde{L}$ is one-dimensional. This defines a rational map

$$\pi: \mathbb{P}^n \dashrightarrow L$$

called **projection from Z** . If coordinates are chosen such that $L = \mathcal{V}_+(x_0, \dots, x_m)$ and $Z = \mathcal{V}_+(x_{m+1}, \dots, x_n)$, then

$$\pi(a_0 : \dots : a_n) = (0 : \dots : 0 : a_{m+1} : \dots : a_n).$$

If $V \subseteq \mathbb{P}^n$ is an irreducible variety with $V \not\subseteq Z$, then $\pi|_V: V \dashrightarrow L$ is again a rational map. \triangle

A **quadric** Q is a variety in \mathbb{P}^n defined by a quadratic polynomial. If $\text{char}(k) \neq 2$, we can assume $Q = \mathcal{V}_+(c_0x_0^2 + \dots + c_r x_r^2)$ with $c_i \in k^\times$, after a change of basis (diagonalization of quadratic forms). For $r \geq 2$ the quadric Q is irreducible (Exercise 75). Q is called **non-degenerate**, if $r = n$.

Theorem 4.4.20. Let k be infinite with $\text{char}(k) \neq 2$, and let $Q = \mathcal{V}_+(q) \subseteq \mathbb{P}^n$ be a non-degenerate quadric. We then have

$$Q \text{ } k\text{-rational} \Leftrightarrow Q(k) \neq \emptyset.$$

Proof. " \Rightarrow " is clear from Example 4.4.14 (iii). For " \Leftarrow " choose $z \in Q(k)$ and a k -hyperplane $H \subseteq \mathbb{P}^n$ with $z \notin H$. The projection from z yields a k -rational map

$$\pi: Q \dashrightarrow H.$$

We now construct an inverse mapping. To this end, for $b \in H$ we intersect the line spanned by z and b with Q . In most cases this will define exactly one point besides z , which is the inverse image of b with respect to π .

So let $z = [v]$ and $b = [w]$. We choose $s \in K$ with $q(sv + w) = 0$ and then set $a = [sv + w]$. We have

$$q(sv + w) = s^2q(v) + 2sb_q(v, w) + q(w) = 2sb_q(v, w) + q(w),$$

where b_q is the symmetric bilinear form defined by q . The equation $0 = q(sv + w)$ can be solved uniquely for s , in case $b_q(v, w) \neq 0$, and yields

$$a = [2b_q(v, w)w - q(w)v].$$

Outside of the k -hyperplane $L = \{[w] \mid b_q(v, w) = 0\}$ we thus obtain a morphism $f: \mathbb{P}^n \setminus L \rightarrow Q$, which defines an inverse rational map $f: H \dashrightarrow Q$ to π , by restriction (H can be chosen with $H \not\subseteq L$). \square

Example 4.4.21. The proof of Theorem 4.4.20 provides us with an explicit birational equivalence. For example, let $q = x_0x_1 + x_2^2 - x_3^2$ and $Q = \mathcal{V}_+(q) \subseteq \mathbb{P}^3$. We choose $z = (1 : 0 : 0 : 0) \in Q$ and $H = \mathcal{V}_+(x_0)$. The map $\pi: Q \dashrightarrow H$ is thus defined by the rule

$$\pi(a_0 : a_1 : a_2 : a_3) = (0 : a_1 : a_2 : a_3).$$

The inverse mapping now has the following form:

$$\begin{aligned} f: H = \mathbb{P}^2 &\dashrightarrow Q \\ (0 : b_1 : b_2 : b_3) &\mapsto (b_3^2 - b_2^2 : b_1^2 : b_1b_2 : b_1b_3), \end{aligned}$$

and defines a rational parametrization of Q . \triangle

Example 4.4.22. There exists a birational equivalence between $\mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{P}^2 . It is defined by the rule

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^2 \\ ((a_0 : a_1), (b_0 : b_1)) &\mapsto (a_0b_0, a_0b_1, a_1b_0) \\ (b_0 : b_2), (b_0 : b_1) &\leftarrow (b_0 : b_1 : b_2). \end{aligned} \quad \triangle$$

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Exercises

Exercise 1. Let R be a ring and $I \subseteq R$ an ideal. Recall the definition of the quotient ring R/I , and show the following:

- (i) I is a radical ideal $\Leftrightarrow R/I$ is reduced.
- (ii) I is a prime ideal $\Leftrightarrow R/I$ is an integral domain.
- (iii) I is a maximal ideal $\Leftrightarrow R/I$ is a field.
- (iv) Maximal ideals are prime, and prime ideals are radical.
- (v) If $\varphi: R \rightarrow S$ is a ring homomorphism and $J \subseteq S$ is an ideal, then $\varphi^{-1}(J)$ is an ideal in R . If J is radical/prime, then so is $\varphi^{-1}(J)$. If J is a maximal ideal, how about $\varphi^{-1}(J)$?

Exercise 2. (i) Show that for every ideal I the radical \sqrt{I} (cf. Def 1.1.1) is again an ideal.

(ii) Prove Lemma 1.1.3.

Exercise 3. Let R be a ring, $S \subseteq R$ a multiplicative subset, and $I \subseteq R$ an ideal. Show the following:

- (i) There is a canonical bijection between ideals of R/I , and ideals in R that contain I .
- (ii) The bijection from (i) preserves radical/prime/maximal ideals.
- (iii) There is a canonical bijection between prime ideals of the localization $S^{-1}R$ and prime ideals of R that are disjoint to S .
- (iv) Is (iii) also true for arbitrary ideals?

Exercise 4. Let R be a ring. Show the following:

- (i) Every prime ideal $\mathfrak{p} \subseteq R$ contains a minimal prime ideal of R .
- (ii) Every minimal prime ideal of R contains only zero divisors.

Hint: examine the localization $R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R$

- (iii) If R is reduced, the set of zero divisors of R is precisely the union of all minimal prime ideals of R .

Exercise 5. Let R be a unique factorization domain and let K be its field of fractions. Show that the only elements in K that are integral over R are the elements of R itself.

Exercise 6. Let R be a ring. Recall the definition of an R -module. An R -module M is called **Noetherian** if every submodule is finitely generated. Show that the direct sum of two Noetherian R -modules is again Noetherian.

Exercise 7. Show that:

- (i) $k[x, y]/(x^2 - y)$ is isomorphic to the polynomial ring $k[t]$ in one variable.
- (ii) $k[x, y]/(xy - 1)$ is not isomorphic to $k[t]$.
- (iii) If k is algebraically closed, then for every quadratic irreducible polynomial $p \in k[x, y]$ the algebra $k[x, y]/(p)$ is isomorphic to $k[x, y]/(x^2 - y)$ or to $k[x, y]/(xy - 1)$.

Exercise 8. Let $p, q \in k[t]$ be univariate polynomials, and set $g := \gcd(p, q)$. Show $\mathcal{V}(p, q) = \mathcal{V}(g)$.

Exercise 9. Let k be an algebraically closed field. Show that every finite subset of k^n is an affine k -variety.

Exercise 10. Let $p_1 = x^2 - yz, p_2 = xz - x$ and $I = (p_1, p_2) \subseteq \mathbb{Q}[x, y, z]$.

- (i) Sketch or plot the real points of $\mathcal{V}(I)$ in \mathbb{R}^3 .
- (ii) Is I a prime ideal?
- (iii) Find three different prime ideals in $\mathbb{Q}[x, y, z]$ that contain I .
- (iv) Find generators for \sqrt{I} .

Exercise 11. Let $I = (p_1, \dots, p_r)$, $J = (q_1, \dots, q_s) \subseteq k[x]$ be ideals. Show the following:

(i) $I + J := \{p + q \mid p \in I, q \in J\}$ is the smallest ideal containing both I and J .

(ii) $I + J = (p_1, \dots, p_r, q_1, \dots, q_s)$.

(iii) $\mathcal{V}(I + J) = \mathcal{V}(I) \cap \mathcal{V}(J)$.

Exercise 12. Let $I = (p_1, \dots, p_r) \subseteq k[x]$ an ideal and y a new variable. Show the following:

(i) $p \in \sqrt{I} \Leftrightarrow 1 \in (p_1, \dots, p_r, 1 - yp) \subseteq k[x, y]$.

(ii) $\sqrt{(x_1x_2, (x_1 - x_2)x_1)} = (x_1)$.

Exercise 13. Let $a, b, c, d \in \mathbb{R}$ and $d \neq 0$. Consider the following ideals:

$$\mathfrak{m}_1 := (x - a, y - b)$$

$$\mathfrak{m}_2 := (x - a, (y - b)^2 + d^2)$$

$$\mathfrak{m}_3 := ((x - a)^2 + d^2, y - (bx + c)).$$

Show that $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are maximal ideals in $\mathbb{R}[x, y]$, and that each maximal ideal of $\mathbb{R}[x, y]$ is of one of that forms.

Exercise 14. Let $V \subseteq \mathbb{C}^n$ be an affine \mathbb{C} -variety. Show that V is an affine \mathbb{R} -variety if and only if V is closed under coordinatewise complex conjugation.

Exercise 15. Let $O \subseteq \mathbb{C}$ be non-empty and open with respect to the Euclidean topology on \mathbb{C} . Show that O is an affine \mathbb{R} -variety if and only if $O = \mathbb{C}$. Then extend the statement to \mathbb{C}^n .

Exercise 16. Let a robotic arm in \mathbb{R}^2 be given by two rods of length 2 and 1, connected by swivel joints, as can be seen in Figure 4.1. The angles α and β can take arbitrary real values.

(i) Show that the set of all points that A can reach in \mathbb{R}^2 is the intersection of an affine \mathbb{R} -variety in \mathbb{C}^2 with \mathbb{R}^2 (i.e. it is the set of real points of this variety).

(ii) Show that the set of all points that B can reach in \mathbb{R}^2 is not the set of real points of an affine variety in \mathbb{C}^2 .

- (iii) Sketch or plot the sets of points that A and B can reach in \mathbb{R}^2 .
- (iv) Show that the set of all points that B can reach in \mathbb{R}^2 is the projection of the real points of an affine variety in \mathbb{C}^4 .
- (v) Determine the set of all positions of A in \mathbb{C}^2 , from which B can reach the positions $(2, 0), (3, 0), (4, 0)$.

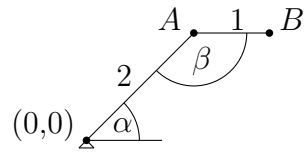


Figure 4.1: Planar robotic arm

Exercise 17. Show that the Zariski topology on $\mathbb{A}^{n+m} = \mathbb{A}^n \times \mathbb{A}^m$ does not coincide with the product topology.

Exercise 18. Let k be a field and K an algebraically closed field extension. Let

$$V = \{(t, t^2, t^3) \mid t \in K\} \subseteq K^3.$$

- (i) Show that V is an affine k -variety, and find defining equations for V .
- (ii) Compute $\mathcal{I}(V)$.
- (iii) Show that V is a subvariety of $\mathcal{V}(xz - y^2)$.

Exercise 19. Let V and W be affine k -varieties with $V \subseteq W$. Show

$$W = V \cup \overline{(W \setminus V)}.$$

Exercise 20. Let R be a ring and denote by $\text{Spec}(R)$ the set of all prime ideals of R . For a subsets $I \subseteq R$ and $V \subseteq \text{Spec}(R)$ we define

$$\mathcal{V}(I) := \{\mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\}$$

$$\mathcal{I}(V) := \bigcap_{\mathfrak{p} \in V} \mathfrak{p}.$$

- (i) Prove an analogue of Hilbert's Nullstellensatz in this setup.

(ii) Show that the sets $\mathcal{V}(I)$ fulfill the axioms for closed sets of a topology.

(iii) If $\varphi: R \rightarrow S$ is a ring homomorphism, then the map

$$\begin{aligned} \text{Spec}(S) &\rightarrow \text{Spec}(R) \\ \mathfrak{q} &\mapsto \varphi^{-1}(\mathfrak{q}) \end{aligned}$$

is continuous with respect to the topologies from (ii).

Exercise 21. Prove Lemma 1.3.6.

Exercise 22. Prove Lemma 1.3.11.

Exercise 23.

(i) Find a set $X \subseteq \mathbb{A}^n$, such that the topological space X (with respect to the Zariski topology) contains an open set $Y \subseteq X$, which is not dense in X .

(ii) Show that each non-discrete set $X \subseteq \mathbb{A}^n$ is not hausdorff (also see Lemma 1.3.3).

Exercise 24. Show that $\text{GL}_n(K)$ is open in the k -Zariski topology.

Exercise 25. Let $V \subseteq \mathbb{A}^3$ be the affine variety defined by the following equations:

$$x^2 - yz = 0 \quad \text{und} \quad xz - x.$$

Show that V has 3 irreducible components and find their prime ideals.

Exercise 26. Let $V = \mathcal{V}(y^2 - z^3, x^2 - z) \subseteq \mathbb{A}^3$. Show that V is a union of two subvarieties (curves), and find defining equation for these curves.

Hint: Consider $(x^4 + x^2z + z^2)(x^2 - z) - (y^2 - z^3) \in (y^2 - z^3, x^2 - z)$.

Exercise 27. Let $V = V_1 \cup V_2 \subseteq \mathbb{A}^3$ be the union of the two curves $V_1 = \mathcal{V}(x^3 - y, x^2 - z)$ and $V_2 = \mathcal{V}(x^3 + y, x^2 - z)$. Further let $W = \mathcal{V}(xy) \subseteq \mathbb{A}^2$.

(i) Show that also W is a union of two strict subvarieties (curves) $W = V_3 \cup V_4$, and find defining equations for V_3 and V_4 .

(ii) Show that there exist polynomial mappings $p_i: V_i \rightarrow \mathbb{A}$ and $q_i: \mathbb{A} \rightarrow V_i$, such that $p_i \circ q_i = \text{id}_{\mathbb{A}}$ and $q_i \circ p_i = \text{id}_{V_i}$, for $i = 1, \dots, 4$.

(iii) Do there exist polynomial mappings $p: V \rightarrow W$ and $q: W \rightarrow V$, such that $p \circ q = \text{id}_W$ and $q \circ p = \text{id}_V$? Sketch a proof of your answer.

Exercise 28. Prove Corollary 1.3.17 for ideals I in an arbitrary noetherian ring A .

Exercise 29. Compute the Zariski closure of the following sets in \mathbb{A}^2 :

- (i) The projection of $\mathcal{V}(xy - 1)$ onto the x -axis.
- (ii) The boundary of the positive orthant in \mathbb{R}^2 .
- (iii) The graph of the sine function

$$\{(x, \sin(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

in \mathbb{R}^2 .

Exercise 30.

- (i) Let R be a ring and $I, J \subseteq R$ ideals in R . Show that

$$(I : J) := \{a \in R \mid aJ \subseteq I\}$$

is an ideal in R , containing I .

- (ii) Compute generators for $(xz, yz) : (z) \subseteq k[x, y, z]$ and determine

$$\mathcal{V}((xz, yz) : (z)) \subseteq \mathbb{A}^3.$$

- (iii) Compute generators for $(y^2 - z^3, x^2 - z) : (x^2 - z, x^3 - y) \subseteq k[x, y, z]$ and determine $\mathcal{V}((y^2 - z^3, x^2 - z) : (x^2 - z, x^3 - y)) \subseteq \mathbb{A}^3$.

Exercise 31. Let $V \subseteq \mathbb{A}^n$ be an affine k -variety.

- (i) Show that V is irreducible if and only if $k[V]$ is an integral domain.
- (ii) Assume V is reducible. Show that there exist polynomials $p, q \in k[x_1, \dots, x_n]$ with

$$V = (V \cap \mathcal{V}(p)) \cup (V \cap \mathcal{V}(q)),$$

with $V \cap \mathcal{V}(p) \subsetneq V$, $V \cap \mathcal{V}(q) \subsetneq V$ and $V \subseteq \mathcal{V}(pq)$.

Exercise 32.

- (i) Let $V = \{(t, t^2, t^3) : t \in \mathbb{C}\} \subseteq \mathbb{C}^3$ and $p_1 = 2x^2 + y^2$, $p_2 = z^2 - y^3 + 3xz$, $q_1 = 2y + xz$, $q_2 = 3y^2$. Show that $p_1 = q_1$ and $p_2 = q_2$ holds in $k[V]$.

(ii) Show that $\mathcal{V}(x^3 + xy^2 - xz, yx^2 + y^3 - yz) \subseteq \mathbb{C}^3$ is reducible.

Exercise 33. Let $V = \mathcal{V}(x^2 - y^2z^2 + z^3) \subseteq \mathbb{C}^3$ and

$$\begin{aligned}\varphi: V &\rightarrow \mathbb{C} \\ (a_1, a_2, a_3) &\mapsto a_3.\end{aligned}$$

(i) Find defining equations for the affine \mathbb{R} -variety

$$\varphi(c)^{-1} = \{a \in V \mid \varphi(a) = c\}$$

where $c \in \mathbb{R}$.

(ii) Show $\varphi(c)^{-1} \cap \mathbb{R}^3 \neq \emptyset$ for all $c \in \mathbb{R}$.

(iii) Sketch or plot $\varphi(c)^{-1}$ for $c \in \{-2, -1, 0, 1, 2\}$.

(iv) Sketch or plot $\psi(c)^{-1}$ for $c \in \{-2, -1, 0, 1, 2\}$, where

$$\begin{aligned}\psi: V &\rightarrow \mathbb{C} \\ (a_1, a_2, a_3) &\mapsto a_1^2 + a_2^2.\end{aligned}$$

Exercise 34. Let K be algebraically closed with prime field k . Show that

$$\{A \in K^{n \times n} \mid A \text{ singular}\}$$

is an irreducible affine k -variety.

Exercise 35. Show that the mappings $p \mapsto p^*$ and $\varphi \mapsto p_\varphi$ from the proof of Theorem 1.4.8 are mutually inverse.

Exercise 36. Prove Theorem 1.4.11.

Exercise 37. Show that the map

$$\begin{aligned}\mathbb{A}^1 &\rightarrow \mathcal{V}(x_1^3 - x_2^2) \subseteq \mathbb{A}^2 \\ r &\mapsto (r^2, r^3)\end{aligned}$$

is a homeomorphism with respect to the Zariski topology.

Exercise 38. Show that the following \mathbb{Q} -varieties are isomorphic:

- (i) $\mathcal{V}(x^2 - y) \subseteq \mathbb{A}^2$ and \mathbb{A}^1
(ii) $\mathcal{V}(x^3 - y, x^2 - z) \subseteq \mathbb{A}^3$ and \mathbb{A}^1

Show that the following \mathbb{Q} -varieties are not isomorphic:

- (iii) $\mathcal{V}(xy - 1) \subseteq \mathbb{A}^2$ and \mathbb{A}^1
(iv) $\mathcal{V}(y^2 - z^3, x^2 - z) \subseteq \mathbb{A}^3$ and $\mathcal{V}(xy) \subseteq \mathbb{A}^2$

Exercise 39. Let

$$V_1 = \mathcal{V}(y - x^2, z + x^3 + 2x) \subseteq \mathbb{C}^3$$

$$V_2 = \mathcal{V}(x^2 + 2xz + 2y^2 + 3y, xy + 2x + z, xz + y^2 + 2y) \subseteq \mathbb{C}^3$$

and $\pi_i: V_i \rightarrow \mathbb{C}; (a_1, a_2, a_3) \mapsto a_1$ for $i = 1, 2$.

- (i) Show that π_1 is bijective and compute its inverse function.
(ii) Show that π_2 is bijective and compute its inverse function.

Exercise 40. Let A, B, C be k -algebras where C is a domain. Show that there is a bijection between $\text{Hom}_k(A \times B, C)$ and

$$\text{Hom}_k(A, C) \cup \text{Hom}_k(B, C)$$

(cf. proof of Theorem 1.4.16).

Exercise 41. Let a closed toggle chain in \mathbb{R}^2 be given, consisting of four rods of lengths a, b, c , and d , connected with swivel joints, as depicted in Figure 4.2.

- (i) Show that the points that A can reach in \mathbb{R}^2 (midpoint of upper rod) are the real points of an affine \mathbb{R} -variety, for $a = b = c = d = 5$, and for $a = 3, b = 9, c = 10, d = 13$.
(ii) Sketch or plot the set of points in \mathbb{R}^2 that A can reach for $a = b = c = d = 5$ and $a = 3, b = 9, c = 10, d = 13$.

Exercise 42. Let an ellipse compass in \mathbb{R}^2 be given, consisting of two slides that can move along the x - and y -axis, respectively. Both slides are joint to a rod of length l via swivel joints, see Figure 4.3.

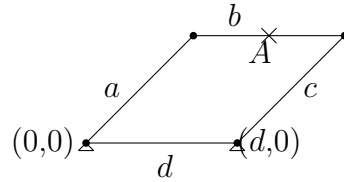


Figure 4.2: Closed toggle chain

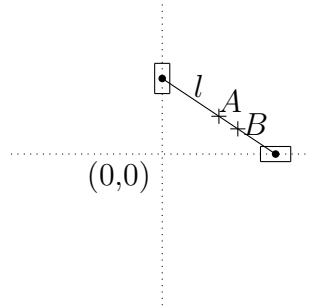


Figure 4.3: Ellipse compass

- (i) Show that the sets of points that A and B can reach in \mathbb{R}^2 (midpoint and quarter point of rod, respectively) are the real points of affine \mathbb{R} -varieties, where we set $l = 2$.
- (ii) Sketch or plot the sets of points that A and B can reach in \mathbb{R}^2 .

Exercise 43. Show that every orthogonal matrix $A \in \mathbb{R}^{2 \times 2}$ with $\det(A) = 1$ is of the form

$$\frac{1}{1+t^2} \begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix}$$

for some $t \in \mathbb{R} \cup \{\infty\}$, where $p(\infty) := \text{LC}(p)$ for $p \in \mathbb{R}[t]$.

Exercise 44. Every element of the group $\text{SE}(2)$ of Euclidean motions can be seen as a mapping

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{1}{1+t^2} \begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (4.1)$$

where $v_1, v_2 \in \mathbb{R}$ and $t \in \mathbb{R} \cup \{\infty\}$. We again consider the ellipse compass from Exercise 42.

- (i) Sketch or plot the set of points that A and B can reach in \mathbb{R}^2 , by proceeding as follows:

Define a coordinate system at the moving rod, and one in the fixed plane. Map the points of the moving system to the fixed system by the mapping (4.1), where v_1 , v_2 and t are yet to be determined. You thus obtain trajectories of the corresponding points, which still depend on the points and the parameters v_1 , v_2 and t . By suitable choice of points, of which the trajectories are known, it is possible to determine two of the three parameters v_1 , v_2 and t . The remaining parameter then parametrizes the trajectories.

- (ii) Let the inverse mechanism to the ellipse compass be given: Two slides are connected by swivel joints to a plane ($= \mathbb{R}^2$). The distance of the slides is $l = 2$. Two rods of lengths $2l = 4$ can slide through the slides. The two rods are connected at a fixed angle of 90 degree in their midpoints, see Figure 4.4. Sketch or plot the set of points that C (intersection point of rods) and D (quarter point of a rod) can reach in \mathbb{R}^2 .

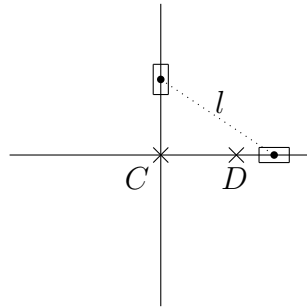


Figure 4.4: Inverse ellipse compass

Exercise 45. Let three rods of lengths l_1 , l_2 and l_3 be given. The rods are connected to a plane ($= \mathbb{R}^2$) with swivel joints at one end, and with a movable platform at the other end, see Figure 4.5. The platform is an isosceles triangle with basis $c = 14$ and height $h_c = 10$. By changing the lengths of the rods, the platform can move. Sketch or plot the position of the platform in \mathbb{R}^2 for $l_1 = 9$, $l_2 = 8$ and $l_3 = 10$, where you can compute the solutions numerically Hint: Use the mapping (4.1).

Exercise 46. Show that there are $\binom{n+d}{n}$ monomials of degree d in the variables x_0, \dots, x_n .

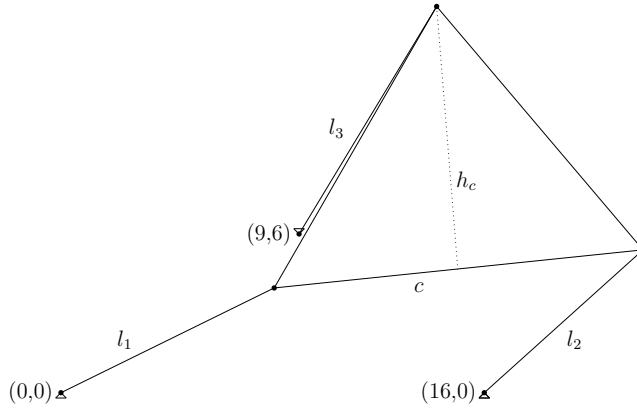


Figure 4.5: Planar 3-RPR mechanism

Exercise 47. Show that the radical of a monomial ideal $I \subseteq k[x]$ is again monomial. Is the converse also true?

Exercise 48. Let $I \subsetneq k[x]$ be a monomial ideal, and

$$C(I) := \{\alpha \in \mathbb{N}^n : \underline{x}^\alpha \notin I\}$$

the set of exponents, whose monomials that are not contained in I . Further set

$$\begin{aligned} e_1 &= (1, 0, \dots, 0), \\ e_2 &= (0, 1, \dots, 0), \\ &\vdots \\ e_n &= (0, \dots, 0, 1), \\ [e_{i_1}, \dots, e_{i_r}] &= \{a_1 e_{i_1} + \dots + a_r e_{i_r} \mid a_j \in \mathbb{N}, 1 \leq j \leq r\}. \end{aligned}$$

for $i_1 < \dots < i_r$.

(i) Show that $\mathcal{V}(I)$ is the union of finitely many coordinate subspaces (i.e. varieties of the form $\mathcal{V}(x_{l_1}, \dots, x_{l_s})$).

(ii) Show

$$\mathcal{V}(x_{l_1}, \dots, x_{l_s}) \subseteq \mathcal{V}(I) \iff [e_{i_1}, \dots, e_{i_r}] \subseteq C(I)$$

where $\{l_1, \dots, l_s\} \uplus \{i_1, \dots, i_r\} = \{1, \dots, n\}$.

Exercise 49.

- (i) Order the following monomials with respect to the lexicographic monomial ordering, and the degree-lexicographic monomial ordering, both in case of $x > y > z$ and $x < y < z$:

$$\{x, y, z, x^2, z^2, x^3\}, \{x^2y^8, x^5yz^4, xyz^3, xy^4\}$$

- (ii) Consider \mathbb{N} with the usual ordering. Between any two numbers there exist only finitely many other numbers. Is the same also true for all monomial orderings on \mathbb{N}^n , if $n \geq 2$?

Exercise 50. Show the following:

- (i) Every monomial ordering \preccurlyeq on \mathbb{N}^n can be extended to a linear ordering \preccurlyeq on \mathbb{Q}^n , with the following property:

$$\alpha \preccurlyeq \beta \Rightarrow \alpha + \gamma \preccurlyeq \beta + \gamma$$

for all $\alpha, \beta, \gamma \in \mathbb{Q}^n$.

- (ii) Let \preccurlyeq be an ordering on \mathbb{Q}^n as in (i). Show that the following set is a hyperplane in \mathbb{R}^n :

$$H = \{x \in \mathbb{R}^n \mid \forall \epsilon > 0 \exists q_+, q_- \in B_\epsilon(x) \cap \mathbb{Q}^n : q_+ \preccurlyeq 0 \preccurlyeq q_-\}.$$

Hint for the dimension: Show $\mathbb{Q}^n \not\subseteq H$, and that for $q \preccurlyeq 0 \preccurlyeq p$ in \mathbb{Q}^n the connecting line always meets H .

- (iii) Compute H from (ii) on case of $\preccurlyeq_{\text{lex}}$.

Exercise 51. Show that the following rule defines a monomial ordering on \mathbb{N}^2 :

$$\alpha \preccurlyeq \beta \quad :\Leftrightarrow \quad \alpha_1 + \alpha_2\sqrt{2} \leq \beta_1 + \beta_2\sqrt{2}.$$

Show that every element from \mathbb{N}^2 has a direct predecessor with respect to \preccurlyeq . Is this true for every monomial ordering?

Exercise 52. Let $k[x]$ be equipped with a lexicographic monomial ordering $\preccurlyeq_{\text{lex}}$. Show that for any ideal $I \subseteq k[x]$, the following are equivalent:

- (i) $\mathcal{V}(I)$ is finite.

(ii) For every $i = 1, \dots, n$ there exists $p_i \in I$ with $\text{LM}(p_i) = x_i^{d_i}$.

Exercise 53.

(i) Compute a normal form of $x^7y^2 + x^3y^2 - y + 1$ modulo $xy^2 - x, x - y^3$, with respect to the lexicographic ordering and with respect to the degree-lexicographic ordering (with $x > y$). What happens if $xy^2 - x, x - y^3$ is replaced by $x - y^3, xy^2 - x$?

(ii) Repeat (i) for $xy^2 - x$ and $xy + 1, y^2 - 1$. What do you notice?

Exercise 54. Prove the converse assertion of Corollary 2.3.4: Let $I \subseteq k[\underline{x}]$ be an ideal and $G = \{g_1, \dots, g_s\}$ a subset of I , such that for every $p \in k[\underline{x}]$, any normal form of p modulo G coincides with the canonical form $\text{cf}_I(p)$. Then G is a Gröbner basis of I .

Exercise 55. Let $I = (p) \subseteq k[\underline{x}]$ be a principal ideal. Show that every Gröbner basis of I contains a constant multiple of p .

Exercise 56. Prove the assertion in Remark 2.3.16(ii).

Exercise 57. Let $g_1, g_2 \in k[x]$ be polynomials for which $\text{LM}(g_1), \text{LM}(g_2)$ are relatively prime. Show that 0 is the only normal form of $S(g_1, g_2)$ modulo g_1, g_2 . What does this mean for the Buchberger Algorithm in general, and in particular when applied to a set of linear polynomials?

Exercise 58. Let $\mathfrak{p} \subseteq k[\underline{x}]$ be a prime ideal. Show that all elements from the reduced Gröbner basis of \mathfrak{p} are irreducible in $k[\underline{x}]$. Is the converse also true?

Exercise 59. Show how $p \in \sqrt{(p_1, \dots, p_s)}$ can be checked with Gröbner bases (cf. Application 2.4.3).

Exercise 60. Find all elements of the finite variety defined by the following 0-dimensional ideals:

$$\begin{aligned} &(x^2y - 1, xy^2 - y), \\ &(x^2 + y, x^4 + 2x^2y + y^2 + 3) \\ &(y^5 - z^4, xz^3 - y^4, xy - z, x^2z^2 - y^3, x^3z - y^2, x^4 - y) \end{aligned}$$

Exercise 61. Use a computer algebra system to determine whether the polynomial p is in the ideal I , and if so, find an ideal representation.

$$(i) \ p = xy^3 - z^2 + y^5 - z^3, I = (y - x^3, x^2y - z).$$

$$(ii) \ p = x^3z - 2y^2, I = (xz - y, xy + 2z^2, y - z).$$

Exercise 62. Consider the ideal

$$I := (x^2 + y^2 + z^2 - 4, x^2 + 2y^2 - 5, xz - 1) \subseteq \mathbb{Q}[x, y, z].$$

Use a computer algebra system to compute generators for $I \cap \mathbb{Q}[x]$, $I \cap \mathbb{Q}[y]$ and $I \cap \mathbb{Q}[z]$.

Exercise 63. Let

$$V := \{(u + v, u^2 + 2uv, u^3 + 3u^2v) \mid u, v \in \mathbb{C}\} \subseteq \mathbb{C}^3.$$

Use a computer algebra system to find generators for the ideal $\mathcal{I}(V)$.

Exercise 64. Let V be a vector space and $f: V \rightarrow V$ an injective linear map with $\mathbb{P}(f) = \text{id}_{\mathbb{P}(V)}$. Show that f is a constant multiple of id_V .

Exercise 65. Let V be a vector space and $W \subseteq V$ a hyperplane. Show that for any fixed $v \in V \setminus W$, the map

$$\begin{aligned} W &\rightarrow \mathbb{P}(V) \setminus \mathbb{P}(W) \\ w &\mapsto [v + w] \end{aligned}$$

is a well-defined bijection.

Exercise 66. For $i = 1, 2$ let $w_i \in \mathbb{R}^2$, $v_i \in \mathbb{R}^2 \setminus \{0\}$ and consider the lines

$$L_i := \{w_i + tv_i \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

If we extend the L_i to projective lines in $\mathbb{P}^2(\mathbb{R})$ via Construction 3.1.8, where do they intersect?

Exercise 67. Prove Lemma 3.2.5.

Exercise 68. Show directly that the radical of a homogeneous ideal is again homogeneous (with ordered index group G).

Exercise 69. Prove Lemma 3.2.11.

Exercise 70. Show that $\sqrt{I^h} = \sqrt{I^h}$ holds for ideals $I \subseteq k[x_1, \dots, x_n]$ (cf. Lemma 3.3.16).

Exercise 71. Consider

$$V := \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^3.$$

Show that $\mathcal{I}(V)$ is homogeneous with respect to some nontrivial \mathbb{Z} -grading of $\mathbb{C}[x, y, z]$, and determine homogeneous generators for this ideal. Show that $\mathcal{I}(V)$ cannot be generated by 2 polynomials.

Exercise 72. Let $V \subseteq \mathbb{A}^n$ be an affine variety with no points at infinity. Show that V is finite.

Exercise 73. Show that a Hausdorff space X is compact if and only if for all Hausdorff spaces Y the projection

$$\pi: X \times Y \rightarrow Y$$

maps closed sets to closed sets (cf. Remark 3.4.3).

Exercise 74. Show that the variety $V := \mathbb{A}^2 \setminus \{(0, 0)\}$ is not affine.

Hint: First compute $\mathcal{O}(V)$. If V was affine, to which simple affine variety would V be isomorphic? Why can't this be?

Exercise 75. Let $Q = \mathcal{V}_+(c_0x_0^2 + \cdots + c_r x_r^2) \subseteq \mathbb{P}^n$ be a quadric with $c_i \neq 0$ for all i . Show that Q is irreducible for all $r \geq 2$. What holds for $r < 2$?

Exercise 76. Consider the three (affine) conic sections

$$V_1 := \mathcal{V}(x^2 + y^2 - 1)$$

$$V_2 := \mathcal{V}(x^2 - y^2 - 1)$$

$$V_3 := \mathcal{V}(x^2 - y)$$

in \mathbb{C}^2 , and let \bar{V}_i denote the projective closure of V_i in $\mathbb{P}^2(\mathbb{C})$. Show that for all $i, j = 1, 2, 3$ there exists a projectivity $\varphi_{ij}: \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$ with

$$\varphi_{ij}(\bar{V}_i) = \bar{V}_j.$$

Why is a similar statement wrong in the affine space?

Exercise 77. Show that a field K is finitely generated over k if and only if $K = k(V)$ holds for an irreducible k -variety V .

Exercise 78. Let $L_1 = \mathcal{V}_+(x_0, x_1)$ and $L_2 = \mathcal{V}_+(x_2, x_3)$ be two skew lines in \mathbb{P}^3 . Show that $L_1 \cup L_2$ cannot be defined with 2 homogeneous equations. Does it work with 3? How many polynomials does one need to generate $\mathcal{I}(L_1 \cup L_2)$?

Exercise 79. (i) Show that a localization of a noetherian ring is again noetherian. (ii) Let R be a ring and \mathfrak{p} a prime ideal in R . Then the localization $R_{\mathfrak{p}}$ has exactly one maximal ideal, namely $\mathfrak{p}R_{\mathfrak{p}}$. Show that

$$R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \cong \text{Quot}(R/\mathfrak{p}).$$

Exercise 80. Let $J = \{\alpha \in \mathbb{N}^n \mid |\alpha| = d\}$, $N = |J|$, and let $Z \subseteq \mathbb{P}^N$ be the projective variety defined by the equations

$$z_{\alpha}z_{\beta} = z_{\delta}z_{\gamma}$$

for $\alpha, \beta, \gamma, \delta \in J$ with $\alpha + \beta = \gamma + \delta$. Now let $\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_r \in J$ such that

$$\sum_i \xi_i = \sum_i \eta_i.$$

Show that for $b = (b_{\alpha})_{\alpha \in J} \in Z$ we then have

$$b_{\xi_1} \cdots b_{\xi_r} = b_{\eta_1} \cdots b_{\eta_r}.$$