ABSTRACT

FOUNDATIONS OF FREE ALGEBRAIC STATISTICS

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Many statistical models admit semialgebraic descriptions, i.e. they can be defined by polynomial inequalities. For instance, two discrete random variables are stochastically independent, if and only if their joint probability matrix has rank one. The set of all probability matrices of rank one, a semialgebraic set, thus constitutes the independence model. *Algebraic statistics* builds upon this insight and analyzes such statistical models with algebraic and geometric methods.

Free semialgebraic geometry examines non-commutative extensions of semialgebraic sets. This is achieved by evaluating polynomials at Hermitian matrices instead of real numbers, and replacing positivity of numbers by positive-semidefiniteness of matrices. Important insight can be gained from this approach, even to the classical commutative case.

The goal of this project is to merge these to approaches, and thereby develop the foundations of *free algebraic statistics*. This is not only a technical show-off, but will lead to deeper insights into problems from theoretical quantum mechanics and quantum information theory.

For example, lifting the inequalities of the independence model to the free setup leads to sets of positive operator valued measures (POVMs), and thus describes independence of measurements in quantum mechanics. Lifting the parametrization of rank one matrices to the free setup instead, leads to such POVMs that admit a tensor-product decomposition additionally. Those objects are used in the theory of quantum non-local games, to describe optimal quantum strategies. In fact the set of all quantum strategies turns out to be the so-called free convex hull of the second mentioned lift. This set is not well understood, and our approach allows to employ methods and results from free semialgebraic geometry and convexity to the problem.

We will examine how the different free lifts relate to each other, describe their free convex hulls and develop algebraic optimization methods in this context. We will not only restrict to the independence model, but extend the results to other structured statistical models. By doing this, our project will lay the first theoretical foundations of a theory of free algebraic statistics.