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The dark side of the moon: structured products from the customer's perspective

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Abstract

Structured financial products have gained more and more popularity in recent years, but nevertheless has their success so far not thoroughly been analyzed. In this article we develop a theoretical framework for the design of optimal structured products and analyze the maximal utility gain for an investor that can be achieved by introducing structured products. We demonstrate that most successful structured products are not optimal for a perfectly rational investor and investigate the reasons that make them nevertheless look so attractive for many investors.

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1 Structured financial products

Structured products, SPs, combine one or more classical assets (stocks, bonds, indices) with at least one derivative into a bundle that has specific interesting features for investors, like capital protection or increased participation. Structured products enable investors with comparatively low budget and knowledge to invest indirectly into derivatives, whereas a direct investment would often not be feasible for them. Banks profit from this market inefficiency caused by the restricted participation in the derivatives market, in that they retain a margin profit when issuing structured products. Structured products are immensely popular in Europe. They are not seen much in the USA, most likely due to regulatory reasons.

Note that in traditional asset market models there is no role for SPs. The classical Capital Asset Pricing Model, CAPM, for example, suggests that an investment in the market portfolio and the risk-free assets would be sufficient to construct optimal investments for all degrees of risk-aversion – this is the famous Two-Fund-Separation, 2FS. Hence the only role for banks would be to offer the market portfolio at minimal cost, e.g. in the form of an exchange traded fund. The existence of structured products depart from this paradigm. Structured products have so far been studied in academic research nearly exclusively from the issuer’s perspective, mostly in the context of option pricing and hedging. Indeed building on the seminal paper of Black and Scholes [1] a new field of finance, called financial engineering, has emerged in which mathematicians and engineers developed more and more elaborate pricing techniques for ever more complicated structured products. Besides this huge technical literature, recently a few empirical studies on the actual market prices [5, 17] of SPs can be found, but the investor’s perspective on structured products is still uncharted territory, somehow “the dark side of

the moon”.

In this article we try to shed some light on this “dark side” and take the investor’s perspective as the starting point for our expedition. In this way we ask whether structured products are an appropriate tool to improve investment performance and what types of products are optimal under normative, but also under behavioral models. In particular, we measure how big the potential improvement of a portfolio can be when adding structured products, but we also demonstrate that the most popular products derive their popularity not from rational, but from behavioral factors like framing, loss aversion and probability mis-estimation.

2 Designing optimal SPs

In the following we introduce a simple two-period model for a structured product, assuming that the market prices can be described by the CAPM.¹ Using results on the co-monotonicity of optimal investments [11] and extending previous results on optimal investments in the strictly concave expected utility setting [9], we obtain qualitative properties that optimal structured products should satisfy under more and more relaxed conditions on the rationality of the investors. We will use these results to show later that the attractiveness of the currently most popular classes of SPs cannot be understood within a rational decision model.

Some of our theoretical results are similar to independent work by Prigent [10] on portfolio optimization and rank-dependent expected utility.

¹Many of our results could be obtained under much weaker assumptions, e.g. monotonicity of the likelihood ratio. We point out when this generalization is possible.

2.1 Basic model

We say that a structured product is *optimal* if its payoff distribution maximizes the given utility of an investor under the constraint that the arbitrage-free price of the product cannot exceed a certain value. It seems appropriate to use a two-period model, since most structured products are sold over the counter and have a fixed maturity: given the ill-liquidity of the secondary markets for structured products, selling them before maturity is usually not recommended and also not frequently done. To obtain intuitive results, we will assume most of the time that the market can be described by the capital asset pricing model. A generalization will be given below. Moreover we assume that the market is complete and that the investor does not hold other assets (or at least considers them as a separate “mental account”).²

If we describe the structured product by its return distribution at maturity depending on the return of the market portfolio, we naturally arrive at an optimization problem for conjoint probability measures, compare [11]:

Let the returns of a SP be given by the probability measure p on \mathbb{R} , and let the return distribution of the market portfolio be given by m on \mathbb{R} . Let their mean and variance be given by $\mathbb{E}(p)$, $\mathbb{E}(m)$ and $\text{var}(p)$, $\text{var}(m)$, respectively.³ Let R be the return of the risk-free asset (i.e. the interest- or risk-free rate). We can describe a financial product by a conjoint probability measure T such that $p = \int_{\mathbb{R}} dT(x, \cdot)$ and $m = \int_{\mathbb{R}} dT(\cdot, y)$; in other words: T describes the return distribution of the product depending on the return of the underlying. It follows from the no-arbitrage condition that in the CAPM all financial

²Most of our results would carry over if the investor holds additionally risk-free assets.

³In this article we will interpret probability measures on the space of possible returns often as random variables by identifying the sample space with the space of possible returns.

products of a given price (for simplicity we set this price to one) satisfy the constraint, known as the Security Market Line, SML,

$$\mathbb{E}(p) - R = \beta_{pm}(\mathbb{E}(m) - R), \quad \text{where } \beta_{pm} = \frac{\text{cov } T}{\text{var}(m)}. \quad (1)$$

We optimize the utility U for the investor over all p satisfying this constraint. In the simplest case, the utility U is an expected utility, i.e.:

$$U(p) := \int_{\mathbb{R}} u(y) dp(y).$$

So far, the class of potential structured products is still huge: T is a probability measure on \mathbb{R}^2 and it only has to satisfy the pricing condition (1). This makes any computation very hard. Fortunately, we do not have to consider this large class of SPs, since the central result of [11] guarantees that every SP which is optimal for *some* decision preference is *co-monotonic* with m , i.e. higher returns for the market cannot result in lower returns for the SP. The decision preference can be chosen freely, in particular expected utility theory or prospect theory are allowed. The only condition is that the decision preference only depends on the return distribution of p (or, e.g., of its difference to the market return). In particular all “state-independent” preferences are admissible.

Due to the co-monotonicity we can write the payoff of every *optimal* SP as a monotonic function of the market return.⁴

The optimization constraint can be simplified substantially in the co-monotonic case, since we can consider the returns of the market portfolio as states of

⁴More precisely it only guarantees that we can write it as a *correspondence* of the market return, i.e. one market return might correspond to more than one return of the SP. This problem cannot occur, e.g., when p is absolutely continuous (which is in all standard models for classical underlyings like stocks or stock market indices the case). We will therefore ignore this technicality and assume that p has this property.

the world which leads to the no-arbitrage condition

$$\int_{\mathbb{R}} y(x)\ell(x)dp(x) = R,$$

where ℓ is the likelihood ratio, which can be written in CAPM as

$$\ell(x) = a - bx,$$

for some positive constants a, b , as has been shown in [2], for example. We sometimes use q for ℓp . From the linearity of the likelihood ratio ℓp the SML can be derived by elementary computations.

The assumption of a CAPM market can be omitted here: let us denote the likelihood ratio of the general case by ℓ , then we can replace the market return m in the above considerations by the *inverted likelihood ratio* $\tilde{\ell}$, defined by $\tilde{\ell}(\mathbb{E}(m) + x) := \ell(\mathbb{E}(m) - x)$. As long as ℓ is a monotonic decreasing function of the market return, the representation of an optimal SP by a monotonic function (or correspondence, to be more precise) of the market return is still possible. Note that the Black and Scholes model is one example besides the CAPM in which ℓp is monotonically decreasing with the market portfolio.

2.2 Optimal SPs in the strictly concave case

Let us assume that the investor has rational preferences, i.e. he follows the expected utility approach by von Neumann and Morgenstern [16]. The case where the utility function u is strictly increasing and strictly concave has been studied previously by Kramkov and Schachermeyer [9]. A straightforward variational approach leads in this case to the optimality condition

$$y(x) = v^{-1}(\lambda\ell(x)), \tag{2}$$

where $v := u'$ and λ is a Lagrange parameter that has to be chosen such that (2) holds.

Note that unless the utility function is quadratic, in the two-period model the 2FS property does not hold for any distribution of returns⁵.

What conditions on y can we derive from (2) in general? In fact, in the most realistic case (when the investor is prudent⁶), y is convex, as the following theorem shows:

Theorem 1. *Let u be a strictly increasing and strictly concave utility function. Assume furthermore that $u''' < 0$ (i.e. the investor is prudent). Then the return of an optimal SP on a CAPM market is strictly convex as a function of the market return. If $u''' = 0$ (i.e. u is quadratic) then the payoff function will be linear. If $u''' > 0$ the payoff function will be strictly concave.*

Proof. Take the second derivative of $y(x) = v^{-1}(\lambda(a + bx))$ with $v := u'$. This gives:

$$y''(x) = -\frac{\lambda^2 b^2 v''(v^{-1}(\lambda(a + bx)))}{(v'(v^{-1}(\lambda(a + bx))))^3}.$$

The denominator is positive, since $v' = u'' < 0$ (u is strictly concave).

Thus y'' is positive if $u''' = v'' < 0$. □

Hence prudent investors prefer SPs that look similar to call options. This result shows by the way that if u is quadratic, the return is a linear function of the market return, i.e. there is no need for a SP at all, which is the classical two-fund separation theorem of the mean-variance portfolio theory.

The following corollary gives a simpler condition under which strict convexity of y holds:

⁵In models with *time continuous* trading, *no transaction costs* and an underlying process that follows a *geometric Brownian motion* the 2FS property would hold for any class of utility functions with constant relative risk aversion.

⁶The importance of prudence is well known from the literature in insurance theory, see for example [4]

Corollary 1. *If the investor has non-increasing absolute risk-aversion, then the payoff function should be strictly convex.*

Proof. Consider the Arrow-Pratt risk measure $r(x) := -u''(x)/u'(x)$. Then

$$r'(x) = -\frac{u'''(x)u'(x) - (u''(x))^2}{(u'(x))^2}.$$

Hence, if $r' \geq 0$ (i.e. r is not increasing) then $u'''(x) \geq \frac{(u''(x))^2}{u'(x)}$. Since u is strictly increasing, this is positive. Thus $v'' = u''' > 0$ and we can apply Theorem 1. \square

Let us now study two examples of specific utility functions, where the payoff function of the optimal structured product can be given explicitly.

Power utility

Let $u(x) := \frac{1}{\alpha}x^\alpha$ with $\alpha \in (0, 1)$. Then $v(x) := u'(x) = x^{\alpha-1}$ and $v^{-1}(z) = z^{\frac{1}{\alpha-1}}$. Therefore, recalling (2), the optimal structured product is given by

$$y(x) = (\lambda \ell(x))^{\alpha-1}.$$

We can compute λ explicitly, if we use the constraint (2):

$$\int (\lambda \ell(x))^{\alpha-1} \ell(x) p(x) dx = R,$$

which can be resolved to

$$\lambda = \frac{R^{\frac{1}{\alpha-1}}}{\left(\int \ell(x)^\alpha p(x) dx\right)^{\frac{1}{\alpha-1}}}.$$

All together we obtain:

$$y(x) = \frac{R \ell(x)^{\alpha-1}}{\int \ell(x)^\alpha p(x) dx}.$$

And in the case of the CAPM we see clearly that the structured product is an increasing and convex function of the underlying:

$$y(x) = \frac{const}{(a - bx)^{1-\alpha}}.$$

Exponential utility

Let $u(x) := -\frac{1}{\alpha}e^{-\alpha x}$. Then $v(x) := u'(x) = e^{-\alpha x}$ and $v^{-1}(z) = -\frac{1}{\alpha} \ln z$.

Thus,

$$y(x) = -\frac{1}{\alpha} \ln(\lambda \ell(x)).$$

Again, we can compute λ explicitly, if we use 2, i.e.

$$\int y(x)\ell(x)p(x) dx = R.$$

The left hand side can be computed as follows:

$$\begin{aligned} \int y(x)\ell(x)p(x) dx &= -\frac{1}{\alpha} \int \ln(\lambda \ell(x))p(x)\ell(x) dx \\ &= -\frac{1}{\alpha} \int (\ln(\lambda) + \ln(\ell(x)))p(x)\ell(x) dx \\ &= -\frac{1}{\alpha} \ln(\lambda) - \frac{1}{\alpha} \int \ln(\ell(x))p(x)\ell(x) dx. \end{aligned}$$

Thus we obtain

$$\lambda = e^{-\alpha R - \int p(x)\ell(x) \ln(\ell(x)) dx}.$$

The optimal structured product is therefore given by

$$y(x) = R - \frac{1}{\alpha} \left(\ln \ell(x) - \int p(x)\ell(x) \ln(\ell(x)) dx \right).$$

And in the case of the CAPM we obtain

$$y(x) = R - \frac{1}{\alpha} (\ln(a - bx) - \text{const}).$$

2.3 Optimal SPs in the general case

The assumption of strict concavity for the utility function is classical, but has been disputed a lot in recent years. Implied utility functions can be computed

from stock market data and often show non-concave regions, compare for example [6] and [3]. Moreover, the most popular descriptive theory for decisions under risk, cumulative prospect theory, predicts non-concavity of u in losses [15]. Other descriptive theories also assume risk-seeking behavior at least for small losses.

Given the empirical and experimental evidence, it seems therefore more likely that u depends on a reference point (e.g. current wealth level) and that it has some strictly convex region for small losses. What would be an optimal SP for an investor described by such a model?

We first show that in order to find an optimal SP, it is sufficient to consider the *concavification* of the value function u , i.e. the smallest function larger or equal u which is concave:

Lemma 1. *Assume that u is concave for large returns and assume that the returns are bounded from below by zero⁷. Let u_c be the concavification of u , then there is an optimal SP for u_c which is also optimal for u .*

Proof. Let y_0 be a point where $u_c(y_0) > u(y_0)$. We prove for any market return x_0 that an optimal SP for u does not have to yield the value y_0 , i.e. $y_0 \notin \text{supp } y$. Suppose the opposite, i.e. $y(x_0) = y_0$ for some x_0 . We can find two points y_1, y_2 with $y_1 < y_0 < y_2$ such that $\lambda y_1 + (1 - \lambda)y_2 = y_0$ with $\lambda \in (0, 1)$, $u(y_1) = u_c(y_1)$, $u(y_2) = u_c(y_2)$ and $u_c(y_0) = \lambda u(y_1) + (1 - \lambda)u(y_2)$. Now we can construct another SP \tilde{y} such that whenever the market return is x_0 , \tilde{y} gives a return of y_1 with probability λ and a return of y_2 with probability $1 - \lambda$. Let us do the same construction for all values y_0 where $u_c(y_0) > u(y_0)$. The new product obviously has a utility which is at least as big as before,

⁷This is always true in assumptions since there is limited liability for structured products, i.e. the most one can lose is the initial investment.

since $\lambda u(y_1) + (1 - \lambda)u(y_2) = u_c(y_0) > u(y_0)$. Moreover, it satisfies the pricing constraint, since it is linear.⁸ \square

For the concavified utility we can now follow the same computations as before. The only difficulty is that the inverse of the derivative is still not everywhere defined, since the derivative can still be constant. This, however, simply corresponds to a jump in y , as can be seen, for instance, by an approximation argument.

2.4 Numerical results

How can we compute optimal structured products numerically? In principle, we just need to evaluate equation (2), however, it is often difficult to compute the Lagrange parameter λ as explicitly as we have done in the examples of a power and of an exponential utility function. Thus, we use for our numerical computation sometimes an iteration method, i.e. we evaluate (2) for a fixed λ , then compute the error of the constraint (2) and correct the λ . We iterate this until the error is sufficiently small.

Let us now give a couple of examples to see how optimal structured products look like.

Power utility

Let us consider the power utility function $u(x) = x^\alpha/\alpha$ (i.e. a typical function with constant relative risk aversion). The optimal structured product for an expected utility investor with this utility function (for $\alpha = 0.8$) on a CAPM market is shown in Fig. 1. We see that the payoff function of the product is strictly convex (which we know already from Theorem 1). It features a

⁸Here we need indeed the CAPM assumption. Otherwise the result would not always hold.

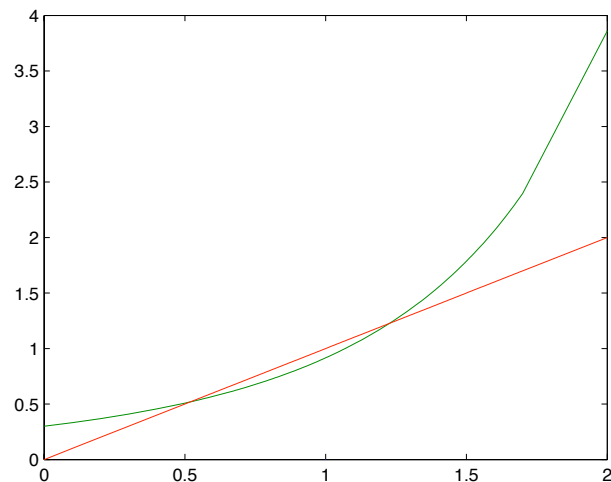


Figure 1: Optimal structured product for an CRRA-investor with $\alpha = 0.8$.

“smooth” capital protection and an increasing participation in gains.

Exponential utility

Let us consider the exponential utility function $u(x) = -e^{-\alpha x}/\alpha$ (i.e. a typical function with constant absolute risk aversion). The optimal structured product for an expected utility investor with this utility function (for $\alpha = 0.5$) on a CAPM market is shown in Fig. 2. We see that the payoff function of the product is only very slightly convex.

Quadratic utility with aspiration level

Let us now consider a non-concave utility function. Non-concave utility functions are a key ingredient of some behavioral decision models, like prospect theory. They can also occur, however, for very rational reasons: an example would be an investor who plans to buy a house in one year and has saved just about enough money for the installment. (We denote his current wealth

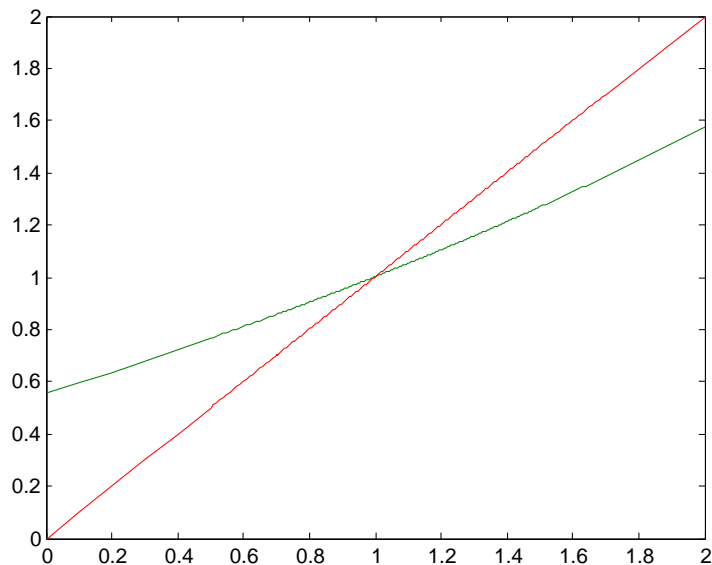


Figure 2: Optimal structured product for an CARA-investor with $\alpha = 0.5$.

level by x_0 .) His utility function, when considering a one-year investment, will now be necessarily have a jump (maybe slightly smoothed by the uncertainty about the house prices) that will make it locally non-concave.⁹ Such an investor will find it naturally attractive to invest in a product with some kind of capital protection, as we will see.

Let us define his utility function as $v(x) = a(x) + u(x)$ with $a(x) = h(x)u_h$, where u_h is the extra utility gained by the house and $h(x)$ is the probability that he is able to afford the house which we set for simplicity as one for $x \geq x_0$ and zero for $x < x_0$, thus inducing a jump in the utility function at x_0 . In reality the probability might be more like a logistic function rather than a precise jump function, thus the overall utility v would look more like a concave–convex–concave function. See Fig. 3 for illustrations.

To keep things simple we choose as u the function $u(x) = x - \alpha x^2$ with

⁹This is similar to the aspiration level in the SP/A model, see [14] for an overview and further references.

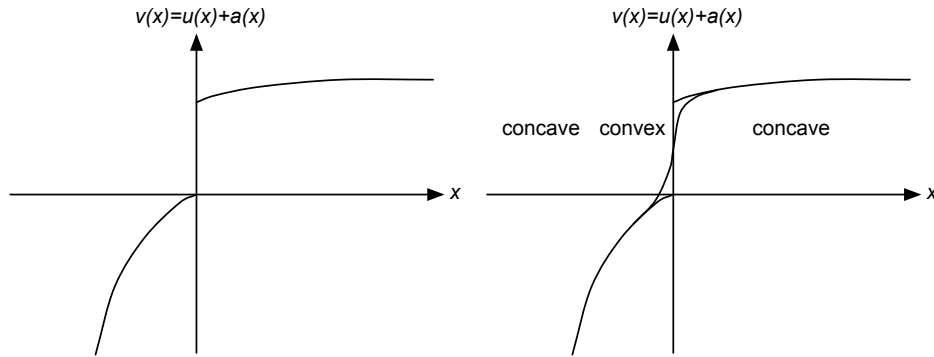


Figure 3: The utility function v of an investor with an aspiration level: his utility increases above a certain threshold, thus making it non-concave with a jump (left). If the precise location of the threshold is uncertain, the effective utility function becomes again continuous, but is still convex around that point (right).

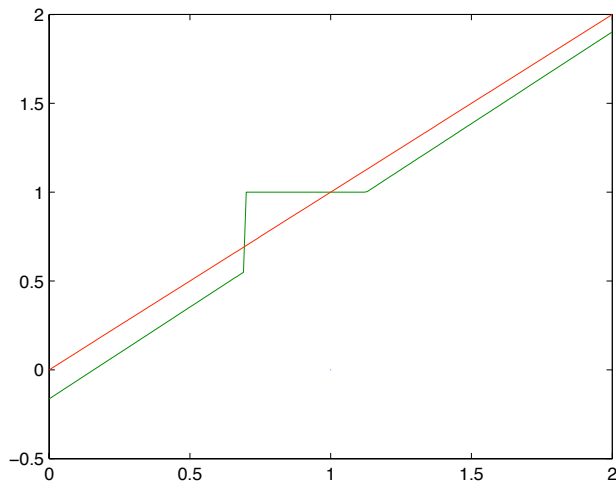


Figure 4: Optimal structured product for an investor with quadratic utility plus an aspiration for $x_0 = 1$.

$\alpha = 0.1$. Moreover we set $x_0 = 1$ and $u_h = 0.02$. We need to compute the concave hull of

$$v(x) = a(x) + u(x) = \begin{cases} x - \alpha x^2, & \text{for } x < 1, \\ 0.02 + x - \alpha x^2, & \text{for } x \geq 1 \end{cases}.$$

To this end, we compute the derivative of u to find the tangential line on u which crosses the point $(1, v(1))$. A straightforward computation using (2) leads to an explicit solution, depending on the Lagrange parameter λ which is then computed numerically by an iteration scheme as outlined above.

The resulting optimal structured product is piecewise affine (as was to be expected when using a quadratic u) and corresponds to a limited capital protection as it can be found in many structured products (compare Fig. 4).

2.5 Utility gain by SPs

We have seen so far that optimal structured products do indeed deliver a payoff that is different to classical portfolios including only the market portfolio and a risk-free asset. The effort of computing such an optimal structured product and hedging it can in praxis not be neglected, therefore the natural question arises whether it is worth it: how big is the potential utility improvement?

To answer this question, we compute the expected utility of an optimal structured product and the expected utility of the optimal mix between the market portfolio and the risk-free asset. We translate both values into certainty equivalent interest rates, i.e. the (hypothetical) risk-free asset that would have the same expected utility as the product respectively the portfolio. We then consider the difference between both certainty rates and compare it with the gain in certainty rates that can be achieved by a classical portfolio over a risk-free investment. In a certain sense, we see how good the

“second order approximation” (the classical two-fund portfolio) is, comparing to the “higher order approximation” (the optimal structured product). Thus we want to answer a couple of questions of practical relevance: is the first (classical) step of improvement big, and the second one (caused by the structured product) negligible? Or are they equally important? Under what circumstances is the additional potential improvement of structured products particularly large?

To keep the analysis simple, we restrict ourselves to the two examples of the previous section: first, we consider a CRRA-utility functions of the form $u(x) = x^\alpha/\alpha$. Later we will study the case of a quadratic utility function with aspiration level.

The optimal structured product for CRRA-utility functions has been already computed (see Section 2.2). Thus we can directly compute the expected utility of y , where we assume a normal distribution with mean $\mu = 1.08$ and standard deviation $\sigma = 0.19$ for the return of the market portfolio. Reporting only this value would be idle: we need to compare it with other utilities. Therefore we compute the improvement, as expressed in terms of certainty equivalent interest rate, over the optimal “classical” portfolio, i.e. the optimal combination of the market portfolio and a risk-free investment. To compute the optimal classical portfolio, we compute the utility of all portfolios with a proportion of θ invested in the market portfolio and $(1 - \theta)$ invested into risk-free assets, where $\theta \in \{0, 0.01, \dots, 0.99, 1\}$. Then we choose the θ yielding the largest expected utility.

The results of this computation are summarized in Table 2.5.

We see that the optimal structured product gives an improvement, but it is not as big as the “first step”, i.e. the improvement induced by the classical mean-variance portfolio theory. The improvement over the classical portfolio

Investment	Certainty equivalent
Fixed interest (4%)	4.00%
Market portfolio (optimal classical)	4.58%
Optimal structured product	4.64%

Table 1: Improvement of a structured product for an investor with a classical CRRA utility as compared to the improvement by classical portfolios.

is only 6 basis points.

Investment	Certainty equivalent
Fixed interest (4%)	4.00%
Optimal classical portfolio (53% market portfolio, 47% fixed interest)	4.24%
Optimal structured product	4.24%

Table 2: Improvement of a structured product for an investor with a classical CARA utility as compared to the improvement by classical portfolios.

The second example, an exponential utility function, can be computed in the same way. Here the improvement is minute (only 0.0004%) and in the precision of the data in Table 2.5 not even visible. This does not come as a big surprise, since the optimal structured product is close to a linear investment (compare Fig. 2).

So far it looks like the improvements of structured products are somewhere between tiny and small, but noticeable. Structured products – much ado about nothing? Are there no situations in which we can generate a decisive improvement of a portfolio with their help?

Let us consider the third example, the quadratic utility with aspiration level.

Here, finally, the improvement due to structured products is considerable as we see in Table 2.5. In fact, the improvement is as big (or bigger) than the “first step improvement” done by the classical mean-variance theory!

Investment	Certainty equivalent
Fixed interest (4%)	4.00%
Market portfolio	3.74%
Optimal classical (8% market portfolio, 92% fixed interest)	4.06%
Optimal structured product	4.30%

Table 3: Improvement of a structured product for an investor with an aspiration level as compared to the improvement by classical portfolios.

To summarize our results it seems that whereas in classical strictly concave settings, the additional amount of risk control due to structured products is not large and, depending on the utility function, sometimes even quite small. As soon as we broaden our horizon and look at situations with partially non-concave utility functions, structured products become much more interesting. In such situations, they can easily improve the portfolio by an amount that is larger than the improvement of the first step from a fixed interest rate to a classical optimal portfolio á la CAPM. However, the banks typically charge at least 1% on SPs hence the utility improvements we found are not worth these costs.

3 Why do people buy structured products?

In this section we want to investigate potential reasons for the attractiveness of structured products, starting with a classical rational approach that we

relax step by step.

3.1 Classical rational investors

Let us assume that the investor follows expected utility theory with a strictly increasing, strictly concave utility function satisfying $u''' < 0$, as before. We have shown in Theorem 1 that a classical rational investor in this sense would only invest into SPs with a strictly convex payoff function.

While there are probably no completely strictly convex SPs on the market, there are types of SPs that come close to this, and they could be very attractive to strictly risk-averse rational investors. Some of the most popular SPs, however, are not convex.

Could this phenomenon be explained by other assets in the investor's portfolio? After all, one of our initial assumptions was that an investor only holds risk-free assets and a structured product.

Let us relax this assumption and allow the investor to hold additionally to the SP a portfolio of classical assets – a combination of the market portfolio and risk-free assets. Suppose the return of the SP is described by a function y which is non-convex. Then the overall portfolio can be described by the function

$$\tilde{y}(x) = \lambda_1 R + \lambda_2 x + (1 - \lambda_1 - \lambda_2)y(x),$$

where $\lambda_1 > 0$ is the proportion of total wealth invested in risk-free assets and $\lambda_2 > 0$ is the proportion of wealth invested in the market portfolio such that $1 - \lambda_1 - \lambda_2 > 0$. According to Theorem 1 the function \tilde{y} is strictly convex, but $\tilde{y}''(x) = y''(x)$, thus we have a contradiction to the assumption that y is non-convex.

We conclude that a non-convex SP for a classical rational investor would only be useful if he already has assets with a strictly convex payoff function in his

portfolio. If we consider such assets as SPs and study the sum of all SPs, the statements about optimal SPs are still true, but apply to the sum of the SPs in the investor's portfolio.

3.2 Investors with PT-type utility function

It is not a tremendous surprise that a classical expected utility model with strictly concave utility function cannot explain the attractiveness of many structured products. After all, there is already ample experimental and empirical evidence in favor of non-classical decision theories like prospect theory. In the next step, we will therefore implement one of the key ingredient of prospect theory, namely the convex-concave structure of the utility function with respect to a reference point that is in itself not fixed, but can, for instance, be the initial value of an asset.

We have seen already that in this case loss-aversion can induce a non-convex payoff function of an optimal structured product with a “plateau” at zero return. This mimics a frequent feature of structured products, namely limited capital protection that is valid only up to a certain amount of losses.

More popular structures, however, like the highly popular barrier reverse convertibles can still not be optimal, since we have the following corollary from the co-monotonicity results in [11]:

Corollary 2. *If the decision model is given by an arbitrary utility function (possibly with reference point set by the initial wealth level), any optimal structured product in a CAPM market (or any market where the likelihood ratio is decreasing as a function of the return of the market portfolio) is a monotonic function (or correspondence) of the market return.*

In other words: the higher the return of the market portfolio at maturity,

the higher the return of the structured product.

This is obviously violated for all types of barrier products or, more generally, all structured products whose return at maturity is not solely defined by the return of the underlying market portfolio at maturity. Thus the modified utility function of prospect theory can only explain the popularity of some of the structured products offered today. The most popular types cannot be understood within this model.

3.3 Investors with probability weighting

In the next step, we add probability weighting and observe its effects on the qualitative features of optimal structured products.

First, we need to distinguish two models of probability weighting:

1. We consider the return distribution of the structured product and apply a probability weighting to it.
2. We apply probability weighting to the return distribution of the underlying.

Both approaches are a priori reasonable, but lead to very different results. In particular, in the first case, the co-monotonicity results [11] still hold, thus the payoff function is still monotonic. In the second case, however, this is no longer the case, thus we can explain, for instance, the attractiveness of constructions that give high payoffs for extreme events (which are overweighted by the probability weighting), in particular straddles.

Nevertheless, the payoff is still (essentially) a function of the underlying:

Theorem 2. *Let $k \geq 2$. Assume that the return of the market portfolio is an absolutely continuous measure with a k -times continuously differentiable (C^k)*

distribution function p which is nowhere equal to zero. For an investor with an arbitrary C^k -utility function (possibly with a reference point depending on the initial wealth) and a probability weighting of the underlying with a probability weighting function $w \in C^{k+1}$, an optimal structured product is a piecewise C^{k-1} -function of the underlying market portfolio.

Proof. Let p_* denote the return distribution of the market portfolio after probability weighting has been applied. More precisely, in cumulative prospect theory we define¹⁰

$$p_*(x) := \frac{d}{dx} \left(w \left(\int_{-\infty}^x p(t) dt \right) \right).$$

Similarly, in prospect theory (see [7, 8, 13]), we define for $\gamma \in (0, 1]$:

$$p_*(x) := \frac{p(x)^\gamma}{\int_{\mathbb{R}} p(x)^\gamma dx}.$$

Now in both cases it is easy to see that $p_* \in C^k$ and is nowhere zero. We define $w_*(x) := p_*(x)/p(x)$. Then the optimal structured product maximizes

$$U(p) := \int_{\mathbb{R}} u(y(x)) p_*(x) dx,$$

subject to

$$\int_{\mathbb{R}} y(x) \frac{\ell(x)}{w_*(x)} p_*(x) dx = 1. \quad (3)$$

From the co-monotonicity result [12], we know that the maximizer y of this problem is a monotonic function (remember that p is absolutely continuous) of $\ell(x)/w_*(x)$. Since w_* is a C^k -function ($k \geq 2$), we can differentiate this expression and arrive at

$$\frac{d}{dx} \left(\frac{\ell(x)}{w_*(x)} \right) = \frac{\ell'(x)w_*(x) - \ell(x)w'_*(x)}{w_*(x)^2},$$

¹⁰We could also use a decumulative function in gains as in the original formulation in [15], but both is mathematically equivalent if we choose w differently in gains and losses.

which is a C^{k-1} -function, thus y is piecewise C^{k-1} . \square

This result implies in particular that y is still a *function* of the market return. Thus, some of the most popular classes of products, namely all products with *path-dependent* payoff (in particular barrier products), still cannot be explained by this model.

3.4 Heterogeneous beliefs and probability mis-estimation

We have seen from the results in the previous sections: if we want to understand why investors buy structured products like barrier reverse convertibles or basket products, we need to take into account other factors than just risk preferences.

Essentially, there seem to be two important reasons that we have ignored so far:

1. Heterogeneous beliefs (“betting”)
2. Probability mis-estimation

Heterogeneous beliefs mean that investors believe that the market will behave differently than the probability distribution p forecasts. This might be wise in some circumstances, but it is probably much more frequently a sign of overconfidence – an all too common characteristics of private investors. Moreover, it seems likely that heterogeneous beliefs can explain the popularity of yield enhancing (call option style) or contrarian (put option style) products, but it seems less likely that it can explain the popularity of complicated constructions like barrier reverse convertibles.

Here a better explanation are probability mis-estimations. The difference to heterogeneity is that here not a difference in beliefs is the main cause, but rather a difficulty to translate quite accurate estimates on the market into

correct estimates on probabilities for certain events that are important for the payoff of a structured product.

As an example we consider barrier products (e.g. barrier reverse convertibles). These are products that have a capital protection that vanishes when the price of the underlying *at some point before maturity* falls below a certain threshold (“barrier”). These products are quite popular on the market, although their payoff at maturity is not a function of the final price of the underlying and thus they are not optimal for any decision model (see above). Can probability mis-estimation explain this popularity? Yes, it can if customers underestimate the probability that the barrier is hit *at some point in time* before maturity with respect to the probability that the prize is below the barrier level *at maturity*.

How different are these two probabilities in reality? In [12] this has been measured using the Dow Jones Industrial Index and assuming an arbitrary issue date between January 1, 1985 and December 31, 2004 and a maturity of one year. Figure 5 shows the probability that the barrier was hit *at some point within a year* and the probability that the prize was below the barrier level *after one year* for various barrier levels. The quotient between these probabilities is depicted in Figure 6. In particular for relatively low barrier levels between 70-80% as they are also quite frequently used in real barrier products, this quotient is quite big.

How do customers estimate these probabilities? Do they see the big difference between the two variants or do they underestimate the difference and could this explain why customers choose the (sub-optimal) down-and-out barrier product?

In an anonymous classroom experiment [12] with a sample of $N = 109$, undergraduate students of economics and finance from the University of Zurich

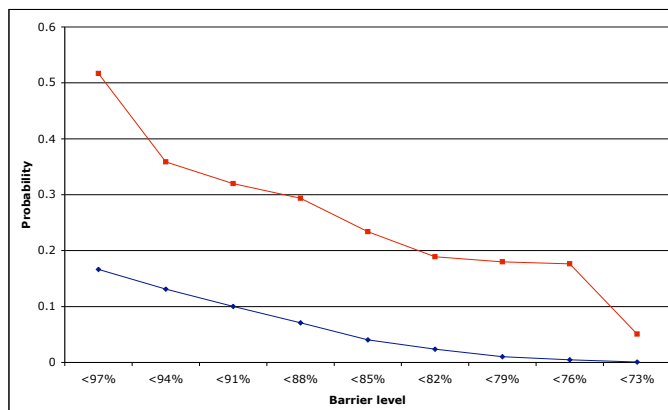


Figure 5: Probability to hit the barrier at maturity (lower line) or at some point before (upper line). The latter corresponds to the probability to lose capital protection in a down-and-out barrier product, see [12].

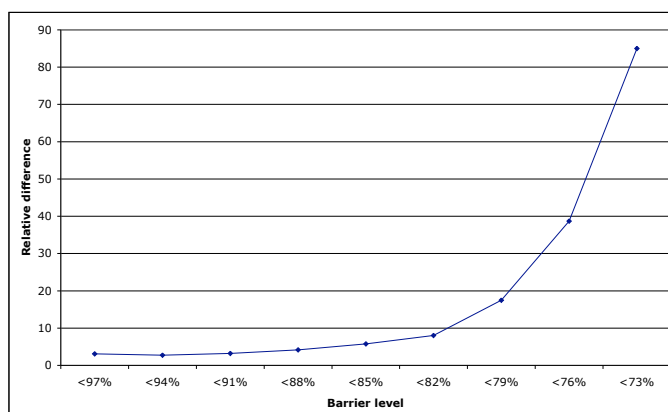


Figure 6: The quotient of the two probabilities is increasing steeply for low barrier levels. This effect is likely to be underestimated by investors, see [12].

have been asked to estimate the probabilities of the events “Dow Jones Index is $X\%$ lower after one year” and “Dow Jones Index is $X\%$ lower *at some point* during one year” for $X = 10$ and $X = 20$, where we specified that we are asking based on the historical data from 1985 to 2004. In other words, the participants had to estimate the probabilities from Figure 5. The choice of the sample implied a fundamental knowledge on stock market development that was needed to comprehend and perform the task.

	Real probabilities		Average estimates	
	(historical data)		by participants	
Probability that DJI is...	-10%	-20%	-10%	-20%
...lower at maturity	10.0%	1.0%	21.8%	14.2%
...lower at some point	32.0%	18.0%	41.4%	27.1%
Relative difference	3.2	17.5	1.9	1.9

Table 4: Real and estimated probabilities for the Dow Jones Index data. The results show a large mis-estimation of the relative difference between the two probabilities, see [12].

The results demonstrate that although the overall estimation of the probabilities was quite good, the relative difference between the two different probabilities was systematically underestimated. 42% of the participants were not even estimating an increase in this relative difference when increasing X from 10 to 20. This leads to a systematic underestimation of the risks involved in certain structured products and a systematic overestimation of the risks involved in structured products without a down-and-out feature. In this way the high popularity of barrier reverse convertibles can be explained by probability mis-estimation.

Another example for this effect are basket products: a recent experimental

study [12] suggests that although subjects are usually quite capable to estimate the volatility and correlation of assets in a basket, they do not consider the correlation effect when estimating the probability that at least one of the assets in the basket breaks a certain barrier. This again increases the subjective attractiveness of certain types of products to them.

We could sum this up by saying that the most popular classes of structured products combine in a clever way prospect theory-like preferences (in particular loss-aversion and risk-seeking behavior in losses) and probability mis-estimation induced by a complicated payoff structure that leads to a systematic underestimation of the probability for unfavorable outcomes.

4 Conclusions

We have seen that structured products can indeed be a useful tool to enhance the performance of a portfolio. Depending on the (rational) risk attitudes of an investor it is usually good to use a product that leads to a strictly convex payoff structure for the risky part of the portfolio. This can be done by hedging against losses of different degrees. We estimate the size of the improvement when comparing to a classical Markowitz-style (mean-variance) investment.

Most popular structured products, however, do not follow this rational guideline, but instead use behavioral factors, like loss-aversion or probability mis-estimation to be attractive in the eyes of potential investors. In particular we could show that the currently most popular products clearly cannot be explained even within the framework of prospect theory, but only when taking into account probability mis-estimation.

Our results suggest a procedure to design tailor-made structured products to

an investor with known risk attitudes.

Currently we are extending this work in several directions.

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