Theory of Plasticity

Theoretical fundamentals and applications to soil mechanics

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Evidence of nonlinear and irreversible behavior

Granular materials exhibit very complex responses under applied loads, even under relatively simple loading conditions.

Examples of axisymmetric loading:

- isotropic compression:
  \[ \sigma_a = p \quad \sigma_r = p \]

- oedometric (1d) compression:
  \[ \sigma_1 = \sigma_a \quad \epsilon_r = 0 \]

- drained triaxial compression:
  \[ \sigma_a > \sigma_r \quad \sigma_r = \text{const.} \]
Clay sample under isotropic compression

\[ p = \frac{\sigma_a + 2\sigma_r}{3} = \sigma_c \]
\[ \epsilon_v = \epsilon_a + 2\epsilon_r \]
Clay sample under drained TX compression

\[ p = \frac{(\sigma_a + 2\sigma_r)}{3} \quad q = \sigma_a - \sigma_r \quad \sigma_r = \sigma_c = \text{const.} \]
Failure locus of a coarse–grained soil

\[ q_f = M p_f \]
The main features of the observed response – **nonlinearity, irreversibility, path–dependence** – must be described by any **constitutive equation** which aims at providing accurate predictions of the soil response under applied loads.

Constitutive equations do not represent universal laws of nature, but rather they define “idealized materials” (**costitutive models**).

When available, constitutive models can be used to predict soil behavior under any possible circumstance, starting from the limited knowledge gathered from a few experimental observations.

The quality of predictions depends crucially on the ability of defining a **suitable idealization** for the real materials, for the class of loading paths of practical interest.
To incorporate the observed non-linearity and path-dependence in a global stress-strain relation, non-linear and non-differentiable functional must be used (Owen & Williams 1969).

Therefore, rather than in global terms, constitutive equations are usually defined in rate form:

\[ \nabla \sigma = F(\sigma, q, d) = C(\sigma, q, \text{dir } d) \]  

rate-ind.

where:

- \( \nabla \sigma \): any objective time rate of Cauchy stress tensor;
- \( d := \text{sym}(\nabla \cdot v) \): rate of deformation tensor;
- \( q \): additional set of state variables (internal variables), taking into account the effects of the previous loading history.
The mathematical theory of plasticity represent the first attempt to provide a rational framework within which formulate the constitutive equations in rate form, taking into account:

- the existence of a well defined limit to admissible stress states in stress space;

- the flowing character of the deformation rates in ultimate failure states, with strain rate depending on current stress rather than stress rate.
The foundations of the classical theory of plasticity can be traced back to the fundamental works of Hill (1950) and Koiter (1960).


For applications to soil mechanics, good references are Desai & Siriwardane (1984), Zienkiewicz et al. (1999), Houlsby & Puzrin (2006), Nova (2010).

The application of plasticity concepts such as failure criterion and flow rule to soil mechanics are as early as the works of Coulomb (1773) and Rankine (1853), referring to the problem of evaluating the earth pressure on retaining structures. These concepts still form the basis of limit equilibrium and limit analysis methods, widely used in current practice.
In the following, it is assumed that displacements and rotations are sufficiently small that the reference and current configurations can be considered almost coincident (small strain assumption).

The material behavior is assumed to be rate-independent (non viscous).

Under such hypotheses, the CE in rate form reduces to:

\[
\dot{\sigma} = C(\sigma, q, \text{dir} \dot{\epsilon}) \dot{\epsilon}
\]

where:

- \(\dot{\sigma}\) : Cauchy stress rate tensor;
- \(\dot{\epsilon} = \text{sym}(\nabla \cdot \dot{u})\) : infinitesimal strain rate tensor.
Assumption 1: kinematic decomposition of $\epsilon$

We assume the following **additive decomposition** of the strain tensor into an **elastic** (reversible) and a **plastic** (irreversible) part:

$$\epsilon = \epsilon^e + \epsilon^p$$

In terms of rates, this yields:

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p$$
Assumption 2: hyperelastic behavior

We postulate the existence of a free energy density function (per unit volume):

$$\psi = \hat{\psi}(\epsilon^e, \alpha)$$

where $\alpha = \text{strain–like} \text{ internal variable}$.  

The Clausius–Duhem inequality thus reads:

$$\mathcal{D} := -\dot{\psi} + \sigma \cdot \dot{\epsilon} \geq 0$$

$$\Rightarrow \quad \mathcal{D} := -\frac{\partial \hat{\psi}}{\partial \epsilon^e} \cdot \dot{\epsilon}^e - \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \sigma \cdot \dot{\epsilon} \geq 0$$
Assumption 2: hyperelastic behavior

Introducing the strain decomposition into the C–D inequality:

\[ -\frac{\partial \hat{\psi}}{\partial \epsilon^e} \cdot (\dot{\epsilon} - \dot{\epsilon}^p) - \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \sigma \cdot \dot{\epsilon} \geq 0 \]

and, rearranging:

\[ \left( \sigma - \frac{\partial \hat{\psi}}{\partial \epsilon^e} \right) : \dot{\epsilon} + \frac{\partial \hat{\psi}}{\partial \epsilon^e} \cdot \dot{\epsilon}^p - \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} \geq 0 \]

For non dissipative processes (\(\dot{\epsilon} = 0, \dot{\alpha} = 0\)) the equality must hold for any possible choice of \(\dot{\epsilon}\). Therefore:

\[ \sigma = \frac{\partial \hat{\psi}}{\partial \epsilon^e} \]
Reduced dissipation inequality

Definition: stress–like internal variable work–conjugate to $\alpha$:

$$q = -\frac{\partial \hat{\psi}}{\partial \alpha}$$

Taking into account the hyperelastic CE and the definition of $q$, the Clausius–Duhem inequality yields the following reduced dissipation inequality:

$$\mathcal{D} := \sigma \cdot \dot{\epsilon}^p + q \cdot \dot{\alpha} \geq 0$$
Assumption 3: elastic domain and yield function

Let the state of the material be described by the generalized stress:

\[ \Sigma = (\sigma, q) \]

Irreversibility of the material response is brought in by requiring that the state of the material belongs to the convex set:

\[ \mathcal{E} = \left\{ \Sigma \in \text{Sym} \times \mathbb{R}^{n_{\text{int}}} \mid f(\Sigma) \leq 0 \right\} \]

For fixed \( q \), the set \( \mathcal{E}_\sigma \) represents the region of admissible states in stress space.

The interior \( \text{int}\{\mathcal{E}_\sigma\} \) is referred to as the elastic domain. The boundary:

\[ \partial \mathcal{E}_\sigma = \left\{ \sigma \in \text{Sym} \mid f(\sigma, q) = 0 \right\} \]

is called the yield surface. The yield surface is convex in stress space.
Assumption 3: elastic domain and yield function

\[ f(\sigma, q) < 0 \]

\[ f(\sigma, q) = 0 \]

\[ f(\sigma, q) > 0 \]

(non admissible)
Assumption 4: principle of maximum dissipation

Let the quantities:

\[ E^p := (\epsilon^p, \alpha) \quad \dot{E}^p := (\dot{\epsilon}^p, \dot{\alpha}) \]

be the generalized plastic strain and generalized plastic strain rate.

From the definitions of generalized stresses and strains, the RDE reduces to:

\[ D := \sigma \cdot \dot{\epsilon}^p + q \cdot \dot{\alpha} = \Sigma \cdot \dot{E}^p \]

**Principle of maximum dissipation.** In plastic loading conditions, for a given generalized plastic strain rate, the actual state of the material is the one, among all admissible states, for which plastic dissipation \( D \) attain its maximum, i.e.:

\[ D(\Sigma, \dot{E}^p) = \Sigma \cdot \dot{E}^p = \max_{\Sigma^* \in \mathcal{E}} \left\{ D(\Sigma^*, \dot{E}^p) \right\} \]
Assumption 4: principle of maximum dissipation

According to a standard result in optimization theory, the actual state is the stationary point of the following unconstrained Lagrangian:

\[
L (\Sigma, \dot{\gamma}) = -\Sigma \cdot \dot{E}^p + \dot{\gamma} f (\Sigma)
\]

subject to the following Kuhn-Tucker optimality conditions:

\[
\dot{\gamma} \geq 0 \quad f (\Sigma) \leq 0 \quad \dot{\gamma} f (\Sigma) = 0
\]

The stationary point of \( L \) are found by setting to zero its first variation w/r to \( \Sigma \):

\[
\frac{\partial L}{\partial \Sigma} = -\dot{E}^p + \dot{\gamma} \frac{\partial f}{\partial \Sigma}
\]

which yields:

\[
\dot{E}^p = \dot{\gamma} \frac{\partial f}{\partial \Sigma}
\]
Associative flow rule and hardening law

- **Associative flow rule:**

\[ \dot{\varepsilon}^p = \dot{\gamma} \frac{\partial f}{\partial \sigma} \]

- **Associative hardening law:**

\[ \dot{\alpha} = \dot{\gamma} \frac{\partial f}{\partial q} \]

- **Complementarity (loading/unloading) conditions in K–T form:**

\[ \dot{\gamma} \geq 0, \ f(\sigma, q) \leq 0, \ \dot{\gamma} f(\sigma, q) = 0 \]
Motivation and historical perspectives
Thermodynamical foundations
Conventional elastoplastic models for soils
Advanced topics: anisotropic hardening plasticity

Associated flow rule: remarks

\[ \dot{E}^p = \dot{\gamma} \frac{\partial f}{\partial \Sigma} \]
\[ \dot{\gamma} \geq 0, \quad f(\Sigma) \leq 0, \quad \dot{\gamma} f(\Sigma) = 0 \]

- A consequence of the K–T complementarity conditions is that plastic flow may take place only if the current state is on the yield surface.

- The generalized plastic flow direction $\dot{E}^p$ is normal to the yield surface. Therefore the flow rule and the hardening law are said to be associative (alternatively, the material is said to obey to the normality rule).
Due to hardening, the yield surface can change in size, shape and position in stress space.

Isotropic hardening/softening
When no internal variables are present, the yield surface remains fixed in stress space. The material is defined perfectly plastic.
Graphical interpretation of PMD

Max dissipation requires that:

$$\forall \Sigma^* \in \mathcal{E} \Rightarrow (\Sigma - \Sigma^*) \cdot \dot{E}^p \geq 0$$

This is possible only if the yield surface is convex and the plastic flow is associative.
Prager’s consistency condition

The K–T complementarity conditions are not sufficient to determine whether plastic flow will actually occur or not for a given stress state on the yield surface.

\[ f(\sigma + d\sigma, q + dq) = 0 \]

[Diagram showing yield surface and stress state]

**Prager’s consistency condition.** In a plastic process, the actual value of the plastic multiplier is obtained by prescribing that, upon plastic loading, the stress state must remain on the yield surface.
Prager’s consistency condition

Mathematically, the consistency condition requires that, if plastic loading is occurring:

\[ \dot{f} = \frac{\partial f}{\partial \Sigma} \cdot \dot{\Sigma} = \frac{\partial f}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial f}{\partial q} \cdot \dot{q} = 0 \]

The quantities \( \dot{\sigma} \) and \( \dot{q} \) can be obtained from the hyperelastic CEs and the flow rule:

\[ \sigma = \frac{\partial \hat{\psi}}{\partial \epsilon^e} \quad \dot{\sigma} = C^e \left( \dot{\epsilon} - \dot{\gamma} \frac{\partial f}{\partial \sigma} \right) \quad C^e := \frac{\partial^2 \hat{\psi}}{\partial \epsilon^e \otimes \partial \epsilon^e} \]

\[ q = -\frac{\partial \hat{\psi}}{\partial \alpha} \quad \dot{q} = -\dot{\gamma} H \frac{\partial f}{\partial q} \quad H := \frac{\partial^2 \hat{\psi}}{\partial \alpha \otimes \partial \alpha} \]

(cont.)
Prager’s consistency condition

Substituting:

\[ \dot{f} = \frac{\partial f}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial f}{\partial q} \cdot \dot{q} \]

\[ = \frac{\partial f}{\partial \sigma} \cdot C^e \left( \dot{\epsilon} - \dot{\gamma} \frac{\partial f}{\partial \sigma} \right) - \dot{\gamma} \frac{\partial f}{\partial q} \cdot H \frac{\partial f}{\partial q} = 0 \]

which yields:

\[ \left\{ \frac{\partial f}{\partial \sigma} \cdot C^e \frac{\partial f}{\partial \sigma} + \frac{\partial f}{\partial q} \cdot H \frac{\partial f}{\partial q} \right\} \dot{\gamma} = \frac{\partial f}{\partial \sigma} \cdot C^e \dot{\epsilon} \]

(cont.)
Prager’s consistency condition

Solving for $\dot{\gamma}$, we have:

$$\dot{\gamma} = \frac{1}{K_p} \frac{\partial f}{\partial \sigma} \cdot C^e \dot{\varepsilon}$$

where:

$$K_p := \frac{\partial f}{\partial \sigma} \cdot C^e \frac{\partial f}{\partial \sigma} + \frac{\partial f}{\partial q} \cdot H \frac{\partial f}{\partial q} = \frac{\partial f}{\partial \sigma} \cdot C^e \frac{\partial f}{\partial \sigma} + H_p$$

Constitutive assumption:

$$K_p > 0$$
Plastic modulus

The quantity:

\[ H_p := \frac{\partial f}{\partial q} \cdot H \frac{\partial f}{\partial q} \]

is called plastic modulus.

- When \( H_p > 0 \) the material is hardening (the elastic domain expands).
- When \( H_p < 0 \) the material is softening (the elastic domain shrinks).
- For \( H_p = 0 \), the material is in a perfectly plastic condition (the elastic domain remains stationary).

Limit to allowable softening:

\[ K_p > 0 \quad \Rightarrow \quad H_p := \frac{\partial f}{\partial q} \cdot H \frac{\partial f}{\partial q} > - \frac{\partial f}{\partial \sigma} \cdot C^e \frac{\partial f}{\partial \sigma} \]
The plastic multiplier cannot be negative. Therefore, from the previous expression we distinguish 3 possible cases:

\[
\begin{aligned}
\frac{\partial f}{\partial \sigma} \cdot C^e \dot{\varepsilon} > 0 & \implies \text{plastic loading process, } \dot{\gamma} > 0 \\
\frac{\partial f}{\partial \sigma} \cdot C^e \dot{\varepsilon} = 0 & \implies \text{neutral loading process, } \dot{\gamma} = 0 \\
\frac{\partial f}{\partial \sigma} \cdot C^e \dot{\varepsilon} < 0 & \implies \text{elastic unloading, } \dot{\gamma} = 0
\end{aligned}
\]

Hence:

\[
\dot{\gamma} = \frac{1}{K_p} \left\langle \frac{\partial f}{\partial \sigma} \cdot C^e \dot{\varepsilon} \right\rangle
\]
Let:

\[ \dot{\sigma}^{\text{tr}} = C^{e} \dot{\epsilon} \quad (\text{trial stress rate}) \quad \Rightarrow \quad \dot{\gamma} = \frac{1}{K_p} \left\langle \frac{\partial f}{\partial \sigma} \cdot \dot{\sigma}^{\text{tr}} \right\rangle \]
**Example: 1–d plasticity**

**Free energy function and hyperelastic behavior.**

We assume an uncoupled, quadratic form:

\[
\hat{\psi} = \frac{1}{2} \varepsilon^e C^e \varepsilon^e + \frac{1}{2} \alpha H \alpha
\]

Thus:

\[
\sigma = \frac{\partial \hat{\psi}}{\partial \varepsilon^e} = C^e \varepsilon^e \quad q = - \frac{\partial \hat{\psi}}{\partial \alpha} = -H \alpha
\]
Example: 1–d plasticity

Yield function:

\[ f(\sigma, q) = |\sigma| - (\sigma_Y + q) = 0 \]
Flow rule and hardening law:

\[ \dot{\varepsilon}^p = \dot{\gamma} \frac{\partial f}{\partial \sigma} = \dot{\gamma} \ \text{sgn}(\sigma) \]

\[ \dot{\alpha} = \dot{\gamma} \frac{\partial f}{\partial q} = -\dot{\gamma} \]

Plastic multiplier:

\[ K_p = \frac{\partial f}{\partial \sigma} C^e \frac{\partial f}{\partial \sigma} + \frac{\partial f}{\partial q} \mathcal{H} \frac{\partial f}{\partial q} = C^e + \mathcal{H} \]

\[ \dot{\gamma} = \frac{1}{K_p} \left\langle \frac{\partial f}{\partial \sigma} C^e \dot{\varepsilon} \right\rangle = \frac{C^e}{C^e + \mathcal{H}} \left\langle \text{sgn}(\sigma) \cdot \dot{\varepsilon} \right\rangle \]
Constitutive equation in rate form.

For $\text{sgn}(\sigma) \cdot \dot{\varepsilon} > 0$ (plastic processes):

$$
\dot{\sigma} = C^e (\dot{\varepsilon} - \dot{\varepsilon}^p) = C^e \dot{\varepsilon} - \dot{\gamma} \left( C^e \frac{\partial f}{\partial \sigma} \right)
$$

$$
= C^e \dot{\varepsilon} - \frac{1}{C^e + \mathcal{H}} \left( C^e \frac{\partial f}{\partial \sigma} \right) \left( \frac{\partial f}{\partial \sigma}, C^e \right) \dot{\varepsilon} = \left( \frac{C^e \mathcal{H}}{C^e + \mathcal{H}} \right) \dot{\varepsilon}
$$
Example: 1–d plasticity

\[
\dot{\sigma} = C^e \dot{\varepsilon} - \frac{1}{C^e + \mathcal{H}} \left( C^e \frac{\partial f}{\partial \sigma} \right) \left( \frac{\partial f}{\partial \sigma}, C^e \right) \dot{\varepsilon} = \left( \frac{C^e \mathcal{H}}{C^e + \mathcal{H}} \right) \dot{\varepsilon}
\]
Non–associative flow rule

There exists a scalar function $g(\sigma, q)$ (plastic potential), such that:

$$\dot{\varepsilon}^p = \dot{\gamma} \frac{\partial g}{\partial \sigma}$$
Non–associative hardening law

There exists a pseudo–vector $\mathbf{h}(\sigma, q) \in \mathbb{R}^{n_{\text{int}}}$ (hardening function) such that:

$$\dot{\gamma} = \dot{\gamma} \mathbf{h}(\sigma, q)$$

For non–associative behavior, consistency condition yields:

$$\dot{\gamma} = \frac{1}{\hat{K}_p} \left\langle \frac{\partial f}{\partial \sigma} \cdot \mathbf{C}^e \dot{\varepsilon} \right\rangle$$

where:

$$\hat{K}_p := \frac{\partial f}{\partial \sigma} \cdot \mathbf{C}^e \frac{\partial g}{\partial \sigma} - \frac{\partial f}{\partial q} \cdot \mathbf{h} > 0$$

$$\hat{H}_p := -\frac{\partial f}{\partial q} \cdot \mathbf{h}$$
Perfect plasticity: Huber–Hencky–von Mises model

\[
f(\sigma) = \text{dev}(\sigma) \cdot \text{dev}(\sigma) - k^2 = 0
\]
Perfect plasticity: Mohr–Coulomb model

\[ f(\tau, \sigma_n) = \tau - c - |\sigma_n| \tan \phi = 0 \]
\[ f(\sigma_1, \sigma_3) = (\sigma_1 - \sigma_3) - 2c \cos \phi - (\sigma_1 + \sigma_3) \sin \phi = 0 \]
Elastic–perfectly plastic models are a rather crude idealization of the actual behavior of geomaterials.

Nonetheless, they are still among the most widely used constitutive models in the FE analysis of practical geotechnical problems, especially in situations where site characterization is problematic.
A radical change in perspective in the application of plasticity to soil mechanics occurred after the work of Roscoe and the Cambridge group, leading to the so-called Critical State Soil Mechanics.

In the language of plasticity, CS models are isotropic hardening models with associative flow rule and non–associative volumetric hardening:

- the internal variables reduce to just one scalar quantity;
- yield function and plastic potential coincide;
- its evolution depends only on volumetric plastic strain rate.

Consequence: at failure conditions, the material “flows” at constant stress and constant volume. Such conditions are called Critical States.
Modified Cam–clay model


Non–linear elastic behavior

\[ \dot{p} = \frac{p}{\kappa^*} \dot{\epsilon}_v \]

\[ \dot{s} = 2G \dot{\epsilon}^e \]

where:

\[ \dot{s} = \text{dev}(\sigma) \]

\[ \dot{\epsilon}^e = \text{dev}(\epsilon^e) \]
Modified Cam–clay model

Yield surface and plastic potential

\[ f(p, q, \theta, p_s) = p(p - p_s) + \left\{ \frac{q}{M(\theta)} \right\}^2 = 0 \]

\[ \frac{\partial f}{\partial p} = 2p - p_s \quad \frac{\partial f}{\partial q} = \frac{2q}{M^2} \quad \frac{\partial f}{\partial p_s} = -p \leq 0 \]
Modified Cam–clay model

Volumetric hardening law
Derived from exptl. observations on isotropic compression of clays.

- Virgin compression line (VCL)
  \[ \epsilon_v = \epsilon_{v0} + \lambda^* \log \frac{p}{p_0} \]

- Unloading/reloading line (URL)
  \[ \epsilon_v = \epsilon_{v0} + \kappa^* \log \frac{p}{p_0} \]

- For the loading cycle ABA':
  \[ \dot{\epsilon}_v = \lambda^* \dot{p}_s \frac{\dot{p}_s}{p_s} \]
  \[ \dot{\epsilon}_v^e = \kappa^* \dot{p}_s \frac{\dot{p}_s}{p_s} \]

(cont.)
Modified Cam–clay model

Volumetric hardening law
Derived from expctl. observations on isotropic compression of clays.

\[ \dot{\varepsilon}_v = \lambda^* \frac{\dot{p}_s}{p_s} ; \quad \dot{\varepsilon}^e_v = \kappa^* \frac{\dot{p}_s}{p_s} \]

Subtracting:

\[ \dot{\varepsilon}_v^p = (\lambda^* - \kappa^*) \frac{\dot{p}_s}{p_s} \]

\[ \Rightarrow \dot{p}_s = \frac{p_s}{\lambda^* - \kappa^*} \dot{\varepsilon}_v^p \]

\[ = \dot{\gamma} \frac{p_s}{\lambda^* - \kappa^*} \frac{\partial f}{\partial p} \]

(cont.)
Modified Cam–clay model

**Volumetric hardening law**

Derived from exptl. observations on isotropic compression of clays.

- Non–associative hardening law:
  \[
  \dot{p}_s = \dot{\gamma} h_s(p, q, \theta, p_s)
  \]

- Comparing with the expression derived for \( \dot{p}_s \) we obtain:
  \[
  h_s = \frac{p_s}{\lambda^* - \kappa^*} \frac{\partial f}{\partial p}
  \]
Modified Cam–clay model

Plastic modulus

\[ \hat{H}_p = -\frac{\partial f}{\partial p_s} h_s = \frac{1}{\lambda^* - \kappa^*} p p_s (2p - p_s) \]
Example of application of CS models

A refined version of MCC: the RMP model (De Simone & T, 2005). Simulation of TX-CD tests on a pyroclastic soil.
Isotropic CS models: remarks

The application of isotropic hardening plasticity to soils – in particular the development of Critical State soil mechanics – marks the transition between classical and modern soil mechanics (a change of paradigm!).

In spite of their limitations, CS models are capable of reproducing qualitatively, and, in most cases quantitatively, the main features of the inelastic response of granular materials:

- irreversibility;
- pressure dependence;
- dilatant/contractant behavior;
- brittle/ductile transition with increasing confining stress.
Main advantages of CS models:

- **simple** mathematical structure;
- **limited number of material constants**, easily linked to the observed response in standard laboratory tests;
- **limited pool of internal variables** (in most cases just one), of scalar nature: easy definition of initial conditions.
Anisotropy in geomaterials

Almost all geomaterials are characterized to a certain extent by the existence of some preferential orientations at the microstructural level.

Specimen of diatomite rock (from Boheler 1987)
Anisotropy in geomaterials

These preferential orientations can be due to the geological processes which led to the formation of the deposit, or be the result of particle rearrangements induced by the applied engineering loads.

In some cases, the existence of particular symmetries at the microstructural level is apparent (i.e., bedding planes).

In other situations, the detection of preferential directions requires the definition of suitable (statistical) indicators which quantify the geometry of grains and the directional properties of their interaction.

For example, in granular materials, such preferential orientation can be associated to the spatial distribution of intergranular contact normals, grain shape and void shape (Oda et al. 1985).
As a consequence of the directional properties of the microstructure, the phenomenological response of geomaterials may be characterized by a more or less marked anisotropy:

The mechanical response of the material (pre–failure stiffness, shear strength) depends on the relative orientation of the applied load w.r. to the microstructure.
Inherent and induced anisotropy

According to Casagrande & Carillo (1944), two different kinds of anisotropic behavior can be distinguished:

a) inherent anisotropy: “[...]
a physical characteristic inherent in the material and entirely independent of the applied strains.”

b) induced anisotropy: “[...]
a physical characteristic due exclusively to the strain associated with the applied stress”.

Some Authors tend to identify inherent anisotropy as the result of the geological processes which led to the formation of the deposit, and of the geometrical properties of the solid grains (e.g., Wong & Arthur 1985).

The above distinction is somewhat arbitrary, as the possibility of altering the microstructural properties of a granular material (or a rock) do depend on the intensity of the applied load.
Influence of anisotropy on soil behavior

An interesting series of experimental data has been obtained in a number of Directional Shear tests on dense sand performed by Wong & Arthur (1985).

Directional Shear Cell (DSC, Wong & Arthur, 1985)
DSC compression tests on dense Leighton Buzzard Sand specimens, tested in plane strain compression with different orientations of the principal directions w.r. to the bedding plane (after Wong & Arthur 1985).

The angle $\delta$ is the inclination between the bedding plane and the max principal stress direction.
Influence of anisotropy on soil behavior
Anisotropy due to previous loading history (induced anisotropy)

DSC compression tests on dense Leighton Buzzard Sand specimens prepared with the bedding plane coincident with the plane of deformation.

Loading program: A) plane strain compression with fixed principal directions, and unloading up to $\sigma_1 = \sigma_3$; B) plane strain compression with fixed principal directions, oriented differently from those of stage A.
Effects of loading history on YS

Stress–path controlled TX tests on dense Aio Sand (Yasufuku et al. 1991).
Effects of loading history on YS

Stress–path controlled TX tests on Pisa clay (Callisto & Calabresi 1998).

![Graph showing stress–path controlled TX tests on Pisa clay](image-url)
Consider a general CE of the form:

$$\sigma = F(d, \xi)$$

in which:

- $\sigma$ = Cauchy stress
- $d$ = applied “load” (either $\epsilon^e$ or $\dot{\epsilon}^p$)
- $\xi$ = structure tensor

This can represent an elastic CE, in which case $d \equiv \epsilon^e$, or a flow rule, in which case $d \equiv \dot{\epsilon}^p$. 

The structure tensor provides (in phenomenological terms) the directional properties of the microstructure which are responsible for the anisotropy. The symmetries of the structure tensor coincide with those of the microstructure.

The principle of Material Frame Indifference requires that:

\[ F(QdQ^T, Q\xi Q^T) = QF(d, \xi)Q^T \quad \forall Q \in \mathcal{O} \]

This implies that \( F \) is an isotropic function of its tensorial arguments. However, this does not mean that \( F \) describes an isotropic mechanical response!
Isotropy

A material is **isotropic** if no changes in the response occur for a given load, upon an arbitrary transformation of the microstructure:

\[ F(d, Q\xi Q^T) = F(d, \xi) \quad \forall Q \in \mathcal{O} \]

It can be proven that for an isotropic material:

\[ \xi = \xi_1 \quad F(d, Q\xi Q^T) = F(d, \xi) \quad F(QdQ^T) = QFQ^T \]

that is, \( \xi \) is an isotropic tensor and \( F \) is an isotropic function w.r. to the load \( d \).
Anisotropy: definition

A material is anisotropic if there exist a transformation of the microstructure which affects the response to a given load:

$$\exists Q \in \mathcal{O} : \quad \sigma^* := F(d, Q\xi Q^T) \neq F(d, \xi) = \sigma$$

Let $S$ be the group of properly orthogonal transformations that leave the response unchanged:

$$\sigma^* := F(d, Q\xi Q^T) = F(d, \xi) = \sigma \quad \forall Q \in S$$

$S$ is defined symmetry group for the material, and corresponds to the group on invariant transformations for the structure tensor.
Symmetry group: particular cases

1. **General anisotropy:**
   \[ S = \{+1, -1\} \]

2. **Orthotropy:**
   \[ S_{\text{Orth}} = \{+1, -1, S_1, S_2, S_3\} \]

3. **Transverse isotropy:**
   \[ S_{\text{Tras}} = \{+1, -1, S_1, S_2, S_3, R_\theta \ (0 \leq \theta \leq 2\pi)\} \]

where:
- \( S_1, S_2, S_3 \) = reflections w.r. to the principal directions of \( d \);
- \( R_\theta \ (0 \leq \theta \leq 2\pi) \) = rotation around the axis of anisotropy.
In the framework of the theory of plasticity, the description of anisotropy requires the introduction of one or more structure tensors (or order $\geq 2$) as material parameters (inherent) or as internal variables (induced).

The structure tensors can affect:

- the elastic behavior of the material;
- the yield function and plastic potential.

Intrinsic anisotropy is typically described by assuming that the structure tensors remains constant during plastic deformation.

Induced anisotropy is typically described by introducing in the set of internal variables one or more symmetric second order tensors as structure tensors, which control the position and the orientation of the yield locus.
Inherent anisotropy can be described by means of an approach first proposed by Hill (1950). In this approach, the structure tensors are employed as metric tensors in the construction of the invariants appearing in the yield function and plastic potential.

**Example.** Model by Nova & Sacchi (1982) for transverse isotropic materials, constructed starting from MCC model:

\[ f(\sigma, k) := Z \cdot CZ - 3k^2 = 0 \quad Z := \frac{1}{\mu} s + (p - k) 1 \]

where \( C \) is the structure metric tensor (in principal anisotropy axis):

\[
C := \begin{bmatrix}
\alpha & 0 & \gamma & 0 & 0 & 0 \\
0 & \alpha & \gamma & 0 & 0 & 0 \\
\gamma & \gamma & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 2\beta & 0 \\
0 & 0 & 0 & 0 & 0 & 2\beta
\end{bmatrix}
\]

\((\alpha, \beta, \gamma = \text{const.})\)
Induced anisotropy is typically described by incorporating a symmetric second order tensor (structure tensor) in the set of internal variables. This tensor controls the position and the orientation of the yield locus.

Most plasticity models for soils with anisotropic hardening can be grouped in two broad classes:

a) **kinematic hardening models**, in which the yield surface can expand, contract and translate in stress space, along the prescribed stress path;

b) **rotational hardening models**, in which the yield surface can expand, contract and rotate in stress space, being oriented by the stress path direction.
Kinematic hardening models

Ref.: Mroz (1967); Iwan (1967).

\[ f(\sigma, \alpha, q_k) = \hat{f}(\hat{\sigma}, q_k) = 0 \]
\[ \hat{\sigma} := \sigma - \alpha \]
Several KH models have been proposed for geomaterials since the late 70. See, e.g., Prevost (1977, 1986); Mroz, Norris & Zienkiewicz (1978, 1979, 1981); Mroz & Pietruszczak (1983a,b); Hashiguchi (1980, 1981, 1985, 1988); Wood and coworkers (1989, 2000); Stallebrass & Taylor (1997)...

The main reason for their development stems from the need to improve the description of cyclic behavior of soils. These model are quite effective in modelling hysteresis and such phenomena as the so-called Bauschinger effect.

KH models have also been advocated to reproduce the observed non-linearity at small strain levels and the effects of recent stress history on the stiffness decay with increasing strains (see Stallebrass 1990).
Rotational hardening models

Ref.: Sekiguchi & Otha (1977); Ghaboussi & Momen (1982).

\[ f(\sigma, \alpha^a, p_0) = \hat{f}(I^a, J^a, Z^a, p_0) \]

\[ I^a := \sigma \cdot \alpha^a \]
\[ (J^a)^2 := (1/2) s^a \cdot s^a \]
\[ Z^a := \sqrt{6} \, \text{tr}(d^a)^3 / [\text{tr}(d^a)^2]^{3/2} \]

\[ s^a := \sigma - \frac{I^a}{3} \alpha^a \]
\[ d^a := \text{dev}(s^a) \]
\[ \alpha^a \cdot \alpha^a = 3 \]
Rotational hardening models

RH models can be subdivided in two main classes:

i) RH models with yield surfaces open towards the positive side of the hydrostatic axis (conical YS). Best suited for sands, where plastic volumetric strains are typically much smaller than deviatoric ones;

ii) RH models with closed YS, suitable for both coarse– and fine–grained soils.

Among the RH models for geomaterials, we mention, e.g., Ghaboussi & Momen (1982); Gajo & Wood (1999a,b); Anandarajah & Dafalías (1985, 1986); Banerjee and coworkers (1985,1986); Whittle and coworkers (1994, 1999); di Prisco (1993a,b); Manzari & Dafalías (2004); Karstunen and coworkers.