Describing Self-Similarity using Category Theory

presenting Ideas of Tom Leinster

Hubert Bottesch
supervised by Tobias Fritz
University of Innsbruck

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Figure 1: Roman Cauliflower

What is self-similarity? There isn't a clear definition for this rather vague term, but in many cases something (often a space) that is self-similar may look like it's made from differnet copies of itself, maybe glued together. As we will see, this can very nicely be described by some "categorical equations". We will be looking at a paper of Tom Leinster [1] in which he formulates his ideas in a precise manner and give a few examples. Also for the interested there is also an overview of this published by Leinster himself [2].

Notation

First let us fix some notation and nomenclature:

- The **sum (coproduct)** of a family $(X_i)_{i\in I}$ of objects of a category is written $\sum_i X_i$. If $X_i = X$ for all i then the sum is written $I \times X$. The sum of a finite family $X_1, \ldots, X_n$ of objects is written as $X_1 + \cdots + X_n$, or as 0 if $n = 0$.

- A **discrete** category is a category where the only morphisms are the identities.
- A **finite** category is a category with only finitely many morphisms (and therefore also finitely many objects).
- A category is **connected** if it is nonempty and can't be written as coproduct (disjoint union) of two nonempty categories.
- A module $M : \mathbf{B} \nrightarrow \mathbf{A}$ is a functor

  $$M : \mathbf{B}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$$

  For objects $a \in A$ and $b \in B$ we write

  $$m : b \nrightarrow a$$

  and mean $m \in M(b, a)$.

1 Discrete equational systems

Before we look at the general case, equational systems, we consider a special case: discrete equational systems. They can be thought of as a system of linear equations, for example

$$\begin{aligned} x_1 &= 2x_1 + 5x_2 + x_3 \\ x_2 &= x_2 \\ x_3 &= 4x_1 + x_2. \end{aligned}$$

Though one doesn't read this as classical equations but rather a "categorification" of such: The **variables** x_i **are spaces**, the **addition is the coproduct** and the **equalities are only isomorphisms**.

For example consider the Cantor set, that one can construct as the limit of the process which removes the inner third of the unit interval and then removes the inner thirds of the remaining pieces and so on:

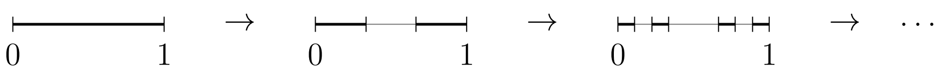


Figure 2: Construction of the Cantor set

It obviously is made of two copies of precisely itself, therefore "solves" the discrete equational system

$$X \cong X + X \cong 2 \times X$$

One can formulate this "solution-property" in a nicer way. For this we need **coalgebras**:

Definition 1. Let $\mathbf{C}$ be a category and G an endofunctor of $\mathbf{C}$ (that is, a functor $C \to C$). A G-**coalgebra** is a pair (X, ξ) where X is an object in $\mathbf{C}$ and $\xi : X \to GX$. A map $(X, \xi) \to (X', \xi')$ of G-coalgebras is a map $f : X \to X'$ in $\mathbf{C}$ such that the evident square commutes:

$$\begin{array}{ccc} X & \xrightarrow{\xi} & GX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gf} \\ X' & \xrightarrow{\xi'} & GX' \end{array}$$

There is a endofunctor $G : \mathbf{Top} \to \mathbf{Top}$ defined by $G(X) = X + X$. Now the universal property of the Cantor set C and the canonical isomorphism $i : C \to C+C$ is, that the pair is the **terminal G-coalgebra**, that is, the **terminal object in the category of G-coalgebras**. This means if we have some G-coalgebra (X, ξ), there is an unique G-coalgebra map $(X, \xi) \to (C, i)$.

It is not a coincidence that i really is a isomorphism. As **Lambek's lemma** will tell, this is alway the case for terminal coalgebras.

1.1 Lambek's Lemma

Lemma 1 (Lambek's Lemma)**.** *Let* $\mathbf{C}$ *be a category and* G *an endofunctor of* $\mathbf{C}$*. If* (I, ι) *is terminal in the category of* G*-coalgebras then* $\iota : I \to GI$ *is an isomorphism.*

Proof. Let (I, ι) be a terminal object in the category of G-coalgebras. We want to show that ι is an isomorphism by finding an inverse.
Now $(GI, G\iota)$ is also a G-coalgebra and by the terminality of (I, ι) there is a unique coalgebra map $f : GI \to I$, which means it is a map f such that this square commutes:

$$\begin{array}{ccc} GI & \xrightarrow{G\iota} & GGI \\ \downarrow{\scriptstyle f} & & {\scriptstyle Gf}\downarrow \\ I & \xrightarrow{\iota} & GI \end{array}$$

This will be the inverse for ι.
Clearly ι is always a coalgebra map between the coalgebras (I, ι) and $(GI, G\iota)$, since this square

$$\begin{array}{ccc} I & \xrightarrow{\iota} & GI \\ \downarrow{\scriptstyle \iota} & & {\scriptstyle G\iota}\downarrow \\ GI & \xrightarrow{G\iota} & GGI \end{array}$$

always commutes.
So why are they inverse to each other? First $f \circ \iota$ is a coalgebra map between (I, ι) and itself, what can be seen by attaching the lower diagram to the top of the upper one above. Since (I, ι) is terminal, there is a unique coalgebra map from (I, ι) to itself, namely the identity. Therefore

$$f \circ \iota = \mathrm{id}_I$$

and then we use that f is a coalgebra map:

$$\iota \circ f = Gf \circ G\iota = G(f \circ \iota) = G\,\mathrm{id}_I = \mathrm{id}_{GI}$$

$\square$

1.2 Defining discrete equational systems

Definition 2. A discrete equational system is a pair (A, M) where A is a set and M is a family $(M_{b,a})_{b,a \in A}$ of sets such that for each $a \in A$, the disjoint union $\sum_{b \in A} M_{b,a}$ is finite.

The idea behind it is, that for a discrete equational system (A, M) and some category (with finite sums) $\mathcal{E}$ there is endofunctor $M \otimes -$ of $\mathcal{E}^A$ defined by

$$(M \otimes X)_a = \sum_{b \in B} M_{b,a} \times X_b.$$

The solution of such a system will be the terminal coalgebra. This notion is really justified by Lambek's lemma, since the terminal coalgebra gives an isomorphism between its space X and $M \otimes X$.

Definition 3. Let (A, M) be a discrete equational system and $\mathcal{E}$ a category with finite sums. An M**-coalgebra** (in $\mathcal{E}$) is a coalgebra for the endofunctor $M \otimes -$ of $\mathcal{E}^A$. A **universal solution** of (A, M) (in $\mathcal{E}$) is a terminal M-coalgebra.

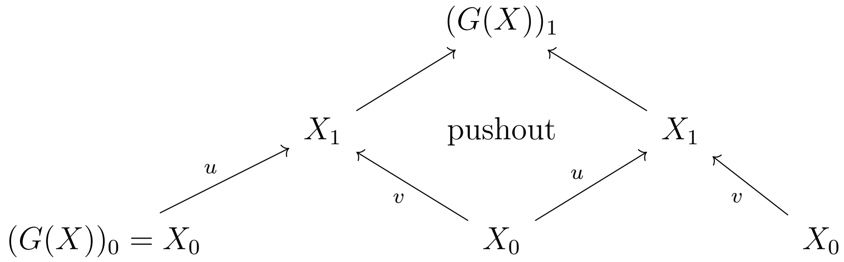
2 Equational Systems

In a discrete equational system each space is written as *disjoint* union of some of the spaces themselves. Now we consider the general case, equational systems, where it is just as before, but we don't constrict ourselves to disjoint unions.

Example 2.1. One simple example for a self similar space which can be characterized by a equational system is simply the unit interval $[0, 1] \subseteq \mathbb{R}$. It has the property that two times itself glued end to end are isomorphic to itself. And in some sense, it is the universal object with that property.

But how does the actual equation look here?
We consider the category of objects (X_0, X_1, u, v) where X_0, X_1 are sets and u, v are injections from X_0 to X_1 with disjoint images. Morphisms are mappings $X_0 \to X_0'$ and $X_1 \to X_1'$ that make the obvious two squares (coming from the two injections) commute.

Again we define an endofunctor on this category, by pushout:



This gives then $(G(X))_1 = [0, 2]$. One can show that the object $([0, 1], \{*\}, 0, 1)$ together with the obvious mapping $(x \mapsto 2x)$ to $([0, 2], \{*\}, 0, 2)$ is the terminal coalgebra. This theorem was proven by **Peter Freyd** in a message on the categories mailing list [3]. One can also use **Top** instead of **Set** and assume u and v are continuous and closed and prove the terminality.

The objects (X_0, X_1, u, v) correspond to objects from $[\mathbb{A}, \mathbf{Set}]$ for the small category

$$\mathbb{A} = \left(0 \rightrightarrows 1 \right).$$

Therefore we search for solutions written as such functors rather than tuples of spaces and mappings.

Because of the conditions on the maps u and v we don't look for solutions in the *whole* functor category $[\mathbb{A}, \mathbf{Set}]$ (or $[\mathbb{A}, \mathbf{Top}]$), but a full subcategory. One generalizes these conditions as follows:

Definition 4. Let $\mathbb{A}$ be a small category. A functor $X : \mathbb{A} \to \mathbf{Set}$ is **nondegenerate** (or componentwise flat) if the functor

$$- \otimes X : [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$$

preserves finite connected limits.

Let U be the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$. A functor $X : \mathbb{A} \to \mathbf{Set}$ is called **nondegenerate** if $U \circ X$ is nondegenerate and, the map Xf is closed for every morphism f in $\mathbb{A}$.

We write $\langle \mathbb{A}, \mathbf{Set} \rangle$ (or $\langle \mathbb{A}, \mathbf{Top} \rangle$) for the full subcategory of $[\mathbb{A}, \mathbf{Set}]$ (or $[\mathbb{A}, \mathbf{Top}]$) formed by the nondegenerate functors.

Our endofunctors $M \otimes -$ act on the subcategory of nondegenerate functors and therefore we need exactly this property of M:

Definition 5. Let $\mathbb{A}$ and $\mathbb{B}$ be small categories. A module $M : \mathbb{B} \nrightarrow \mathbb{A}$ is called **nondegenerate** if $M(b, -) : \mathbb{A} \to \mathbf{Set}$ is nondegenerate for each $b \in B$.

We want to confine ourselves to finite gluings on the right side of the system, therefore it makes sense to reqire the module to have the following property:

Definition 6. A presheaf $Y : \mathbb{B}^{\mathrm{op}} \to \mathbf{Set}$ is **finite** if its category of elements is finite. A module $M : \mathbb{B} \nrightarrow \mathbb{A}$ is **finite** if for each $a \in \mathbb{A}$, the presheaf $M(-, a)$ is finite.

When $\mathbb{A}$ and the sets $M(b, a)$ are finite this certainly holds.

All in all this is now everything we need to define a equational system in this sense and what the solutions for such systems may be.

Definition 7. An **equational system** is a small category $\mathbb{A}$ together with a finite nondegenerate module $M : \mathbb{A} \nrightarrow \mathbb{A}$.

Later on one shows how the module $M : \mathbb{A} \nrightarrow \mathbb{A}$ induces a endofunctor $M \otimes -$ of $[\mathbb{A}, \mathbf{Set}]$ which again restricts to a endofunctor of $\langle \mathbb{A}, \mathbf{Set} \rangle$.

Definition 8. Let $(\mathbb{A}, M)$ be an equational system. An M**-coalgebra** in **Set** (respectively, **Top**) is a coalgebra for the endofunctor $M \otimes -$ of $\langle A, \mathbf{Set} \rangle$ (respectively, $\langle \mathbb{A}, \mathbf{Top} \rangle$). A **universal solution** of $(\mathbb{A}, M)$, in **Set** or **Top**, is a terminal object in the category of M-coalgebras.

3 Examples

We will look at a few examples of self similar objects

Example 3.1. In our characterization of the unit interval it was formed out of two copies of itself, glued at one of their endpoints. This means we take the module M as

$M(b,a)$	0	1	$\to b$
0	1	$\emptyset$	
1	3	2	
$\to a$			

.

Example 3.2. One of the prettiest examples that lead to nice "general" M are Julia sets, which are often fractal in nature. They can be defined for every holomorphic map on a Riemann surface as the subset where the map behaves unstably under iteration. In this case, as ist is the best studied case, Tom Leinster looks at a rational fuction $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, namely $f(z) = \left(\frac{2z}{(1+z^2)}\right)^2$.

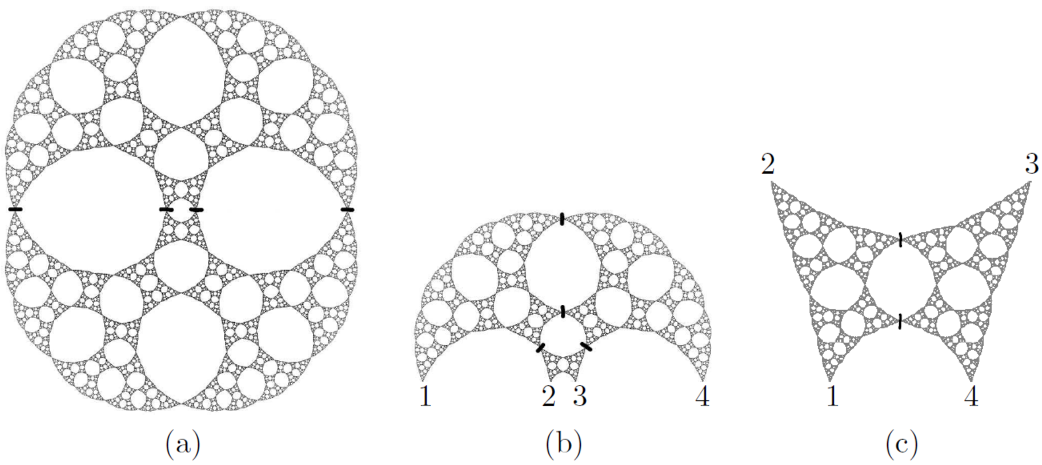
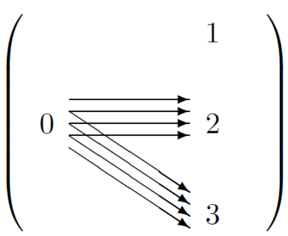


Figure 3: (a) Julia set of f, (b) upper half, (c) middle subset rescaled

For the single points space used for the gluings we write I_0 and include the equation $I_0 = I_0$. Then we see I_2 and I_3 have four distinguished points used for gluings. This then leads to choosing

$\mathbb{A} =$ 

.

$M(b,a)$	0	1	2	3	$\to b$
0	1	$\emptyset$	$\emptyset$	$\emptyset$	
1	4	$\emptyset$	2	$\emptyset$	
2	8	$\emptyset$	2	1	
3	6	$\emptyset$	$\emptyset$	2	
$\to a$					

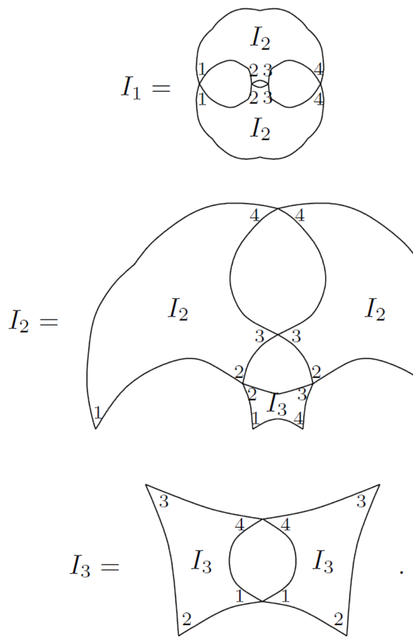


Figure 4: How the subsets are gluings of themselves

Example 3.3. Another example would be the **Sierpinski triangle**. It is made by splitting an equilateral trangle into four such and removing the inner one. This process is then repeated for the reamaining ones over and over.

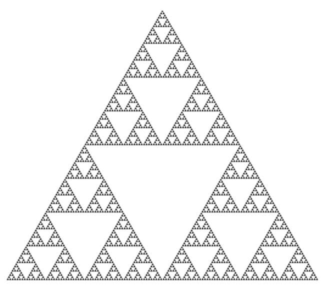


Figure 5: Sierpinski Triangle

We see that it is made of three copies of itself glued at the vertices. The three copies are all glued on exactly two points to the others. Calling the one point space for the gluings I_0 again and the triangle itself I_1, we get following equational system:

$$\mathbb{A} = \left(0 \overset{p_1}{\underset{p_2}{\rightrightarrows}} 1 \right)$$

$M(b,a)$	0	1	$\to a$
0	1	$\emptyset$	
1	3	3	
$\to b$			

Example 3.4. An interesting, or perhaps especially not interesting, example would be Koch's snowflake. It is constructed by taking a line and replacing the middle third by two lines of the same length. Then reapeating that process over and over.

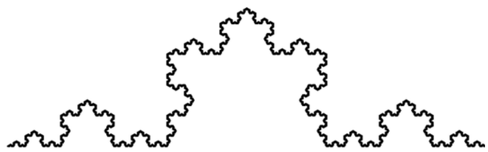


Figure 6: Koch's Snowflake

It is therefore made of four of it's own copies, glued at their endpoints. But actually the topology underlying the snowflake is simple. Any simple line, for example the interval $[0, 1]$ again, would also fulfill that condition. Meaning, while the Koch snowflake is *one* universal solution of the resulting system, any simple line also is and therefore there is nothing special about the topology of the snowflake.

4 Outlook

Now that the notion of a equational system has been defined, one can ask whether a given one can be solved or what the solution is in general. In the paper of Leinster he manages to prove three main things:

- For a given system there is a **explicit condition** that **determines whether a solution exists**.
- If this condition is fulfilled, one can **construct the solution**.
- Given a coalgebra, one can **check if it is the universal solution**.

References

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