Enriched categories

Mirte van der Eyden

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Abstract

In these notes we define enriched categories, motivated by several examples. Along the way, monoidal categories and monoidal closed categories are defined as well. Assuming some basic category theoretic knowledge, these notes are aimed to give a brief but consistent introduction to enriched category theory.

1 Introduction

Category theory consists of a very abstract framework in which similarities between mathematical objects and their relations in many possible fields are made precise. By means of commuting diagrams, functors, natural transformations and universal objects we can capture a lot of the underlying structure that sets, groups, vector spaces, measurable spaces, topological spaces and much more structures have in common. Enriched categories make it possible to see even more of the underlying structures in these various mathematical objects, which sometimes even leads to new insights. In homotopy theory for example, the usual unenriched categories do not have enough structure to describe all homotopical phenomena [1].

Let us start by illustrating the idea of enriched categories by a first example.

Example 1 (Linear maps and vector spaces)**.** Think of the category $\mathbf{Vect}_K$ of vector spaces over a field K. The morphisms in this category are linear maps. Given two vector spaces V and W, since this is an example of a locally small category, all morphisms between V and W form a set, denoted $\mathrm{Hom}(V, W)$. By just considering this as a set however, we throw away some information that we have about the morphisms between V and W. They do not just form a set, but actually have a *richer* structure: they form a vector space. This is the idea of an enriched category, $\mathbf{Vect}_K$ is enriched over, in this case, itself. △

There are of course many more examples of this type, and there is a lot of very rich structure to be explored in this way. In these notes we introduce the definition of an enriched category (Section 2), give a selection of insightful examples (Section 3) and highlight examples of self-enrichment of categories following from the definition of a closed monoidal category (Section 4).

2 Towards the definition

The goal of this section is to give the definition of an enriched category. In order to arrive there, we will introduce *monoidal categories* along the way.

First some notes on notation and terminology. As in the example above, we consider *enriched* categories $\mathbf{C}$ in which the 'hom-sets' are not just sets but potentially have more structure: i.e. they are objects in some *enriching* category, often denoted $\mathcal{V}$. From now an we denote the collection of morphisms between objects X and Y in category $\mathbf{C}$ as $\mathbf{C}(X, Y)$ instead of $Hom(X, Y)$, since it is (in general) no longer just a set.

The enriching category cannot just be any category, it needs a special structure. Given three objects U, V and W that are objects in $\mathbf{C}$. Given a morphism $f : U \to V \in \mathbf{C}(U, V)$ and a morphism $f : V \to W \in \mathbf{C}(V, W)$, we know by definition of a category that compositions of all morphisms exist, so the morphism $g \circ f : U \to W \in \mathbf{C}(U, W)$ should also exist in the enriching category. Viewing all the objects of the form $C(-,-)$ now as objects in the enriching category $\mathcal{V}$, there is some way of combining two objects in $\mathcal{V}$ and map in into a third one, i.e. there has to be a relation of the form

$$C(U,V) \times C(V,W) \to C(U,W).$$

A first requirement for this, is that there exist a notion of a product of two objects in the enriching category, i.e. it has to be a *monoidal* category.

Definition 2 (Monoidal category)**.** *A monoidal category* $(\mathcal{V}, \otimes, \star)$ *consists of:*

(i) *a category* $\mathcal{V}$*,*

(ii) *as monoidal product a bifunctor* $- \otimes - : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$*,*

(iii) *a monoidal unit* $\star$

(iv) *three natural isomorphisms expressing associativity and unitality of the monoidal product:*

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z, \quad \lambda : \star \otimes X \cong X \quad \rho : X \otimes \star \cong X, \tag{1}$$

which satisfy the following coherence relations:

1. *For all* W, X, Y, Z *the pentagon diagram commutes:*

$$\begin{array}{ccccc} W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{\alpha_{W,X,Y\otimes Z}} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha_{W\otimes X,Y,Z}} & ((W \otimes X) \otimes Y) \otimes Z) \\ \downarrow{\scriptstyle id_W \otimes \alpha_{X,Y,Z}} & & & & \uparrow{\scriptstyle \alpha_{W,X,Y} \otimes id_Z} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\alpha_{W,X\otimes Y,Z}} & & (W \otimes (X \otimes Y)) \otimes Z \end{array}$$

2. *For all* X, Y *the triangle diagram commutes:*

$$\begin{array}{ccc} X \otimes (\star \otimes Y) & \xrightarrow{\alpha X,\star,Y} & (X \otimes \star) \otimes Y \\ & {\scriptstyle id_X \otimes \lambda_Y} \searrow \quad \swarrow {\scriptstyle \rho_Z \otimes id_Y} & \\ & X \otimes Y & \end{array}$$

Because of these conditions, we will never have to take care of bracketing in a monoidal category. Note that a *symmetric* monoidal category has the extra condition that the monoidal product is commutative, i.e. there is an extra natural isomorphism

$$X \otimes Y \cong Y \otimes X.$$

All of the examples we will see will be symmetric monoidal categories, but for our purposes this does not matter.

Example 3 (Vector spaces as monoidal categories)**.** Continuing along the lines of Example 1, we can now show that $\mathbf{Vect}_K$ is a monoidal category. First this means that in $\mathbf{Vect}_K$ there is a way of combining two vector spaces to form a new one. This monoidal product exists in the form of the tensor product. The monoidal unit is given by the field K, since the objects are invariant under tensor products with elements from K. It is left to the reader to check that $(\mathbf{Vect}_K, \otimes, K)$ indeed satisfies the pentagon and triangle diagrams and is thus a monoidal category.

Note that even though it was the case in our current example and we chose therefore the same notation, the monoidal product is not necessarily the tensor product.

Example 4 (Examples of monoidal categories)**.** Here follows a list of some other monoidal categories and their monoidal product and monoidal unit.

- $(\mathbf{Set}, \times, \{1\})$ where $\times$ is the catesian pruduct and $\{1\}$ the terminal object of **Set**, the one-element set.
- $(\mathbf{Top}, \times, \{1\})$ where $\times$ is the catesian pruduct and $\{1\}$ the terminal object of **Top**, the one-point space.

- $(\mathbf{Mod}_R, \otimes_R, R)$ where $\mathbf{Mod}_R$ is the category of modules over a commutative ring R (see Example 8 for more details). In particular, $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ and $(\mathbf{Vect}_K, \otimes_K, K))$ (Example 3) are instances of this example.

△

We denote an enriched category $\mathbf{C}$ that is enriched over $\mathcal{V}$ a $\mathcal{V}$-category.

Definition 5 (Enriched category)**.** *Let $\mathcal{V}$ be a monoidal category. A $\mathcal{V}$-category $\boldsymbol{C}$ consists of:*

(i) a collection of objects, denoted C,

(ii) for each ordered pair of objects $X, Y \in C$, an object $\boldsymbol{C}(X, Y) \in \mathcal{V}$,

(iii) for each ordered triple $X, Y, Z \in C$, a morphism $\circ : \boldsymbol{C}(Y, Z) \otimes \boldsymbol{C}(X, Y) \to \boldsymbol{C}(X, Z) \in \mathcal{V}$,

(iv) for each object $X \in C$ a morphism $id_X : \star \to \boldsymbol{C}(X, X)$,

such that

1. *for all $W, X, Y, Z \in C$, the composition in $\boldsymbol{C}$ is associative, meaning the following diagram commutes*[1]*:*

$$\begin{array}{ccc} C(Y,Z) \otimes C(X,Y) \otimes C(W,X) & \xrightarrow{1 \otimes \circ} & C(Y,Z) \otimes C(W,Y) \\ \downarrow{\scriptstyle \circ \otimes 1} & & \downarrow{\scriptstyle \circ} \\ C(X,Z) \otimes C(W,X) & \xrightarrow{\quad \circ \quad} & C(W,Z) \end{array}$$

2. *for all $X, Y \in C$ the following diagram commutes:*

$$\begin{array}{ccccc} C(X,Y) \otimes \star & & & & \star \otimes C(X,Y) \\ \downarrow{\scriptstyle 1 \otimes id_X} & \searrow{\scriptstyle \cong} & & \swarrow{\scriptstyle \cong} & \downarrow{\scriptstyle id_Y \otimes 1} \\ C(X,Y) \otimes C(X,X) & \xrightarrow[\circ]{} & C(X,Y) & \xleftarrow[\circ]{} & C(Y,Y) \otimes C(X,Y) \end{array}$$

The isomorphisms in the above diagram are exactly the natural isomorphisms specified in Definition 2 by the fact that $\mathcal{V}$ is monoidal.

3 Examples of enriched categories

In this section we describe three examples of enriched categories: the category of vector spaces (Example 6), enrichment over preorders (Example 7) and modules as enriched categories (Example 8).

Example 6 (Vector spaces as an enriched category)**.** Given the symmetric monoidal category $\mathcal{V} = (\mathbf{Vect}_K, \otimes, K)$ (Example 3), we can define $\mathbf{Vect}_K$ to be a $\mathcal{V}$-category. First, let us verify in more detail the claim we made already in Example 1: given two vector space V and W, then $\mathbf{Vect}_K(V, W)$ has a vector space structure as well. Given two linear maps $f, g : V \to W \in \mathbf{Vect}_K(V, W)$, we can define addition pointwise as follows:

$$(f + g)(x) = f(x) + g(x),$$

such that $(f + g)$ is again a linear map, i.e. it behaves well under under scalar multiplication with any element $k \in K$

$$k(f + g)(x) = kf(x) + kg(x) = f(kx) + g(kx) = (f + g)(kx),$$

and under addition

$$(f + g)(x + y) = f(x + y) + g(x + y) = f(x) + f(y) + g(x) + g(y) = (f + g)(x) + (f + g)(y).$$

Similarly, for every $c \in K$, it is easy to check that $cf \in \mathbf{Vect}_K(V, W)$. Now $\mathbf{Vect}_K$ is a $\mathcal{V}$-category if we have composition (iii) and and identity morphisms (iv) (Definition 5).

[1] note that we have excluded the associativity natural isomorphism (1) from the diagram

The reason we need as a monoidal product the tensor product, instead of the usual cartesian product is the following. We want the composition map

$$\circ : \mathbf{Vect}_K(Y,Z) \times \mathbf{Vect}_K(X,Y) \to \mathbf{Vect}_K(X,Z)$$

to be again a morphism in $\mathbf{Vect}_K$, i.e. a linear map.

If the monoidal product were the cartesian product, it would mean that

$$c(f,g) \mapsto cf \circ cg.$$

However, we would instead want that

$$c(f,g) \mapsto cf \circ g.$$

Using the bilinearity of the tensor product, the composition map

$$\circ : \mathbf{Vect}_K(Y,Z) \otimes \mathbf{Vect}_K(X,Y) \to \mathbf{Vect}_K(X,Z) \in \mathcal{V},$$

ensures that the composition map is linear and therefore is a valid morphism in $\mathcal{V}$.

Together with (iv) a morphism $\mathrm{id}_X : K \to \mathbf{Vect}_K(X,X)$ that we define to map any $c \in K$ to the scalar multiplied with identity morphism, cI_X on X, for all X, all diagrams in Definition 5 commute. Since all elements in $C(Y,Z) \otimes C(X,Y) \otimes C(W,X)$ are linear combinations of elementary tensors like $f \otimes g \otimes h$, and all morphisms are linear maps, we only have to check commutativity of the diagrams for elementary tensors.

1. For any T, U, V, W, for any $f \in C(V,W), g \in C(U,V), h \in C(T,U)$, the following commutes

$$\begin{array}{ccc} f \otimes g \otimes h & \xmapsto{1 \otimes \circ} & f \otimes g \circ h \\ \downarrow{\scriptstyle \circ \otimes 1} & & \downarrow{\scriptstyle \circ} \\ f \circ g \otimes h & \xmapsto{\circ} & (f \circ g) \circ h = f \circ (g \circ h) \end{array}.$$

2. For any $f \in C(U,V)$, for any $c \in K$,

$$\begin{array}{ccccc} f \otimes c & & & & c \otimes f \\ \downarrow{\scriptstyle 1 \otimes \mathrm{id}_U} & \searrow{\scriptstyle \cong} & & {\scriptstyle \cong}\swarrow & \downarrow{\scriptstyle \mathrm{id}_V \otimes 1} \\ f \otimes cI_U & \xmapsto{\circ} & cf & \xleftarrow{\circ} & cI_V \otimes f \end{array}.$$

Example 7 (Enriching over preorders)**.** Let us consider a category $\mathbf{C}$ that has the special property that between any two objects X, Y in C, there is at most one arrow in each direction (so for every $X, Y \in C$, $C(X,Y)$ viewed as a set is the one element set). We will consider two examples of enriching $\mathbf{C}$ over symmetric monoidal preorders.

(i) Consider the two element set $\{0,1\}$, which is a category that is a preorder, with two elements 0 and 1 and the morphism $0 \leq 1$. $\mathcal{V} = (\{0,1\}, \wedge, 1)$ is a symmetric monoidal category, with as monoidal product the logical conjunction operator '$\wedge$' , and monoidal unit 1 ('true'). Therefore, we can enrich over this category. The specification of the composition law and identity morphism of a $\mathcal{V}$-category $\mathbf{C}$ look like

(iii) $C(X,Y) \wedge C(Y,Z) \Rightarrow C(X,Z)$ for all $X, Y, Z \in C$

(iv) $1 \Rightarrow C(X,X)$ for all $X \in C$,

Enriching $\mathbf{C}$ over $\mathcal{V}$ means to every arrow in $\mathbf{C}$ a value 0 or 1, or a truth value if you want, is assigned. It can be seen that $\mathbf{C}$ enriched over $\mathcal{V}$ is a preorder, since the two conditions of a preorder are exactly the same as the ones given above. Given two elements $X, Y \in C$, if $X \leq Y$, $C(X,Y) = 1$, otherwise, $C(X,Y) = 0$. Note that the converse is also clearly true: every preorder can be viewed as a category enriched over $\{0,1\}$ (so also $\{0,1\}$ itself, which relates to it being a closed monoidal category as will be defined in Section 4).

(ii) Consider the interval $[0, \infty]$. This choice of enriching category is also an example of a preorder, where for historical reasons there is an arrow from $A \in \mathcal{V}$ to $B \in \mathcal{V}$ whenever $A \geq B$ (as opposed to the previous example and the usual convention for preorders). It is a symmetric monoidal category

$\mathcal{V} = ([0, \infty], +, 0)$, with addition as monoidal product and 0 as the monoidal unit. Then, a $\mathcal{V}$-category $\mathbf{C}$ consists of the following composition law and identity morphism:

$$\mathcal{C}(X, Y) + \mathcal{C}(Y, Z) \geq \mathcal{C}(X, Z) \tag{2}$$
$$0 \geq \mathcal{C}(X, X) \tag{3}$$

Since $\mathcal{C}(X, X) \in [0, \infty]$, (3) ensures that $\mathcal{C}(X, X) = 0$. The first condition will probably be very familiar: it is the triangle inequality. Actually, this makes the $\mathcal{V}$-enriched category $\mathbf{C}$ a *generalized metric space*. It does not fulfil all properties of a metric space, since the following are missing:

- if $C(X, Y) = 0$ then $X = Y$,
- $C(X, Y) < \infty$,
- $C(X, Y) = C(Y, X)$.

However, a generalized metric space still has a quite intuitive understanding. For example, the values $C(X, Y) \in [0.\infty]$ can be, instead of as distances, thought of as the work it takes to go from X to Y, where X is potentially situated high on a mountain and Y in the valley, which makes the work asymmetric. These generalized metric spaces and their connection to $[0, \infty]$-enriched categories was first explored by Lawvere in 1973 [2].

Example 8 (Left modules)**.** Let us consider the category of modules over a fixed, commutative ring R, denoted $\mathbf{Mod}_R$. A R-module is a group $(M, +)$ together with an operation $\cdot : M \times R \to M$ that is associative, transitive, unital and linear in R. For each pair of objects $X, Y \in \mathbf{Mod}_R$, the set $\mathbf{Mod}_R(X, Y)$ is has the structure of an Abelian group, in other words $\mathbf{Mod}_R(X, Y)$ is an object in the category $\mathbf{Ab}$. Let us check this statement, and find the monoidal structure of $\mathbf{Ab}$ as part of an enriching category $\mathcal{V} = (\mathbf{Ab}, ?, ?)$.

First, we check that for every $X, Y \in \mathbf{C}$, $\mathbf{Mod}_R(X, Y)$ is an Abelian group. The null-homomorphism from X to Y, sending all elements in X to $0 \in Y$ is the identity element. Furthermore, the sum of two homomorphisms from X to Y can be defined by using point-wise addition in the module Y, in an associative and commuting way. Finally, every element has an inverse element, since every element in the R-module has an inverse element (since M is a group). For every homomorphism f between X and Y, there is also a homomorphism $-f$ that maps all elements to the inverse of the image under f in Y. This way, $f + -f$ is exactly the null-homomorphism so $-f$ is the inverse of f.

Now, in order to see that we have the desired monoidal structure, we have to specify the monoidal product and monoidal unit. The monoidal product should be such that composition in $\mathbf{Mod}_R$ distributes nicely over the addition of morphisms, meaning that given three objects $X, Y, Z \in \mathbf{Mod}_R$, and morphisms $f_1, f_2 \in \mathbf{Mod}_R(X, Y)$ and $g_1, g_2 \in \mathbf{Mod}_R(Y, Z)$, it should hold that

$$(g_1 + g_2) \circ (f_1 + f_2) \cong g_1 \circ f_1 + g_1 \circ f_2 + g_2 \circ f_1 + g_2 \circ f_2.$$

In particular, this means that it should be possible to add any number of morhpisms, and composition behaves well. Since $f + f + f$ can also be seen as just $3f$, and the inverse of f is $-f$, this condition is equivalent to saying that the composition should be $\mathbb{Z}$-bilinear. The reason that $\mathbb{Z}$ pops up is not all that surprising. Every Abelian group is a $\mathbb{Z}$-module and therefore the elements of every object in $\mathbf{Ab}$ seen as homomorphisms should be $\mathbb{Z}$-linear and composition should be $\mathbb{Z}$-bilinear. Moreover, it is interesting to note that the group $\mathbb{Z}$ plays another important role in the category $\mathbf{Ab}$ as it is the representing object of the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$. The set of homomorphisms $\mathbb{Z} \to S$ is naturally isomorphic to the elements of the set S.

We thus have a monoidal product

$$- \otimes_{\mathbb{Z}} - : \mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$$

that is the $\mathbb{Z}$-bilinear tensor product, allowing a composition map

$$\mathbf{Mod}_R(Y, Z) \otimes_{\mathbb{Z}} \mathbf{Mod}_R(X, Y) \to \mathbf{Mod}_R(X, Z)$$

that can be defined analogous to Example 6. The monoidal unit is $\mathbb{Z}$, since tensoring on the left or right with $\mathbb{Z}$ is naturally isomorphic to the identity functor. From all this we conclude that $\mathbf{Mod}_R$ is enriched over $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

Many more examples can be found for example in [1].

4 Closed categories and self-enrichment

As a final section of these notes it is worth to mention a specific type of examples of monoidal categories that leed to self-enrichment. These are *closed* monoidal categories.

The intuitive idea of a closed monoidal category is that all the hom-objects are again objects in the same category. For example, for **Set**, all the hom-sets are again sets and therefore present as objects in the same category **Set**. We then call these hom-objects *internal homs*. If all internal homs exist, the category is closed. More formally this is defined as follows.

Definition 9 (Closed monoidal categories)**.** *A closed monoidal category is a monoidal category* $(\mathcal{V}, \otimes, \star)$ *such that for every object* $Y \in \mathcal{V}$*, the functor* $F_Y : \mathcal{V} \to \mathcal{V}$ *that is defined by tensoring with* Y*,*

$$F_Y : X \mapsto X \otimes Y,$$

has a right adjoint, denoted

$$G_Y : Z \mapsto \underline{\mathcal{V}}(Y, Z).$$

This means that there is a natural isomorphism

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, \underline{\mathcal{V}}(Y, Z)),$$

that is natural in both X *and* Z*.*

Note that the notation of the right adjoing $\underline{\mathcal{V}}(Y, Z)$ is not chosen by accident as the object $\underline{\mathcal{V}}(X, Y)$ is called the internal hom of X and Y.

When $\mathcal{V}$ is a closed monoidal category and every functor F_Y has an adjoint, one can construct a unique bifunctor

$$\underline{\mathcal{V}}(-, -) : \mathcal{V}^{\text{op}} \times \mathcal{V} \to \mathcal{V},$$

such that for all $X, Y, Z \in \mathcal{V}$,

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, \underline{\mathcal{V}}(Y, Z)),$$

natural in all three variables. This bifunctor is called the *internal Hom functor*. $\mathcal{V}$ is in this case enriched over itself, where the structure of a $\mathcal{V}$-category is given by the following data:

- the objects in $\mathcal{V}$,
- for every $X, Y \in \mathcal{V}$ the internal hom of X and Y: $\underline{\mathcal{V}}(X, Y)$,
- for every $X, Y, Z \in V$ the composition map

  $$\underline{\mathcal{V}}(X, Y) \otimes \underline{\mathcal{V}}(Y, Z) \to \underline{\mathcal{V}}(X, Z),$$

  which is under the given adjunction $(- \otimes X \dashv \underline{\mathcal{V}}(X, -))$ adjunct to

  $$\underline{\mathcal{V}}(Y, Z) \otimes \underline{\mathcal{V}}(X, Y) \otimes X \xrightarrow{1 \otimes \epsilon} \underline{\mathcal{V}}(Y, Z) \otimes Y \xrightarrow{\epsilon} Z,$$

  where for all X the *evaluation* maps $\epsilon : \underline{\mathcal{V}}(X, Y) \otimes X \mapsto Y$ are the counits of the adjunction.
- for each $X \in \mathcal{V}$ the morphism $\star \to \underline{\mathcal{V}}(X, X)$ which corresponds by adjunction to the natural isomorphism $\star \otimes X \cong X$.

A closed monoidal category where the monoidal product is the catesian product, is called a *catesian closed* category.

Example 10. In **Set**, there is a bijection from every fuction of two variables $f : (a, b) \mapsto c$ to a function $\bar{f} : a \mapsto (b \mapsto c)$ that maps a to another function, that maps b to c. This shows that indeed there is an isomorphism

$$Hom(A \times B, C) \cong Hom(A, C^B).$$

All hom-objects C^B, i.e. the homsets $Hom(B, C)$, thus exist in **Set**, and **Set** is a closed mononidal and self-enriched category.

Example 11 (Modules over commutative rings)**.** The category of modules is a closed symmetric monoidal category. This means in particular that **Ab** and $\mathbf{Vect}_K$ are closed symmetric monoidal categories and therefore self-enriched, as we have already seen in Section 3.

Example 12 (2-category)**.** The category of categories **Cat** is cartesian closed. Functors form again a category where natural transformations are the morphisms. A **Cat**-enriched category **Cat** is called a 2-category. This is the starting point of *higher category theory*.

References

[1] E. Riehl *categorical homotopy theory* 2014

[2] F. W. Lawvere *Metric spaces, generalized logic, and closed categories* 1973