Calibrating CAT bonds for Mexican earthquakes

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Abstract

The study of natural catastrophe models plays an important role in the prevention and mitigation of disasters. After the occurrence of a natural disaster, the reconstruction can be financed with catastrophe bonds (CAT bonds) or reinsurance. This paper examines the calibration of a real parametric CAT bond for earthquakes that was sponsored by the Mexican government. The calibration of the CAT bond is based on the estimation of the intensity rate that describes the earthquake process from the two sides of the contract, the reinsurance and the capital markets, and from the historical data. The results demonstrate that, under specific conditions, the financial strategy of the government, a mix of reinsurance and CAT bond, is optimal in the sense that it provides coverage of USD 450 million for a lower cost than the reinsurance itself. Since other variables can affect the value of the losses caused by earthquakes, e.g. magnitude, depth, city impact, etc., we also derive the price of a hypothetical modeled-index loss (zero) coupon CAT bond for earthquakes, which is based on the compound doubly stochastic Poisson pricing methodology from Baryshnikov et al. (2001) and Burnecki and Kukla (2003). In essence, this hybrid trigger combines modeled loss and index trigger types, trying to reduce basis risk borne by the sponsor while still preserving a non-indemnity trigger mechanism. Our results indicate that the (zero) coupon CAT bond price increases as the threshold level increases, but decreases as the expiration time increases. Due to the quality of the data, the results show that the expected loss is considerably more important for the valuation of the CAT bond than the entire distribution of losses.

Keywords: CAT bonds, Reinsurance, Earthquakes, Doubly Stochastic Poisson Process, Trigger mechanism

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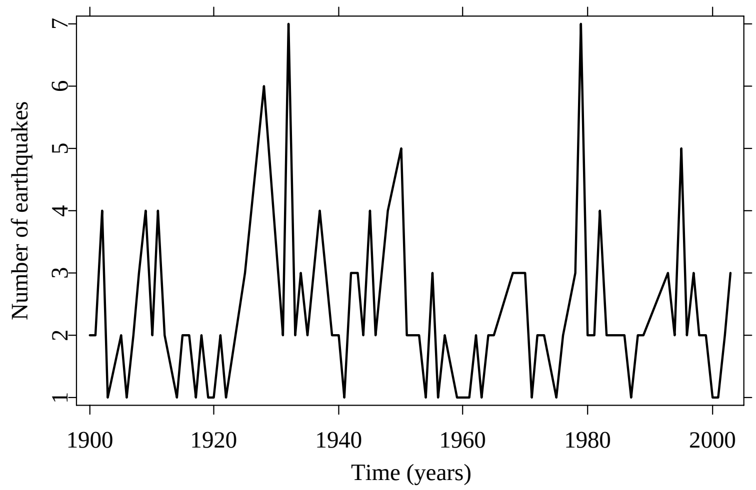
JEL classification: G19, G29, N26, N56, Q29, Q54

1 Introduction

By its geographical position, Mexico finds itself under a great variety of natural phenomena which can cause disasters, like earthquakes, eruptions, hurricanes, burning forest, floods and aridity (dryness). In case of disaster, the effects on financial and natural resources are huge and volatile. In Mexico the first risk to transfer is the seismic risk, because although it is the less recurrent, it has the biggest impact on the population and country. For example, an earthquake of magnitude 8.1 Mw Richter scale that hit Mexico in 1985, destroyed hundreds of buildings and caused thousand of deaths. The Mexican insurance industry officials estimated payouts of four billion dollars. Figure 1 depicts the number of earthquakes higher than 6.5 Mw occurred in Mexico during the years 1900-2003.

Figure 1: Number of earthquakes occurred in Mexico during 1900-2003.



Source: Own representation.

After the occurrence of a natural disaster, the reconstruction can be financed with catastrophe bonds (CAT bonds) or reinsurance. For insurers, reinsurers and other corporations CAT bonds are hedging instruments that offer multi year protection without the credit risk present in reinsurance by providing full collateral for the risk limits offered throught the transaction. For investors CAT bonds offer attractive returns and reduction of portfolio risk, since CAT bonds defaults are uncorrelated with defaults of other securities.

Baryshnikov et al. (1998, 2001) present an arbitrage free solution to the pricing of CAT bonds under conditions of continous trading and according to the statistical characteristics of the dominant underlying processes. Instead of pricing, Anderson et al. (2000) devoted to the CAT bond benefits by providing an extensive relative value analysis. Others, like Croson and Kunreuther (2000) focus on the CAT management and their combination with reinsurance. Lee and Yu (2002) analyze default risk on CAT bonds and therefore their pricing methodology is focused only on CAT bonds that are issued by insurers. Also under an arbitrage-free framework, Vaugirard (2003) valuate catastrophe bonds by Monte Carlo simulation and stochastic interest rates. Burnecki and Kukla (2003) correct and apply the results of Baryshnikov et al. (1998) to calculate non-arbitrage prices of a zero coupon and coupon CAT bond.

As the study of natural catastrophe models plays an important role in the prevention and mitigation of disasters, the main motivation of this paper is the analysis of pricing CAT bonds. In particular, we examine the calibration of a parametric CAT bond for earthquakes that was sponsored by the Mexican government and issued by the special purpose CAT-MEX Ltd in May 2006. The calibration of the CAT bond is based on the estimation of the intensity rate that describes the earthquakes process from the two sides of the contract: from the reinsurance market that consists of the sponsor company (the Mexican government) and the issuer of reinsurance coverage (in this case Swiss Re) and from the capital markets, which is formed by the issuer of the CAT bond (CAT-MEX Ltd.) and the investors. In addition to these intensity estimates, the historical intensity rate is computed to conduct a comparative analysis between the intensity rates to know whether the sponsor company is getting protection at a fair price or whether the reinsurance company is selling the bond to the investors for a reasonable price. Our results demonstrate that the reinsurance market estimates a probability of an earthquake lower than the one estimated from historical data. Under specific conditions, the financial strategy of the government, a mix of reinsurance and CAT bond is optimal in the sense that it provides coverage of USD 450 million for a lower cost than the reinsurance itself.

Since a modeled loss trigger mechanism takes other varibles into account that can affect the value of the losses, the pricing of a hypothetical CAT bond with a modeled-index loss trigger for earthquakes in Mexico is also examined in this paper. These new approach is also fundamentally driven by the desire to minimize the basis risk borne by the sponsor, while remaining non-indemnity based. Due to the missing information of losses, different loss models are proposed to describe the severity of earthquakes and the analytical distribution is fitted to the loss data that is formed with actual and estimated losses. We found that the best process governing the flow of earthquakes is described by the homogeneous Poisson process. Formerly estimating the frequency and severity of earthquakes, the modeled loss is connected with an index CAT bond, using the compound doubly stochastic Poisson pricing methodology from Baryshnikov et al. (2001). and Burnecki and Kukla (2003). This methodology and Monte Carlo simulations are applied to the studied data to find (zero) coupon CAT bond prices for earthquakes in Mexico. The threshold level and the maturity time are also computed. Furthermore, the robustness of the modeled loss with respect to the CAT bond prices is analyzed. Because of the quality of the data, the results show that there is no significant impact of the choice of the modeled loss on the CAT bond prices. However, the expected loss is considerably more important for the evaluation of a CAT bond than the entire distribution of losses.

Our paper is structured as follows. In the next section we discuss fundamentals of CAT bonds and how this financial instrument can transfer seismic risk. Section 3 is devoted to the calibration of the real parametric CAT bond for earthquakes in Mexico. Section 4 presents the valuation framework of a modeled-index CAT bond fitted to earthquake data in Mexico. Section 5 summarizes the article and suggests a possible extension. All quotations of money in this paper will be in USD and therefore we will omit the explicit notion of the currency.

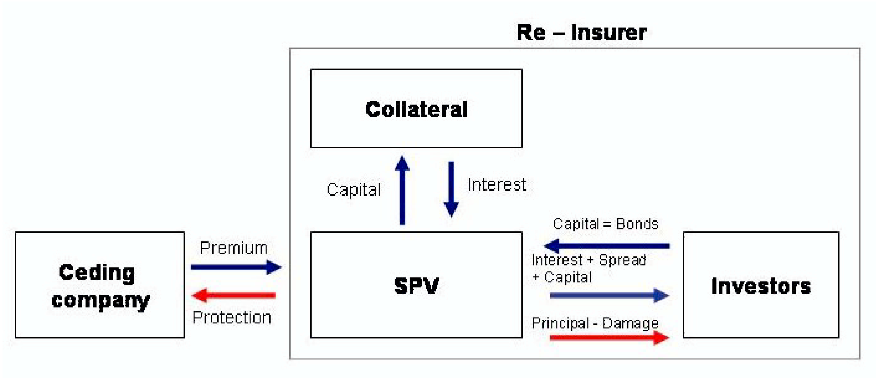
2 CAT bonds

In the mid-1990's catastrophe bonds (CAT bonds), also named as *Act of God* or *Insurance-linked bond*, were developed to ease the transfer of catastrophe based insurance risk from insurers, reinsurers and corporations (sponsors) to capital market investors. CAT bonds are bonds whose coupons and principal payments depend on the performance of a pool or index of natural catastrophe risks, or on the presence of specified trigger conditions. They protect sponsor companies from financial losses caused by large natural disasters by offering an alternative or complement to traditional reinsurance.

The transaction involves four parties: the sponsor or ceding company (government agencies, insurers, reinsurers), the special purpose vehicle SPV (or issuer), the collateral and the investors (institutional investors, insurers, reinsurers, and hedge funds). The basic structure is shown in Figure 2. The sponsor sets up a SPV as an issuer of the bond and a source of reinsurance protection. The issuer sells bonds to capital market investors and the proceeds are deposited in a collateral account, in which earnings from assets are collected and from which a floating rate is payed to the SPV. The sponsor enters into a reinsurance or derivative contract with the issuer and pays him a premium. The SPV usually gives quarterly coupon payments to the investors. The premium and the investment bond proceeds that the SPV received from the collateral are a source of interest or coupons paid to investors. If there is no trigger event during the life of the bonds, the SPV gives the principal back to the investors with the final coupon or the generous interest, otherwise the SPV pays the ceding according to the terms of the reinsurance contract and sometimes pays nothing or partially the principal and interest to the investors.

There is a variety of trigger mechanisms to determine when the losses of a natural catastrophe should be covered by the CAT bond. These include the indemnity, the industry index, the pure parametric, the parametric index, the modeled loss

Figure 2: Cash flows diagram of a CAT bond. In case of event (red arrow), no event (blue arrow).



Source: Own representation.

and the hybrid trigger. Each of these mechanisms shows a range of levels of basis risks and transparency to investors.

The *Indemnity trigger* involves the actual loss of the ceding company. The ceding company receives reimbursement for its actual losses from the covered event, above the predetermined level of losses. This trigger closely replicates the traditional reinsurance, but it is exposed to catastrophic and operational risk of the ceding company. With an *Industry index trigger*, the ceding company recovers a proportion of total industry losses in excess of a predetermined point to the extent of the remainder of the principal. The *Pure parametric index* payouts are triggered by the occurrence of a catastrophic event with certain defined physical parameters, for example wind speed and location of a hurricane or the magnitude or location of an earthquake. The *Parametric index trigger* uses different weighted boxes to reflect the ceding company's exposure to events in different areas. In a *Modeled loss trigger* mechanism, after a catastrophe occurs the physical parameters of the catastrophe are used by a modelling firm to estimate the expected losses to the ceding company's portfolio. Instead of dealing with the company's actual claims, the transaction is based on the estimates of the model. If the modeled losses are above a specified threshold, the bond is triggered. A *Hybrid trigger* uses more than one trigger type in a single transaction.

The pricing of CAT bonds reveals some similarities to the defaultable bonds, but CAT bonds offer higher returns because of the unfixed stochastic nature of the catastrophe process. The similarity between catastrophe und default in the log-normal context has been commented on Kau and Keenan (1990).

2.1 Seismology in Mexico

Mexico has a high level of seismic activity due to the interaction between the Cocos plate and the North American plate. This zone along the Middle America Trench suffers large magnitude events with a frequency higher than any other subduction zone in the world. These events can cause substantial damage in Mexico City, due to a phenomenon known as the *Mexico City effect*. The Mexico City soil, which consists mostly of reclaimed, water-saturated lakebed deposits, amplifies 5 to 20 times the long-period seismic energy, RMS (2006). Due to this effect and the high concentration of exposure in Mexico City, seismic risk is on the top of the list for catastrophic risk in Mexico.

Historically, the Cocos plate boundary produced the 1985 Michoacan earthquake of magnitude 8.1 Mw Richter scale. It destroyed hundreds of buildings and caused thousand of deaths in Mexico City and other parts of the country. It is considered the most damaging earthquake in the history of Mexico City. The Mexican insurance industry officials estimated payouts of four billion dollars. In the last decades, other earthquakes have reached the magnitude 7.8 Mw Richter scale.

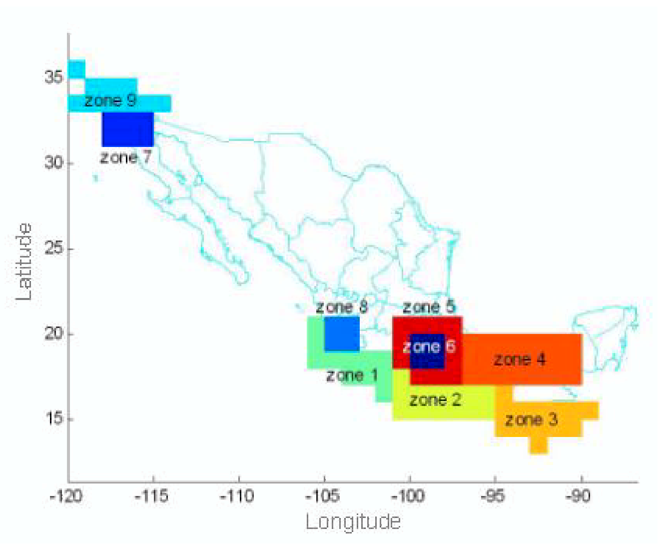
For earthquakes, the Mexican insurance market has traditionally been highly regulated, with limited protection provided to homeowners and reinsurance by the government. Today, after the occurrence of an earthquake, the reconstruction can be financed by transferring the risk to the capital markets with catastrophic (CAT) bonds that would pass the risk on to investors. The first successful CAT bond against earthquakes losses in California was issued in 1997 by Swiss Re and the first CAT bond by a non-financial firm was issued in 1999 in order to cover earthquake losses in Tokyo region for Oriental Land Company Ltd., the owner of Tokyo Disneyland. Also for the first time since 2003, a non-(re)insurance sponsor, the government of Mexico, elected to access the CAT bond market directly. Froot (2001) described other transactions on the market for catastrophic risk and Clarke et al. (2007) give a catastrophe bond market update.

3 Calibrating a Mexican Parametric CAT Bond

In 1996, the Mexican government established the Mexico's Fund for Natural Disasters (FONDEN) in order to reduce the exposure to the impact of natural catastrophes and to recover quickly as soon as they occur. However, FONDEN is funded by fiscal resources which are limited and have been insufficient to meet the government's obligations. Faced with the shortage of the FONDEN's resources and the high probability of earthquake occurrence, in May 2006 the Mexican government sponsored a parametric CAT bond against earthquake risk. The decision was taken because the instrument design protects and magnifies, with a degree of transparency, the resources of the trust. The CAT bond payment is based on some physical parameters of the underlying event (e.g. the magnitude Mw), thereby there is no justification of losses. The parametric CAT bond helps the government with emergency services and rebuilding after a big earthquake.

The CAT bond was issued by a special purpose Cayman Islands CAT-MEX Ltd. and structured by Swiss Reinsurance Company (SRC) and Deutsche Bank Securities. The 160 million CAT bond pays a tranche equal to the London Inter-Bank Offered Rate (LIBOR) plus 235 basis points. The CAT bond is part of a total coverage of 450 million provided by the reinsurer for three years against earthquake risk and with total premiums of 26 million. The payment of losses is conditional upon confirmation by a leading independent consulting firm which develops catastrophe risk assessment. This event verification agent (Applied Insurance Research Worldwide Corporation - AIR) modeled the seismic risk and detected nine seismic zones, see Figure 3. Given the federal governmental budget plan, just three out of these nine zones were insured in the transaction: zone 1, zone 2 and zone 5, with coverage of 150 million in each case, SHCP (2004). The CAT bond payment would be triggered if there is an *event*, i.e. an earthquake higher or equal than 8 Mw hitting zone 1 or zone 2, or an earthquake higher or equal than 7.5 Mw hitting zone 5.

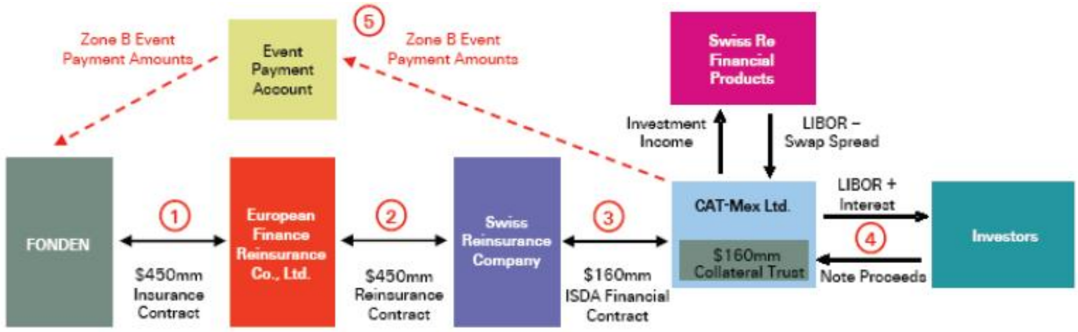
Figure 3: Map of seismic regions in Mexico. Insured zones: 1,2,5.



Source: SHCP, 2004:8.

The cash flows diagram for the mexican CAT bond are described in Figure 4. CAT-MEX Ltd. is a special purpose that issues the bond that is placed among investors and invests the proceeds in high quality assets within a collateral account. Simultaneous to the issuance of the bond, CAT-MEX Ltd. enters into a reinsurance contract with SRC. The proceeds of the bond will also serve to provide SRC coverage for earthquakes in Mexico in connection with an insurance agreement that FONDEN has entered with the European Finance Reinsurance Co. Ltd., an indirect wholly-owned subsidiary of SRC. A separate Event Payment Account was established with the Bank of New York providing FONDEN the ability to receive loss payments directly from CAT-MEX Ltd., subject to the terms and conditions of the insurance agreement. In case of occurrence of a trigger event, an earthquake with a certain magnitude in any of the three defined zones in Mexico, SRC pays the covered insured amount to the government, which stops paying premiums at that time and investors sacrifices their full principal and coupons.

Figure 4: The cash flows diagram for the mexican CAT bond.



Source: SHCP.

Assuming perfect financial market, the calibration of the parametric CAT bond is based on the estimation of the intensity rate that describes the flow process of earthquakes from the two sides of the contract: from the reinsurance and the capital markets.

Let $\mathcal{F}_t$ be an increasing filtration with time $t \in [0, T]$. The arrival process of earthquakes or the number of earthquakes in the interval $(0, t]$ is described by the process $N_t, t \geq 0$. This process uses the times T_i when the ith earthquake occurs or the times between earthquakes $\tau_i = T_i - T_{i-1}$. The earthquake process N_t in terms of τ_i's is defined as:

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}(T_n < t) \tag{1}$$

Since earthquakes can strike at any time during the year with the same probability, the traditional approach in seismology is to model earthquake recurrence as a random process, in which the earthquakes suffer the *loss of memory property* $\mathrm{P}(X > x + y | X > y) = \mathrm{P}(X > x)$, where X is a random variable. Nevertheless it is possible to predict, on average, how many events will occur and how severe they will be. The arrival process of earthquakes N_t can be characterized with a Homogeneous Poisson Process (HPP), with intensity rate $\lambda > 0$ if N_t is a point process governed by the Poisson law and the waiting times τ_i are exponentially distributed with intensity λ. Hence, the probability of occurrence of an earthquake is:

$$\mathrm{P}(\tau_i < t) = 1 - \mathrm{P}(\tau_i \geq t) = 1 - e^{-\lambda t} \tag{2}$$

In fact, we are interesting in the occurrence of the first event. We define the first waiting time as the *stopping time* equal to $\tau = \min\{t : N_t > 0\}$, with cdf $F_\tau(t) = \mathrm{P}(\tau < t) = \mathrm{P}(N_t > 0) = 1 - e^{-\lambda t}$ and $f_\tau(t) = \lambda e^{-\lambda t}$.

Let the random variable $J = 450 \cdot 1(\tau < 3)$ with density function $f_\tau(t)$ be the payoff of the covered insured amount to the government in case of occurrence of an event over a three year period $T = 3$. Denote H as the total premium paid by the government equal to 26 million. Suppose a flat term structure of continuously compounded discount interest rates and a HPP with intensity λ_1 to describe the arrival process of earthquakes. Under the non-arbitrage framework, a compounded discount *actuarially fair insurance price* at time $t = 0$ in the reinsurance market is defined as:

$$\begin{aligned} H &= \mathrm{E}\left[Je^{-\tau r_\tau}\right] \\ &= \mathrm{E}\left[450 \cdot \mathbf{1}(\tau < 3)e^{-\tau r_\tau}\right] \\ &= 450 \int_0^3 e^{-r_t t} f_\tau(t) dt \\ &= 450 \int_0^3 e^{-r_t t} \lambda_1 e^{-\lambda_1 t} dt \end{aligned} \tag{3}$$

i.e. the insurance premium is equal to the value of the expected discounted loss from earthquake. Substituing the values of H and assuming an annual continously compounded discount interest rate $r_t = \log(1.0541)$ constant and equal to the LIBOR in May 2006, we get:

$$26 = 450 \int_0^3 e^{-\log(1.0541)t} \lambda_1 e^{-\lambda_1 t} dt \tag{4}$$

where $1 - e^{-\lambda_1 t}$ is the probability of occurrence of an event. The estimation of the intensity rate from the reinsurance market λ_1 is equal to 0.0214. That means that the premium paid by the government to the insurance company considers a probability of occurrence of an event in three years equal to 0.0624 or the insurer expects 2.15 events in one hundred years.

For computing the intensity in the capital markets λ_2, we suppose that the contract structure defines a coupon CAT bond that pays to the investors the principal P equal to 160 million at time to maturity $T = 3$ and gives coupons C every 3 months during the bond's life in case of no event. If there is an event, the investors sacrifice their principal and coupons. These coupon bonds offered by CAT-MEX Ltd. pay to the investors a fixed spread rate z equal to 235 basis points over LIBOR. We consider the annual discretely compounded discount interest rate $r_t = 5.4139\%$ to be constant and equal to LIBOR in May 2006. The fixed coupons payments C have a value (in USD million) of:

$$C = \left(\frac{r_t + z}{4}\right) P = \left(\frac{5.4139\% + 2.35\%}{4}\right) 160 = 3.1055 \tag{5}$$

Let the random variable G be the investors' gain from investing in the bond, which consists of the principal and coupons. Moreover, assume that the arrival process of earthquakes follows a HPP with intensity λ_2. Under an arbitrage free scenario, the discretely discount *fair bond price* at time $t = 0$ is given by:

$$\begin{aligned} P &= \mathrm{E}\left[G\left(\frac{1}{1+r_\tau}\right)^{\tau}\right] \\ &= \mathrm{E}\left[\sum_{t=1}^{12} C \cdot \mathbf{1}(\tau > \frac{t}{4})\left(\frac{1}{1+r_t}\right)^{\frac{t}{4}} + P \cdot \mathbf{1}(\tau > 3)\left(\frac{1}{1+r_t}\right)^{3}\right] \\ &= \sum_{t=1}^{12} C e^{-\lambda_2 \frac{t}{4}}\left(\frac{1}{1+r_t}\right)^{\frac{t}{4}} + P e^{-3\lambda_2}\left(\frac{1}{1+r_t}\right)^{3} \end{aligned} \tag{6}$$

In this case, the investors receive 12 coupons during 3 years and its principal P at maturity $T = 3$. Hence, substituting the values of the principal $P = 160$ million and the coupons $C = 3.1055$ million in equation (6), it follows:

$$160 = \sum_{t=1}^{12} 3.06\left(\frac{e^{-\lambda_2}}{1.0541}\right)^{\frac{t}{4}} + \frac{160 e^{-3\lambda_2}}{(1.0541)^3} \tag{7}$$

Solving the equation (7), the intensity rate from the capital market λ_2 is equal to 0.0241. In other words, the capital market estimates a probability of occurrence of an event equal to 0.0699, equivalently to 2.4 events in one hundred years.

Additionally to the estimation of the intensity rate for the reinsurance and the capital markets, the historical intensity rate that describes the flow process of earthquakes λ_3 is calculated. The data was provided by the National Institute of Seismology in Mexico (SSN). It describes the time t, the depth d, the magnitude Mw and the epicenters of 192 earthquakes higher than 6.5 Mw occurred in the country during 1900 to 2003. Earthquakes less than 6.5 Mw were not taken into account because of their high frequency and low loss impact. Table 1 shows that almost 50% of the earthquakes has occurred in the insured zones, mainly in zone 2.

Let Y_i be i.i.d. random variables, indicating the magnitude Mw of the ith earthquake at time t. Define $\bar{u}$ as the threshold for a specific location. The estimation of the historical λ_3 is based on the *intensity model*. This model assumes that there exist i.i.d. random variables ε_i called *trigger events* that characterize earthquakes with magnitude Y_i higher than a defined threshold $\bar{u}$ for a specific location,

Table 1: Frequency of the earthquake location for the 1900-2003 earthquake data.

Zone	Frequency	Percent	% Cumulative
1	30	16%	16%
2	42	22%	38%
5	18	9%	47%
Other	102	53%	100%

Source: Own calculations.

i.e. $\varepsilon_i = \mathbf{1}(Y_i \geq \bar{u})$. Then the trigger event process B_t is characterized as:

$$B_t = \sum_{i=1}^{N_t} \varepsilon_i \tag{8}$$

where N_t is an HPP describing the arrival process of earthquakes with intensity $\lambda > 0$. B_t is a process which counts only earthquakes that trigger the CAT bond's payoff. However, the dataset contains only three such events, what leads to the calibration of the intensity of B_t be based on only two waiting times. Therefore in order to compute λ_3, consider the process B_t and define p as the probability of occurrence of a trigger event conditional on the occurrence of the earthquake. Then the probability of no event up to time t is equal to:

$$\begin{aligned} \mathrm{P}(B_t = 0) &= \mathrm{P}(N_t = 0) + \mathrm{P}(N_t = 1)(1-p) + \mathrm{P}(N_t = 2)(1-p)^2 + \dots \\ &= \sum_{k=0}^{\infty} \mathrm{P}(N_t = k)(1-p)^k = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{(-\lambda t)}(1-p)^k \\ &= \sum_{k=0}^{\infty} \frac{\{\lambda(1-p)t\}^k}{k!} e^{(-\lambda t)} e^{-\lambda(1-p)t} e^{\lambda(1-p)t} = e^{-\lambda p t} = e^{-\lambda_3 t} \end{aligned} \tag{9}$$

by definition of the Poisson distribution and since $\sum_{k=0}^{\infty} \frac{\{\lambda(1-p)t\}^k}{k!} e^{-\lambda(1-p)t} = 1$. Now the calibration of the λ_3 can be decomposed into the calibration of the intensity of all earthquakes with a magnitude higher than 6.5 Mw and the estimation of the probability of the trigger event.

Since the historical data contains three earthquakes with magnitude Mw higher than the defined thresholds by the modelling company, the probability of occurrence of the trigger event is equal to $p = \left(\frac{3}{192}\right)$. The estimation of the annual intensity is obtained by taking the mean of the daily number of earthquakes times 360 i.e. $\lambda = (0.005140)(360) = 1.8504$. Consequently the annual historical intensity rate for a trigger event is equal to $\lambda_3 = \lambda p = 1.8504\left(\frac{3}{192}\right) = 0.0289$. This means that approximately 2.89 events are expected to occur in the insured areas of the country within one hundred years.

Table 2 summarizes the values of the intensities rates $\lambda' s$ and the probabilities of occurrence of a trigger event in one and three years. Whereas the reinsurance

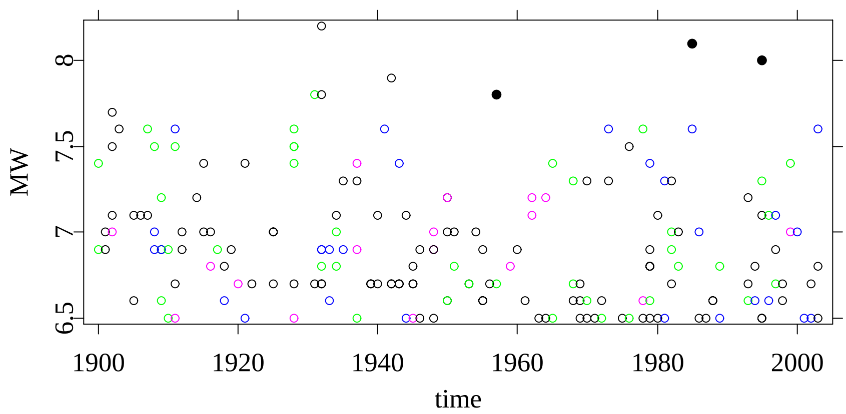
Table 2: Calibration of intensity rates: the intensity rate from the reinsurance market λ_1, the intensity rate from the capital market λ_2 and the historical intensity rate λ_3

	λ_1	λ_2	λ_3
Intensity	0.0214	0.0241	0.0289
Probability of event in 1 year	0.0212	0.0238	0.0284
Probability of event in 3 year	0.0624	0.0699	0.0830
No. expected events in 100 years	2.1482	2.4171	2.8912

Source: Own calculations.

market expects 2.15 events to occur in one hundred years, the capital market anticipates 2.22 events and the historical data predicts 2.89 events. Observe that the value of the λ_3 depends on the time period of the historical data, it is estimated from the years 1900 to 2003 and it is not very accurate since it is based on three events only. For a different period, λ_3 might be smaller than λ_1 or λ_2. The magnitude of earthquakes above 6.5 Mw that occurred in Mexico during 1990 to 2003 are illustrated in Figure 5. It also indicates earthquakes that occurred in the insured zones and trigger events. Apparently the difference

Figure 5: Magnitude of trigger events (filled circles), earthquakes in zone 1 (black circles), earthquakes in zone 2 (green circles), earthquakes in zone 5 (magenta circles), earthquakes out of insured zones (blue circles).



Source: Own representation.

between the intensity rates λ_1, λ_2 and λ_3 seems to be insignificant, but for the government it has a financial and social repercussion since the intensity rate of the flow process of earthquakes influences the price of the parametric CAT bond that will help the government to obtain resources after a big earthquake.

The absence of the public and liquid market of earthquake risk in the reinsurance market might explain the small difference between λ_1 and λ_2, since just limited information is available. This might cause the pricing in the reinsurance market to be less transparent than pricing in the capital markets. Another argument to this difference might be because contracts in the capital market are more expensive than contracts in the reinsurance market: the associated risk or default or the cost of risk capital (the required return necessary to make a capital budgeting project) in the capital markets is usually higher than that in the reinsurance market. A CAT bond presents no credit risk as the proceeds of the bond are held in a SPV, a transaction off the insurer's balance sheet. The estimation of λ_3 is not very precise since it is based on the time period of the historical data, but for interpretations we suppose that λ_3 is the real intensity rate describing the flow of process of earthquakes.

Particularly after a catastrophic event occurred, the reinsurance market suffers from a shortage of capital but this gives reinsurance firms the ability to gain more market power that will enable them to charge higher premiums than expected. Our estimation of intensity rates, contrary to the theory predictions, show that the Mexican government paid total premiums of 26 million that is 0.75 times the real actuarially fair one (34.605 million), which is obtained by substituting the historical intensity λ_3 in equation (4). At first glance, it appears that either the government saves 8.605 million in transaction costs from transferring the seismic risk with a reinsurance contract or that reinsurer is underestimating the occurrence probability of a trigger event. This is, however, not a valid argument because the actuarially fair reinsurance price assumes that the coverage payout depends only on the loss of the insured event. In reality, the reinsurance market and the coverage payouts are exposed to other risks that can affect the value of the premium, e.g. the credit risk. Considering this fact, the probability that the resinsurer will default over the next three years could be approximately equal to the price discount that the government gets in the risk transfer of earthquake risk ($\approx 24.86\%$).

However, the best explanation of the low premiums for covering the seismic risk might be the mix of the reinsurance contract and the CAT bond. Since the 160 million CAT bond is part of a total coverage of 450 million, the reinsurance company transfers 35% of the total seismic risk to the investors, who effectively are betting that a trigger event will not hit specified regions in Mexico in the next three years. If there is no event the money and interests are returned to the investors, otherwise the reinsurer must pay to the government 290 million from the reinsurance part and 160 million from the CAT bond to cover the insured loss of 450 million. The value of the premium for 290 million coverage with intensity rate of eaqrthquake λ_1 is $\int_0^3 290\lambda_1 e^{-t(r_t+\lambda_1)}dt = 16.755$. Therefore the total premium of 26 million might consist of 16.755 million premium from the reinsurance part and the CAT bond and 9.245 million for transaction costs or the management added value or for coupon payments. This government's financial strategy is optimal in the sense that it provides coverage of 450 million against seismic risk

Table 3: Descriptive statistics for the variables time t, depth d, magnitude Mw and loss X of the loss historical data.

Descriptive	t	d	Mw	X ($ million)
Minimum	1900	0	6.5	10.73
Maximum	2003	200	8.2	0
Mean	1951	39.54	6.93	1443.69
Median	1950	33	6.9	0
Sdt. Error	-	39.66	0.37	105.16
25% Quantile	1928	12	6.6	0
75% Quantile	1979	53	7.1	0
Skewness	-	1.58	0.92	13.19
Kurtosis	-	5.63	3.25	179.52
Nr. obs.	192	192	192	192
Distinct obs.	82	54	18	24

Source: Own calculations.

for a lower cost than the reinsurance itself, which has an actuarially fair premium equal to 34.605 million. However, this financial strategy of the government does not eliminate completely the costs imposed by market imperfections.

4 Pricing modeled-index CAT bonds for mexican earthquakes

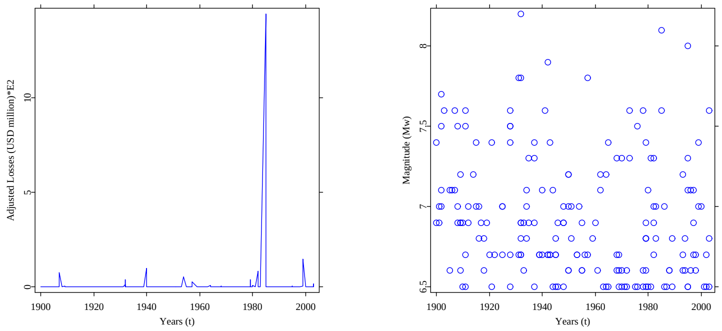
Since the value of the losses can be affected by different variables, e.g. not only by the magnitude Mw of the earthquake but also by the depth d of the earthquake, the impact on cities $I(0,1)$, etc., under the assumptions of non-arbitrage and continuous trading, we examine the pricing of a CAT bond for earthquakes with a modeled-index loss trigger mechanism. In essence, this hybrid trigger combines modeled loss and index trigger types, trying to reduce basis risk borne by the sponsor, while remaining a non-indemnity trigger mechanism. Besides, this time, the payout of the bond will be based on historical and estimated losses. We applied the pricing CAT bond methodology of Baryshnikov et al. (2001) and Burnecki and Kukla (2003) to Mexican earthquake data from the National Institute of Seismology in Mexico (SSN) and to its corresponding loss data that we built. In order to calibrate the pricing model we first have to fit both the distribution function of the incurred losses $F(x)$ and the process N_t governing the flow of earthquakes.

4.1 Severity of mexican earthquakes

The historical losses of earthquakes occurred in Mexico during the years 1900 - 2003 were adjusted to the population growth, the inflation and the exchange rate (peso/dollar) and were converted to USD of 1990. The annual Consumer

Price Index (1860-2003) was used for the inflation adjustment and the Average Parity Dollar-Peso (1821-1997) was used for the exchange rate adjustment, both provided by the U.S. Department of Labour. For the population adjustment, the annual population per Mexican Federation (1900-2003) provided by the National Institute of Geographical and Information Statistics in Mexico (INEGI) was used. Table 3 describes some descriptive statistics for the variable time t, depth d, magnitude Mw and adjusted loss X of the historical data. From 1900 to 2003, the data considers 192 earthquakes higher than 6.5 Mw and 24 of them with financial adjusted losses, see Figure 6. The peaks mark the occurrence of two outliers: the 8.1 Mw earthquake in 1985 and the 7.4 Mw earthquake in 1999. The earthquake in 1932 had the highest magnitude in the historical data (8.2 Mw), but its losses are not big enough compared to the other earthquakes.

Figure 6: Plot of adjusted losses (left panel) and the magnitude Mw (right panel) of earthquakes occurred in Mexico during the years 1900-2003.



Source: Own representation.

We observe that when all the historical adjusted losses are taken in account, they are directly proportional to the time t and the magnitude Mw, and inversely proportional to the depth d. However, when the outliers are excluded, the adjusted losses are inversely proportional to the time, magnitude and depth. Considering this, we model the losses of mexican earthquakes in terms of logarithm ($\log(x)$) by means of the linear regression. Under the selection criterion of the highest coefficient of determination r^2, the linear regression loss models that fit better the historical earthquake loss data are:

$$\log(x) = -27.99 + 2.10Mw + 4.44d - 0.15I(0,1) - 1.11\log(Mw) \cdot d$$

For the case without the outlier of the earthquake in 1985:

$$\log(x) = -7.38 + 0.97Mw + 1.51d - 0.19I(0,1) - 0.52\log(Mw) \cdot d$$

For the case without the outliers of the earthquakes in 1985 and 1999:

$$\log(x) = 1.3037 + 0.4094Mw + 0.2375d + 0.1836I(0,1) - 0.2361\log(Mw) \cdot d$$

Table 4: Coefficients of determination and standard errors of the linear regression models*

r^2_{LR1}	r^2_{LR2}	r^2_{LR3}	SE_{LR1}	SE_{LR2}	SE_{LR3}
0.226	0.151	0.129	2.8698	2.8302	2.8383

*Applied to the adjusted loss data (r^2_{LR1}, SE_{LR1}), without the outlier of the earthquake in 1985 (r^2_{LR2}, SE_{LR2}) and without the outliers of the earthquakes in 1985 and 1999 (r^2_{LR3}, SE_{LR3}).
Source: Own calculations.

where $I(0,1)$ indicates the impact of the earthquake on Mexico city. Table 4 displays the coefficients of determination and standard errors for each of the proposed linear regresion models of the historical adjusted loss data r^2_{LR1}, SE_{LR1}, without the observation of the earthquake in 1985 r^2_{LR2}, SE_{LR2} and for the data without the outliers of the earthquakes in 1985 and 1999 r^2_{LR3}, SE_{LR3}. After selecting the best models, we apply the Expectation - Maximum algorithm (EM) with linear regression to the historical and estimated losses to fill the missing data of losses, Howell (1998). See Figure 7.

In order to find an accurate loss distribution that fits the loss data, we compared the shapes of the empirical $\hat{e}_n(x)$ and the theoretical mean excess function $e(x)$. Given a loss random variable X, the mean excess function (MEF) is the expected payment per insured loss with a fixed amount deductible of x i.e. the mean excess function restricts a random variable X given that it exceeds a certain level x, Hogg and Klugman (1984):

$$e(x) = \mathrm{E}(X - x | X > x) = \frac{\int_x^\infty 1 - F(u) du}{1 - F(x)} \tag{10}$$

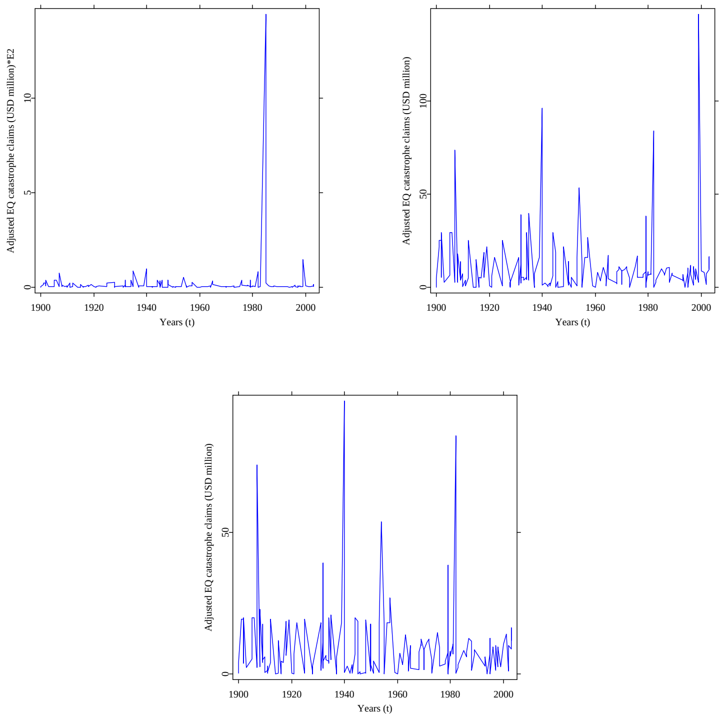
The empirical mean excess function is defined as:

$$\hat{e}_n(x) = \frac{\sum_{x_i > x} x_i}{\# i : x_i > x} - x$$

The left panel of Figure 8 shows an increasing pattern for the $\hat{e}_n(x)$, pointing out that the distribution of losses have heavy tails i.e. it indicates that the Lognormal, the Burr or the Pareto distribution are candidates to be the analytical distribution of the loss data. Whereas eliminating the outlier of the earthquake in 1985 from that modeled loss data, the $\hat{e}_n(x)$ shows a decreasing pattern, indicating that Gamma, Weibull or Pareto could model adequately, see right panel of Figure 8.

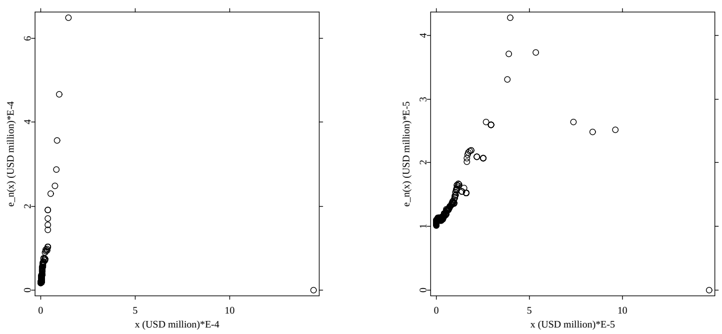
To test whether the fit is adequate, the empirical $F_n(x) = \frac{1}{n} \# \{i : x_i \leq x\}$ is compared with the fitted $F(x)$ distribution function. To this end the Kolmogorov Smirnov, the Kuiper statistic, the Cramér-von Mises and the Anderson Darling non-parametric tests are applied. The test of the fit procedure consists of the null hypothesis: *the distribution is suitable* $\{H_0 : F_n(x) = F(x; \theta)\}$, and the alternative: *the distribution is not suitable* $\{H_1 : F_n(x) \neq F(x; \theta)\}$, where θ is a vector

Figure 7: Historical and modeled losses of earthquakes occurred in Mexico during 1900-2003 (upper left panel), without the outlier of the earthquake in 1985 (upper right panel), without outliers of the earthquakes in 1985 and 1999 (lower panel)



Source: Own representation.

Figure 8: The empirical mean excess function $\hat{e}_n(x)$ for the modeled loss data with (left panel) and without the outlier of the earthquake in 1985 (right panel).



Source: Own representation.

Table 5: Parameter estimates by A^2 minimization procedure and test statistics for the modeled loss data.*

Distrib.	Log-normal	Pareto	Burr	Exponential	Gamma	Weibull
Parameter	$\mu = 1.456$ $\sigma = 1.677$	$\alpha = 2.199$ $\lambda = 12.53$	$\alpha = 3.354$ $\lambda = 17.33$ $\tau = 0.895$	$\beta = 0.132$	$\alpha = 0.145$ $\beta = -0.0$	$\beta = .214$ $\tau = .747$
Kolmogorov S.	0.185	0.142	0.150	0.149	0.299	0.157
(D test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
Kuiper	0.308	0.265	0.278	0.245	0.570	0.298
(V test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
Cramér-von M.	1.447	0.879	0.987	0.911	6.932	1.16
(W^2 test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)
Anderson D.	10.490	6.131	6.018	10.519	35.428	6.352
(A^2 test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)

*In parenthesis, the related p-values based on 1000 simulations. Source: Own calculations.

of known parameters. The fit is accepted when the value of the test is less than the corresponding critical value C_α, given a significance level α.

The estimated parameters of the modeled loss data (via A^2 statistic minimization, (D'Aagostino and Stephens (1986)) and the corresponding *edf* test statistics are shown in Table 5. It also shows the corresponding p-values based on 1000 simulated samples. Observe that all the tests reject the fit for all the distributions. However, for other loss models the A^2 statistic pass the Burr distribution at the 2%, 1% and 1% level respectively. Table 6 displays the estimated parameters, the hypothesis testing and p-values based on 1000 simulated samples of the modeled loss data without the outlier of the earthquake in 1985. The exponential distribution with parameter $\beta = 0.120$ passes all the tests at the 0.8% level, except the A^2 statistic. Likewise, the Pareto distribution passes two tests at 0.6% and 1.2% level, but with unacceptable fit in the A^2 statistic. All the remaining distributions give worse fits. However, in other loss models without the outlier of 1985 earthquake the Gamma distribution passes all the test statistics and the A^2 statistics at the $0.6\%, 6\%, 5.6\%, 1.8\%$ level respectively.

We also compute the limited expected value function to find the best fit of the earthquake-loss distribution. For a fixed amount deducible of x, the limited expected value function characterizes the expected amount per loss retained by the insured in a policy, Hogg and Klugman (1984):

$$l(x) = \mathrm{E}\left\{\min(X, x)\right\} = \int_0^x y dF(y) + x\left\{1 - F(x)\right\}, x > 0 \tag{11}$$

where X is the loss amount random variable, with cdf $F(x)$. The empirical estimate is given by:

$$\hat{l}_n(x) = \frac{1}{n}\left(\sum_{x_j < x} x_j, + \sum_{x_j \geq x} x\right)$$

Besides curve-fitting purposes, the limited expected function is very useful be-

Table 6: Parameter estimates by A^2 minimization procedure and test statistics for the modeled loss data without the outlier of the 1985 earthquake.*

Distrib.	Log-normal	Pareto	Burr	Exponential	Gamma	Weibull
Parameter	$\mu = 1.493$ $\sigma = 1.751$	$\alpha = 2.632$ $\lambda = 17.17$	$\alpha = 1.8e7$ $\lambda = 9.5e7$ $\tau = 0.770$	$\beta = 0.120$	$\alpha = 0.666$ $\beta = .070$	$\beta = 0.194$ $\tau = .770$
Kolmogorov S.	0.116	0.077	0.070	0.081	0.070	0.070
(D test)	(< 0.005)	(< 0.005)	(0.001)	(0.084)	(< 0.005)	(0.008)
Kuiper	0.215	0.133	0.126	0.138	0.121	0.126
(V test)	(< 0.005)	(0.006)	(< 0.005)	(0.008)	(< 0.005)	(< 0.005)
Cramr-von M.	0.702	0.168	0.166	0.202	0.147	0.166
(W^2 test)	(< 0.005)	(0.012)	(< 0.005)	(0.152)	(0.006)	(< 0.005)
Anderson D.	6.750	3.022	1.617	4.732	1.284	1.617
(A^2 test)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)	(< 0.005)

*In parenthesis, the related p-values based on 1000 simulations. Source: Own calculations.

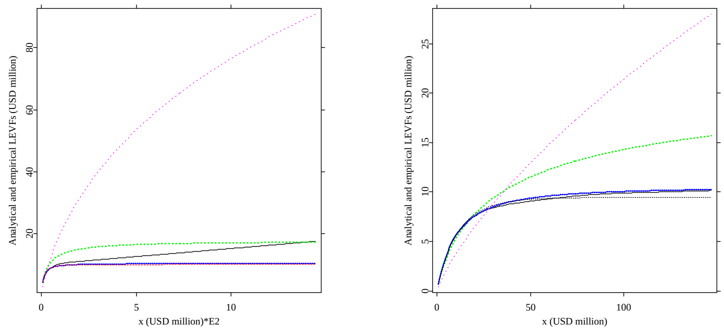
cause it emphasizes how different parts of the loss distribution function contribute to the premium. Figure 9 presents the empirical and analytical limited expected value functions for the analyzed data set with (left panel) and without the earthquake in 1985 (right panel). The closer they are, the better they fit and the closer the mean values of both distributions are. The graphs give explanation for the choice of the Burr, Pareto, Gamma and Weibull distributions. Hence, the prices of the CAT bonds will be based on these distributions.

4.2 Frequency of mexican earthquakes

In this section we focus on efficient simulation of the arrival point process of earthquakes N_t. We first look for the appropriate shape of the approximating distribution. One can achieve that examining the empirical mean excess function $\hat{e}_n(t)$ for the waiting times of the earthquake data, see left panel of Figure 10. The empirical mean excess function plot shows an increasing starting and a decreasing ending behaviour, implying that the exponential, Gamma, Pareto and Log-normal distribution could be possible candidates to fit the arrival process of earthquakes. However, for a large t, the tails of the analytical distributions fitted to the earthquake data are different from the tail of the empirical distribution. The analytical mean excess functions $e(t)$ increase with time. See right panel of Figure 10.

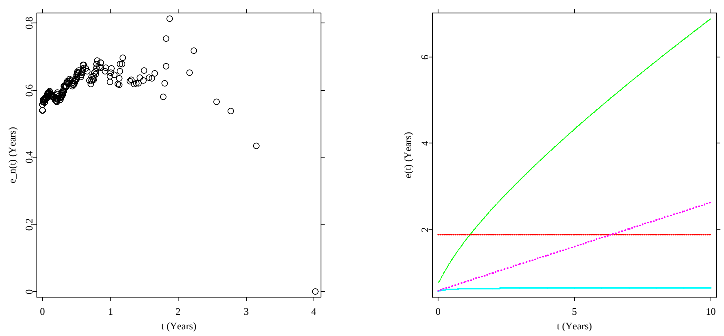
Another way to model the claim arrival process of earthquakes is by a renewal process, where one estimates the parameters of the candidate analytical distributions via the A^2 minimization procedure and tests the Goodness of fit. The estimated parameters and their corresponding p-values based on 1000 simulations are illustrated in Table 7. Observe that the exponential, Pareto and Gamma distributions pass all the tests at a very high level. The Gamma distribution passes the A^2 test with the highest level (88%).

Figure 9: The empirical $\hat{l}_n(x)$ (black solid line) and analytical $l(x)$ limited expected value function for the log-normal (green dashed line), Pareto (blue dashed line), Burr (red dashed line), Weibull (magenta dashed line) and Gamma (black dashed line) distributions for the modeled loss data with (left panel) and without the outlier of the 1985 earthquake (right panel).



Source: Own representation.

Figure 10: The empirical mean excess function $\hat{e}_n(t)$ for the earthquakes data (left panel) and the mean excess function $e(t)$ for the log-normal (green solid line), exponential (red dotted line), Pareto (magenta dashed line) and Gamma (cyan solid line) distributions for the earthquakes data (right panel).



Source: Own representation.

Table 7: Parameter estimates by A^2 minimization procedure and test statistics for the earthquake data.*

Distrib.	Log-normal	Exponential	Pareto	Gamma
Parameter	$\mu = -1.158$ $\sigma = 1.345$	$\beta = 1.880$	$\alpha = 5.875$ $\lambda = 2.806$	$\alpha = 0.858$ $\beta = 1.546$
Kolmogorov S. (D tests)	0.072 (0.005)	0.045 (0.538)	0.035 (0.752)	0.037 (0.626)
Kuiper (V test)	0.132 (< 0.005)	0.078 (0.619)	0.067 (0.719)	0.064 (0.739)
Cramr-von M. (W^2 test)	0.212 (< 0.005)	0.062 (0.451)	0.031 (0.742)	0.030 (0.730)
Anderson D. (A^2 test)	2.227 (< 0.005)	0.653 (0.253)	0.287 (0.631)	0.190 (0.880)

*In parenthesis, the related p-values based on 1000 simulations. Source: Own calculations.

If the claim arrival process of earthquakes is modelled with an HPP, the intensity is independent of time and the estimation of the annual intensity is obtained by taking the mean of the daily number of earthquakes times 360, i.e. $\lambda = (0.005140)(360)$ equal to 1.8504 earthquakes higher than 6.5 Mw per year. Comparing this annual intensity with the annual intensity of the renewal process modelled with an exponential distribution equal to 1.88 indicates that the earthquakes arrival process can be correctly model with the HPP.

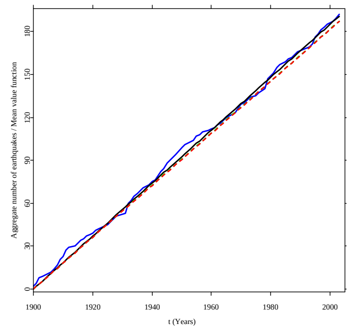
In order to check for a better estimate, we also model the arrival process of earthquakes with a Non-homogeneous Poisson Process (NHPP). This time, the expected value is equal to $\mathrm{E}(N_t) = \int_0^t \lambda_s ds$, where the intensity rate is dependent of time λ_s and it can be fitted in some parametric functions by least squares. We tested different polynomial functions to model the intensity λ_s of the earthquake data, but the constant intensity $\lambda_s^1 = 1.8167$ with a coefficient of determination $r^2 = 0.99$ and standard error $SE = 2.33$ was the best fit. This result shows that the HPP describes well the arrival process of earthquakes and confirms the theory of time independence of earthquakes. Earthquakes can strike at any time during the year with same probability, they do not show seasonality as other natural events do. Figure 11 depicts the accumulated number of earthquakes and the mean value functions $\mathrm{E}(N_t)$ of the HPP with intensity rates $\lambda_s = 1.8504$ and $\lambda_s^1 = 1.8167$.

4.3 Pricing Modeled-Index CAT bonds

An index CAT bond is priced by means of the compound doubly stochastic Poisson pricing methodology from Baryshnikov et al. (2001), according to the statistical characteristics of the dominant underlying processes. The pricing of CAT bonds relies on a few stochastic assumptions:

(A1) There is a doubly stochastic Poisson process N_s, i.e. a Poisson process conditional on an stochastic intensity process λ_s with $s \in [0, T]$, describing the

Figure 11: The accumulated number of earthquakes (solid blue line) and mean value functions $\mathbf{E}(N_t)$ of the HPP with intensity $\lambda_s = 1.8504$ (solid black line) and $\lambda_s^1 = 1.8167$ (dashed red line).



Source: Own representation.

flow of a particular catastrophic natural event in a specified region.

(A2) The financial losses $\{X_k\}_{k=1}^{\infty}$ caused by these catastrophic events t_i are independent and i.i.d random variables with cdf $F(x)$.

(A3) The process N_s and X_k are assumed to be independent. Then, the countinuous and predictable aggregate loss process is:

$$L_t = \sum_{i=1}^{N_t} X_i \tag{12}$$

(A4) A continuously compounded discount interest rate r describing the value at time s of 1 USD paid at time $t > s$ by:

$$e^{-R(s,t)} = e^{-\int_s^t r(\xi)d\xi}$$

(A5) A threshold time event $\tau = \inf\{t : L_t \geq D\}$, that is the moment when the aggregate loss L_t exceeds the threshold level D. Baryshnikov et al. (2001) defines the threshold time as a doubly stochastic Poisson process $M_t = 1(L_t > D)$, with a stochastic intensity depending on the index position:

$$\Lambda_s = \lambda_s \{1 - F(D - L_s)\} \mathbf{1}(L_s < D) \tag{13}$$

Under these assumptions, assume a zero coupon CAT bond that pays a principal amount P at time to maturity T, conditional on the threshold time $\tau > T$. Let P be a predictable process $P_s = \mathrm{E}(P|\mathcal{F}_s)$, i.e. the payment at maturity is independent from the occurrence and timing of the threshold D. Consider that in case of occurrence of the trigger event the principal P is fully lost.

The non arbitrage price of the zero coupon CAT bond V_t^1 associated with the threshold D, earthquake flow process N_s with intensity λ_s, a loss distribution function F and paying the principal P at maturity is thus given by, Burnecki and Kukla (2003):

$$
\begin{aligned}
V_t^1 &= \mathrm{E}\left[Pe^{-R(t,T)}\left(1-M_T\right)|\mathcal{F}_t\right] \\
&= \mathrm{E}\left[Pe^{-R(t,T)}\left\{1-\int_t^T \lambda_s\left\{1-F(D-L_s)\right\}\mathbf{1}(L_s<D)d_s\right\}|\mathcal{F}_t\right] \quad (14)
\end{aligned}
$$

i.e. the price of a zero coupon CAT bond is equal to the expected discounted value of the principal P contingent on the threshold time $\tau > T$. Here the compounded Poisson process is used to incorporate the various characteristics of the earthquake process, where the rates at which earthquakes occur and the impact of their occurrence are regarded as doubly stochastic Poisson processes.

Similarly, under the same assumptions that the zero coupon bonds, a coupon CAT bond V_t^2 that pays the principal P at time to maturity T and gives coupon C_s until the threshold time τ is given by, Burnecki and Kukla (2003):

$$
\begin{aligned}
V_t^2 &= \mathrm{E}\left[Pe^{-R(t,T)}\left(1-M_T\right)+\int_t^T e^{-R(t,s)}C_s\left(1-M_s\right)ds|\mathcal{F}_t\right] \\
&= \mathrm{E}\left[Pe^{-R(t,T)}+\int_t^T e^{-R(t,s)}\left\{C_s\left(1-\int_t^s \lambda_\xi\left\{1-F(D-L_\xi)\right\}\right.\right.\right. \\
&\qquad \left.\left.\left.\mathbf{1}(L_\xi<D)d\xi\right)-Pe^{-R(s,T)}\lambda_s\left\{1-F(D-L_s)\right\}\mathbf{1}(L_s<D)\right\}d_s|\mathcal{F}_t\right] \quad (15)
\end{aligned}
$$

These coupons bonds usually pay a fixed spread z over LIBOR that reflects the value of the premium paid for the insured event, and LIBOR reflects the gain for investing in the bond.

Following this pricing methodology, we obtain the values of a (zero) coupon CAT bond for earthquakes at $t = 0$. We consider that the continuously compounded discount interest rate $r = \log(1.054139)$ is constant and equal to the LIBOR in May 2006, $P = 160$ million and the expiration time $T \in [0.25, 3]$ years. Define now the threshold $D \in [100, 135]$ million, corresponding to the 0.7 and 0.8-quantiles of the three yearly accumulated modeled losses, i.e. approximately three payoffs are expected to occur in one hundred years (see Table 8).

After applying 1000 Monte Carlo simulations, the price of the zero coupon CAT bond at $t = 0$ is calculated with respect to the threshold level D and expiration time T. The Burr and Pareto distribution are considered as loss distributions for the modeled loss data, while the Gamma, Pareto and Weibull distribution are studied for the modeled loss data without the outlier of the earthquake in 1985. For all the cases the arrival process of earthquakes follows an HPP with constant intensity $\lambda_s = 1.8504$. The simulations show that the price of the zero coupon CAT bond decreases as the expiration time increases, because the

Table 8: Quantiles of 3 years accumulated modeled losses.

Quantile	3 years accumulated loss
10%	18.447
20%	23.329
30%	32.892
40%	44.000
50%	61.691
60%	80.458
70%	**109.11**
80%	**119.86**
90%	142.72
100%	1577.6

Source: Own calculations.

Table 9: Minimum and maximum of the differences in the zero coupon CAT bond prices (in % of principal)*

	Min. (% Principal)	Max. (% Principal)
Diff. ZCB Burr-Pareto	-2.640	0.614
Diff. ZCB Gamma-Pareto	0.195	4.804
Diff. ZCB Pareto-Weibull	-4.173	-0.193
Diff. ZCB Gamma-Weibull	-0.524	1.636

*For the Burr-Pareto distributions of the modeled loss data and for the Gamma-Pareto, Pareto-Weibull, Gamma-Weibull distributions of the modeled loss data without the outlier of the earthquake in 1985.
Source: Own calculations.

occurrence probability of the trigger event increases. However, the bond price increases as the threshold level increases, since one expects a trigger event with low probability. When D = 135 USD million and T = 1 year, the CAT bond price $160e^{-\log(1.054139)} \approx 151.78$ million is equal to the case when the threshold time $\tau = \inf\{t : L_t > D\}$ is greater than the maturity T with probability one.

Although the prices are pretty similar, we observe that the loss distribution function influences the price of the CAT bond, see Table 9. When we consider the modeled loss data, the zero coupon bond price with respect to expiration time T and threshold level D is higher and less volatile in the case of the Pareto distribution (Std. deviation = 10.08) than the Burr distribution (Std. deviation = 10.6). While for the modeled loss data without the outlier of the earthquake in 1985, the Gamma distribution leads to higher prices than the Weibull and Pareto distributions and whose standard deviations are 8.83, 10.44 and 9.05 respectively.

For a coupon CAT bond, we consider the assumptions of the zero coupon bond and a spread rate z equal to 235 basis points over LIBOR. The bond has quarterly annual coupons $C_{\frac{t}{4}} = \left(\frac{LIBOR+235bp}{4}\right) 160 =$ 3.1055 million. After 1000 Monte Carlo simulations, the price of the coupon CAT bond at $t = 0$ with respect to the threshold level D and expiration time T is computed for the Burr, Pareto, Gamma and Weibull distribution of the modeled loss data with and without the outlier of the earthquake in 1985. Note in Table 10 that the coupon CAT bond

Table 10: Minimum and maximum of the differences in the (zero) coupon CAT bond prices (in % of principal)*

	Min. (% Principal)	Max. (% Principal)
Diff. ZCB-CB Burr	-6.228	-0.178
Diff. ZCB-CB Pareto	-5.738	-0.375
Diff. ZCB-CB Gamma	-7.124	-0.475
Diff. ZCB-CB Pareto (no outlier '85)	-5.250	-0.376
Diff. ZCB-CB Weibull	-5.290	-0.475
Diff. CB Burr-Pareto	-1.552	0.809
Diff. CB Gamma-Pareto	0.295	6.040
Diff. CB Pareto-Weibull	-3.944	-0.295
Diff. CB Gamma-Weibull	-0.273	3.105

*For the Burr-Pareto distributions of the modeled loss data and the Gamma-Pareto, Pareto-Weibull, Gamma-Weibull distributions of the modeled loss data without the outlier of the earthquake in 1985.
Source: Own calculations.

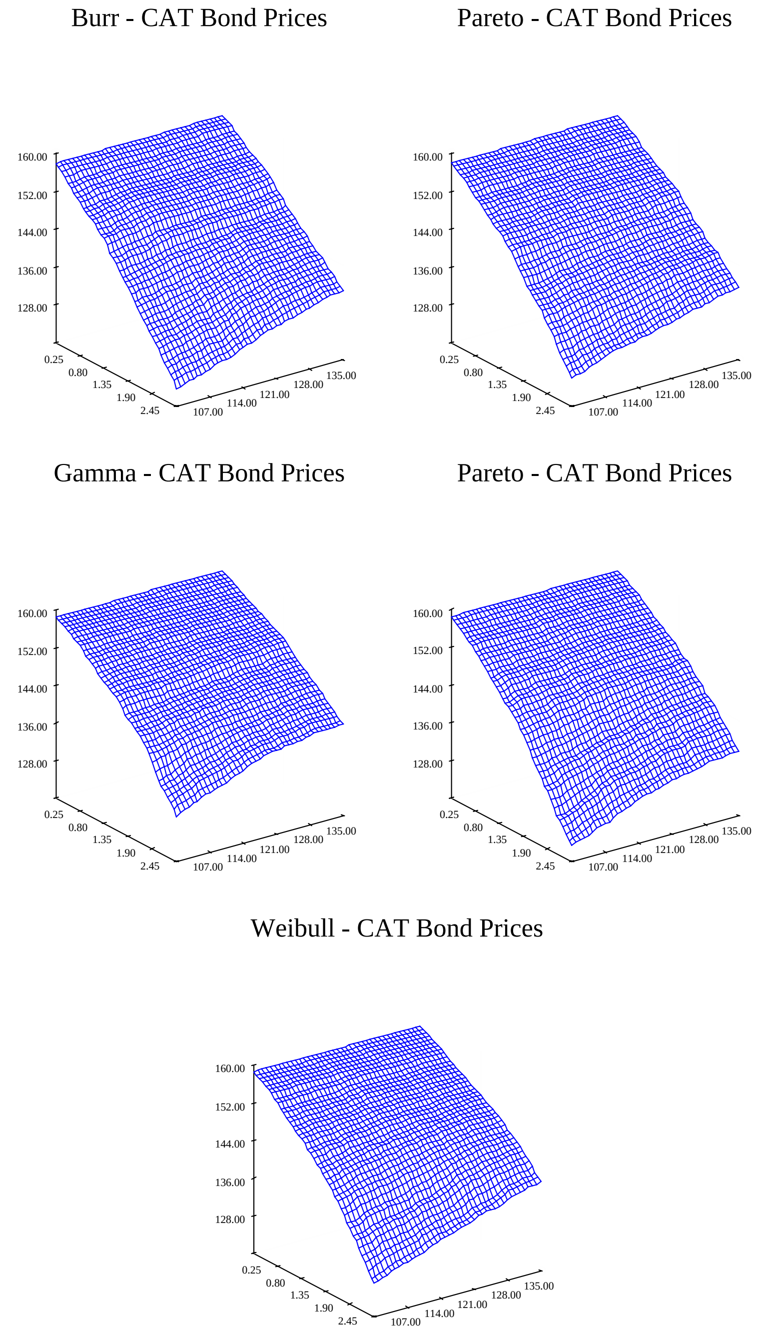
prices are higher than the zero coupon CAT bond prices. Figures 12 indicate that for all the distributions the price of the coupon CAT bond value increases as the threshold level D increases. But, increasing the expiration time T leads to lower coupon CAT bond price because the probability of a trigger event increases and more coupon payments are expected to be received.

Figure 13 illustrates the difference in distributions of the coupon CAT price with respect to expiration time T and threshold level D. Concerning to the loss distribution function for the modeled loss data, the Pareto distribution also leads to higher prices than the Burr distribution and lower standard deviation (equal to 8.15 and 8.31 respectively). While for the modeled loss data without the outlier of the earthquake in 1985, the Gamma distribution offered higher prices and lower standard deviation (6.39) than the Weibull and Pareto distributions (equal to 8.62 and 7.24 respectively).

In order to verify the robustness of the modeled loss with the prices of the zero and coupon CAT bonds we compare the bond prices calculated from different loss models with the bond prices simulated from the pricing algorithm. Let $\hat{P}^*$ be the reference price or the (zero) coupon CAT bond prices of the best loss model and let $\hat{P}_i$ with $i = 1 \ldots m$ be the (zero) cupon CAT bond price from the ith loss model, with $\hat{P}^* \neq \hat{P}_i$. The same seed of the pseudorandom number generator in 1000 Monte Carlo simulations is used to generate $\hat{P}^*$ and $\hat{P}_i$. Furthermore, let $\hat{P}_j$ with $j = 1 \ldots n$ be the algorithm CAT bond price obtained in the jth simulation of 1000 trajectories of the (zero) coupon CAT bond of the best loss model and which did not use the same seed for their generation.

To check if the type of the model has strong impact on the prices, we compute the *mean of absolute differences* (MAD) i.e. the mean of the differences of the bond prices $\hat{P}_i$ with the reference bond prices $\hat{P}^*$ and the mean of the differences of the algorithm bond prices $\hat{P}_j$ with the reference bond prices $\hat{P}^*$. If the MAD's are similar then the type of the model has no influence on the prices of the (zero)

Figure 12: Coupon CAT bond prices (vertical axis) with respect to the threshold level (horizontal right axis) and expiration time (horizontal left axis) in the Burr-HPP (upper left side), Pareto-HPP (upper right side), Gamma-HPP (middle left side), Pareto-HPP (middle right side) and Weibull-HPP (lower side) cases for the modeled loss data with and without the outlier of the earthquake in 1985.

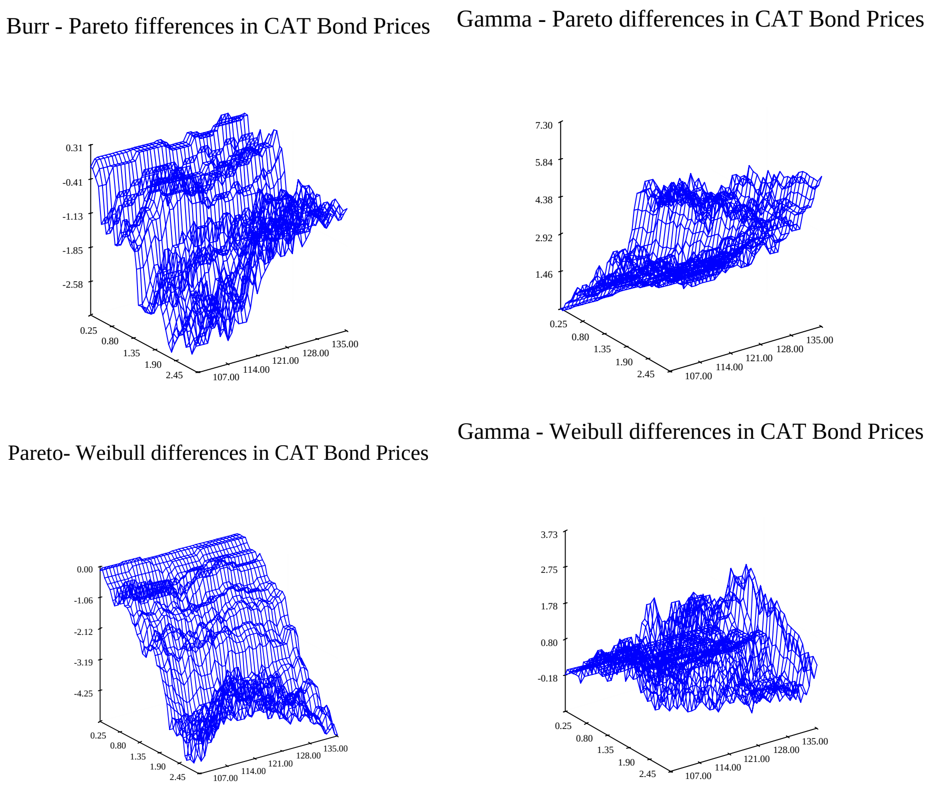


Source: Own representation.

coupon CAT bond:

$$\sum_{i=1}^{m} \frac{\hat{P}_i - \hat{P}^*}{m} \simeq \sum_{j=1}^{n} \frac{\hat{P}_j - \hat{P}^*}{n}, m > 0, n > 0 \tag{16}$$

Figure 13: Difference in the coupon CAT bond price (vertical axis) in the Burr-Pareto (upper left side), the Gamma-Pareto (upper right side), the Pareto-Weibull (lower left side) and the Gamma-Weibull (lower right side) distributions under an HPP, with respect to the threshold level (horizontal right axis) and expiration time (horizontal left axis).



Source: Own representation.

Table 11: Percentages in terms of $\hat{P}^*$ of the MAD and the MAVRD of the (zero) coupon CAT bond prices*

	T	D	$\hat{P}^*$	(%) MAD_A	(%) MAD_B	(%) $MAVRD_A$	(%) $MAVRD_B$
ZCCB	1	100	148.576	0.283	0.975	0.329	0.265
	1	120	149.637	0.203	0.663	0.270	0.228
	1	135	149.637	0.619	0.802	0.619	0.183
	2	100	133.422	1.577	2.334	1.577	0.566
	2	120	137.439	0.823	1.306	0.823	0.375
	2	135	138.873	0.884	1.161	0.930	0.358
	3	100	114.866	4.666	5.316	4.666	0.859
	3	120	123.177	2.409	2.958	2.409	0.640
	3	135	125.766	2.468	2.817	2.468	0.520
CCB	1	100	151.236	0.513	1.152	0.556	0.257
	1	120	152.306	0.398	0.853	0.419	0.216
	1	135	152.920	0.383	0.601	0.405	0.178
	2	100	139.461	0.966	2.131	0.966	0.475
	2	120	142.950	0.731	1.585	0.774	0.395
	2	135	145.141	0.337	0.827	0.556	0.354
	3	100	124.831	2.412	3.421	2.412	0.823
	3	120	131.508	1.844	2.590	1.844	0.708
	3	135	134.324	2.071	2.474	2.071	0.600

*from the different loss models ($MAD_A, MAVRD_A$) and one hundred simulations of 1000 trajectories of the coupon CAT bond prices from the algorithm ($MAD_B, MAVRD_B$) with respect to expiration time T and threshold level D.
Source: Own calculations.

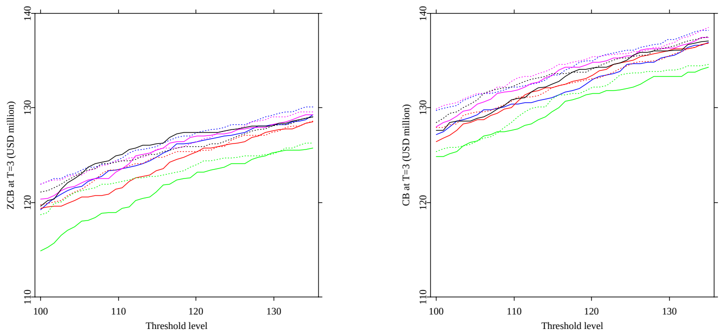
In terms of relative differences, if the means of the absolute values of the relative differences (MAVRD) are similar then the model has no impact on the zero and coupon CAT bond prices:

$$\sum_{i=1}^{m} \frac{1}{m} \left| \frac{\hat{P}_i - \hat{P}^*}{\hat{P}^*} \right| \simeq \sum_{j=1}^{n} \frac{1}{n} \left| \frac{\hat{P}_j - \hat{P}^*}{\hat{P}^*} \right|, m > 0, n > 0 \tag{17}$$

Table 11 shows the percentages in terms of the reference prices $\hat{P}^*$ of the MAD and the MAVRD of the (zero) coupon CAT bond prices from different loss models ($MAD_A, MAVRD_A$) and from the algorithm ($MAD_B, MAVRD_B$), with respect to expiration time T and threshold level D. The prices from the algorithm are generated with one hundred simulations of 1000 trajectories of the (zero) coupon CAT bond prices. We find that most of the percentages of the MAD are similar (the difference is less than 1%) meaning that the loss models do not have impact on the (zero) coupon CAT bond prices. Although the percentages of the $MAVRD_A$ differ from the percentages of the $MAVRD_B$ in the zero coupon CAT bonds prices when $T = 2$ years and $D = 100$ million or $T = 3$ years with $D = 100, 120, 135$ million, the rest of the percentages of the MAVRD remain similar (the difference in percentages is above 0% and less than 2%). These similarities also hold for the percentages of the $MAVRD$ of the coupon CAT bond prices (the difference in percentages is less than 1.5%), meaning no significant influence of the loss models on the coupon CAT bond prices.

An explanation of the previous results is the quality of the original loss data,

Figure 14: The zero coupon (left panel) and coupon (right panel) CAT bond prices at time to maturity $T = 3$ years with respect to the threshold level $D \in [100, 135]$ million. The CAT bond prices under the Burr distribution (solid lines), the Pareto distribution (dotted lines) and under different loss models (different color lines)



Source: Own representation.

where 88% of the data is missing. In our data analysis, the expected loss is considerably more important for the CAT bond prices than the entire distribution of losses. This was due to the nonlinear character of the loss function and the dependence of different variables that affect the price of the CAT bond. For example, an earthquake with strength two Mw higher than the average strength might do more or less than twice the damage of an earthquake of average strength. Figure 14 presents the (zero) coupon CAT bond prices at time to maturity $T = 3$ with respect to the threshold level D, under the Burr and Pareto distribution for different loss models. The bond prices are more dispersed under different loss models with the same distribution assumption than under different distribution assumptions with the same loss model. This confirms the importance of the expected losses over the distribution of losses.

The relevance of the modeled-index loss trigger mechanism is that it considers different variables that influence the underlying risk. Because of the quality of the data, the previous empirical study showed that the modeled loss did not have influence on the CAT bond prices. However, for a given severity and frequency of earthquake risk, this analysis may be useful in determining how a CAT bond will be priced relative to an *expected level*.

5 Conclusion

Mexico has a high level of seismic activity due to the interaction between the Cocos plate and the North American plate. In the presence of this, the Mexican government has turned to the capital markets to cover costs of potential earthquake catastrophes, issuing a CAT bond that passes the risk on to investors. This paper examines the calibration of a real parametric CAT bond that was sponsored by the Mexican government and derives the price of a hypothetical modeled-index loss CAT bond for earthquakes.

Under the assumption of perfect markets, the calibration of the bond is based on the estimation of the intensity rate that describes the flow process of earthquakes from the two sides of the contract: from the reinsurance and the capital markets. Additionally, we estimate the historical intensity rate using the *intensity model* that accounts only earthquakes that trigger the CAT bond's payoff. However, the dataset contained only three such events, what leads to the decomposition of the calibration of the historical intensity rate into the calibration of the intensity of all earthquakes with a magnitude higher than 6.5 Mw and the estimation of the probability of the trigger event. The intensity rate estimates from the reinsurance λ_1 and capital market λ_2 are approximately equal but they deviate from the historical intensity rate λ_3. Assuming that the historical intensity rate would be the adequately correct one, the best argument to the low premiums for covering the seismic risk of 450 million might be the financial strategy of the government, a mix of reinsurance and CAT bond, where 35% of the total seismic risk is transferred to the investors.

This paper also derives the price of a hypothetical CAT bond for earthquakes with a modeled-index loss trigger mechanism, which takes other variables into account that can affect the value of losses, e.g. the physical characteristics of an earthquake. We price a modeled-index CAT bond price by means of a compound doubly stochastic Poisson process, where the trigger event depends on the frequency and severity of earthquakes. We observe that the (zero) coupon CAT bond prices increased as the threshold level D increased, but decreased as the expiration time T increased. This is mainly because the probability of a trigger event increases and more coupon payments are expected to be received. Because of the quality of the data, different loss models reveal no impact on the CAT bond prices and the expected loss is considerably more important for the evaluation of the modeled-index CAT bond than the entire distribution of losses.

The CAT bond's spread rate is reflected by the intensity rate of the earthquake process in the parametric trigger, while for the modeled loss trigger mechanism the spread rate is represented by the intensity rate of the earthquake process and the level of accumulated losses L_s. Without doubt, the availability of information and the quality of the data provided by research institutions attempting earthquakes has a direct impact on the accuracy of this risk analysis and for the evaluation of CAT bonds.

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