Low-energy properties of three resonantly interacting particles

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The scattering and bound states of a system of three resonantly interacting particles are analyzed. The characteristics of the scattering and the bound states are shown to be independent of the particular nature of the two-particle forces; instead, these characteristics are governed simply by the fact that resonances occur. The resonances lead to an effective long-range interaction among the three particles. The physical characteristics of the system obey a scaling law. Explicit equations are derived for the particle pair scattering length, for the particle pair scattering cross section at and above the threshold for disintegration of the three-particle system, and for the ternary-collision amplitude at zero energy. The properties of the system are compared with those of a system of two resonantly interacting particles.

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1. INTRODUCTION

The low-energy resonant scattering of two particles can be described by a single parameter: the scattering length. The scattering length is independent of the particular nature of the forces acting between the particles and in this sense is universal. The physical reason for this universality is that the wavelength is larger than the range of the forces, so that the structural details of the forces are unimportant. The theory of resonant scattering is a good framework for understanding the properties of the two-nucleon system at low energies.[1]

Nevertheless, this approach has not been used extensively to analyze three-particle systems, e.g., the three-nucleon system,[2,3] apparently because the general physical properties of three resonantly interacting particles are not understood nearly as well as those of the two-particle system. It was only comparatively recently, for example, that it was shown that a series of weakly bound states appears in the three-particle spectrum when the two-particle scattering length is large.[4,5]

In this paper we will study in detail the properties of the three-particle system under these physical conditions, i.e., under the conditions

$$a \gg r_0, \quad E \ll 1/r_0^2, \tag{1}$$

where a is the two-particle scattering length, r_0 is the range of the forces, E is the energy of the three particles, and the particles are assumed to have a unit mass. We will show that these properties are governed by an effective interaction of range $|a|$ among the three particles. This effective interaction results from the resonant nature of the two-particle forces. The properties of the three-particle system are universal under the conditions (1), since the effectice interaction is independent of the details of the two-particle forces.

It is this effective interaction which generates the state spectrum mentioned above. This interaction has the simple structure $1/R^2$ ($R^2 \sim r_{12}^2 + r_{23}^2 + r_{31}^2$), and this circumstance has interesting consequences for the low-energy characteristics of the three-particle system. We will show that a scaling law holds, relating the values of the various characteristics at different energies and different scattering lengths. Furthermore, the variation of the characteristics with the energy or the scattering length can be calculated explicitly, in terms of a few parameters. We will carry out these calculations for the particle–pair scattering length, for the particle–pair scattering cross section at and above the threshold for disintegration of the three-particle system, and for the ternary-collision amplitude at zero energy. All these properties exhibit characteristic oscillations.

We will compare the low-energy properties of systems of two and three resonantly interacting particles. Although the three-particle properties are considerably more complicated, the situation is fundamentally the same: There is a universality in both cases, and the properties are characterized by a single parameter. We are particularly interested in the variation of the three-particle properties with the range of the forces; this question necessarily arises because, for example, the number of levels in the spectrum of weakly bound states is an explicit function of r_0, varying in proportion to $\ln(|a|/r_0)$ (Refs. 4 and 5). The physical situation at $r_0 \to 0$ is also governed by the $1/R^2$ law. In the limit $r_0 = 0$, our results reduce to the known results for this case.

We assume spinless particles of identical mass. Since the physical mechanism remains the same in the more complicated cases, our results are of wider applicability and can be extended to these more complicated cases, as has been done for the weakly bound levels of the three-particle system.[6]

2. THREE-PARTICLE INTERACTION

The situation regarding the three-particle discrete spectrum under the conditions (1) has been clarified previously. As it turns out, there is necessarily (i.e., for all short-range forces) a rich spectrum of 0^+ levels.[4,5,7] The number of these levels initially increases with increasing two-particle attraction, then goes to infinity when the two-particle bound state appears, and finally decreases.

This unusual behavior leads to the conclusion that the two-particle resonance should also be important in the low-energy scattering of three particles. Our purpose in this paper is to examine this question and to construct a common description of the discrete spectrum and the scattering processes.

The physical fact which primarily determines the three-particle properties under the conditions (1) is the appearance of a long-range force, with a range on the order of $|a|$, among the three particles. This effect was discussed in detail in Ref. 5, where it was shown that the long-range behavior of the force at $r_0 \ll R \ll a$ is of the form $1/R^2$ (R is defined in the Introduction). The sign and magnitude of the force depend on the quantum numbers of the three-particle state. We emphasize that the long-range behavior is universal in the sense that it is independent of the particular nature of the forces. The only factor determining this behavior is the very fact that a resonance with a scattering length a occurs in the two-particle system. The low-energy properties of the three-particle system should thus also be universal.

The concept of the long-range interaction is central to this discussion, so we will review its basic features.[5,6] It is not difficult to see why the long-range interaction arises. If, for example, a slow particle is approaching a bound pair the exchange (the transfer of a particle from the pair to the incident particle) begins at a distance on the order of the pair radius a. It is quite possible for this exchange interaction to be attractive, as we can see from a second example: the exchange of a light particle between two heavy particles. It is well known that the exchange in this case causes the heavy particles to attract each other if the spatial configuration of the light particle is chosen appropriately. In the case of three identical particles, the physical situation is similar: Attraction occurs in the configuration most favorable for exchange—in the symmetric 0^+ state. In other states there is a repulsion. Finally, the functional form of the long-range interaction—the $1/R^2$ law—is also a natural result. It holds when the scale of distances between particles satisfies $r_0 \ll r_{ik} \ll a$. In a first approximation we can then assume $r_0 = 0$ and $a = \infty$; then we are left with no dimensionless parameters to characterize the forces. Under these conditions the $1/R^2$ interaction is the simplest combination (of the necessary dimensionality and symmetry) of the dimensionless quantities which remain: the relative separations of the particles.

We can see that there is an important difference between the symmetric 0^+ state and the others. For these other states, the physical situation is simply a low-energy scattering by a repulsive $1/R^2$ barrier of range $|a|$. It is difficult for the particles to approach each other to small distances R, and in a first approximation in the parameters in (1) we are quite safe in setting $r_0 = 0$. In the 0^+ state, in contrast, the particles attract each other over a large volume; they easily approach each other to small distances; and they form many bound levels. In this case we cannot set r_0 equal to zero, because this procedure would lead to the collapse of the three-particle system in the $1/R^2$ field. Figure 1 illustrates the three-particle interaction.

Below we will need the exact form of the $1/R^2$ law for

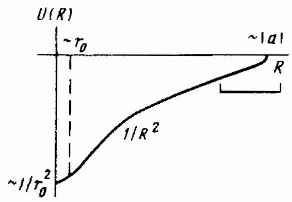

FIG. 1. Attraction of three resonantly interacting particles. The attraction profile at $R > r_0$ is governed by the resonances and is independent of the details of the particular two-particle forces. The interaction in the region indicated by the bracket is parametrized in Section 4.

the 0^+ states; it is[5] $|s_0|^2/R^2$, where $|s_0| \approx 1.01$. The number of three-particle levels in this field is $(|s_0|/\pi) \times \ln(|a|/r_0)$.

3. SCALING LAW

With this physical picture in mind we turn to the three-particle scattering processes. Figure 2 shows the possible scattering channels along with the bound-state region. Here $K = \pm |E|^{1/2}$, where the sign of the square root is chosen[1)] to be the same as the sign of E. The region $a^{-1} < 0$ corresponds to a virtual level in the two-particle system, while the region $a^{-1} > 0$ corresponds to a bound level. The a^{-1} axis can also be understood as the coupling-constant axis for the two-particle force, g, since the deviation δg from the value at which the two-particle bound state forms is proportional to a^{-1}. Above the boundary of the three-particle continuum, but still with $K < 0$, the only possibility is elastic scattering of one particle by a pair. At $K > 0$, the channel with three free particles is open.

The physical quantities (the scattering phase shifts and cross sections) depend on the variables a^{-1} and K. We will show that detailed information on this behavior can be found from the physical arguments above, without solving the three-particle equations. We are concerned primarily with the symmetric 0^+ state, which is the most interesting and the most complicated case.

The properties of the phase shifts and cross sections are conviently formulated in terms of polar coordinates on the (a^{-1}, K) plane: the radius H and the angle ξ. The first property may be called a scaling law: At a fixed ξ, a change in H by a factor $e^{\pi/|s_0|}$ is simply a scale transformation of the quantities with a coefficient $e^{\pi/|s_0|}$. For example, the particle-pair scattering length changes

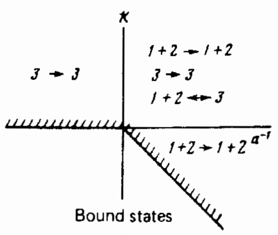

FIG. 2. Classification of three-particle processes. The hatching shows the boundary of the three-particle continuum. The numbers denote the scattering processes which are possible for the given energy of the three particles, $E = K^2 \text{sign} K$, and for the given two-particle scattering length a.

by a factor $e^{-\pi/|s_0|}$, the binding energy changes by a factor $e^{2\pi/|s_0|}$, and so forth. This property is a generalization of the scattering properties established in Ref. 5. It follows from dimensionality considerations and is a consequence of the fact that the $1/R^2$ interaction contains no dimensionless parameters.

Before demonstrating the validity of this scaling law we recall that, in principle, the procedure for solving the problem for the 0^+ state is the same as for the motion of a particle in an attractive $1/R^2$ field. Specifically, some boundary R_0 is chosen such that

$$R_0 \ll a, \quad KR_0 \ll 1, \quad R_0 \gg r_0. \tag{2}$$

Then the wave function in the outer region ($R > R_0$) is found from the solution of the Schrödinger equation and the boundary condition on the logarithmic derivative at $R = R_0$. The logarithmic derivative, which reflects the properties of the interaction at $R < R_0$, is independent of a and K and is a parameter of the theory.

Now returning to the scaling law, we take the wave function in the outer region for some radius H_1 and some value of ξ, and we change the scale for the variable R by some factor, say ν. The new function again satisfies the Schrödinger equation in the outer region, for the value $H_2 = H_1\nu^{-1}$ and for the same value of ξ. The only dimensionless parameters in the outer region are a and K, so that a change in the length scale by a factor ν is equivalent to changes in a and K^{-1} by the same factor. Here H is changed by a factor ν^{-1}, while ξ remains constant.

The wave function obtained in this manner, however, does not satisfy the boundary condition, since this boundary condition is again imposed at R_0, not at νR_0. Then the wave function is not a solution of our problem. Only when ν is such that the size of the region $R > R_0$ is a half-wave (or some integral number of half-waves) can the boundary condition again be satisfied. To determine what the value of ν must be, we note that the R variation of the wave function at $R \sim R_0$ is[5]

$$\sin(|s_0| \ln HR + \theta), \tag{3}$$

where θ is some phase angle.[2)] According to the arguments above, this angle can be assumed constant, equal to $\theta(H_1)$, as ν varies. The region in question is equal to a half-wave in size if

$$|s_0| \ln H_2R_0 - |s_0| \ln H_1R_0 = |s_0| \ln \nu^{-1} = -\pi,$$

i.e., if

$$\nu = e^{\pi/|s_0|}.$$

With this value of ν we thus again have a solution for our problem; this new solution can be seen to differ from the original solution by the transformation of the length scale.

We have thus shown that a change in H by a factor $e^{\pi/|s_0|}$ reduces to a scale transformation with a coefficient $e^{\pi/|s_0|}$, i.e., a scaling law. For identical particles, $e^{\pi/|s_0|} \approx 22.5$.

The significance of this scaling law is that it determines the general structure of the behavior of the physical quantities in region (1). It follows from this law

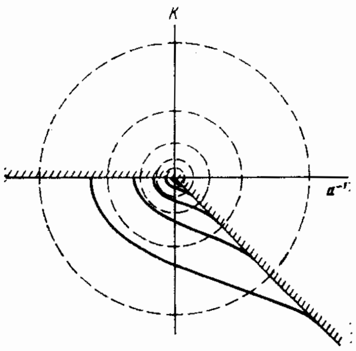

FIG. 3. Scaling law for the physical characteristics of the three-particle system. In adjacent rings, the scattering cross sections, the binding energies, and other characteristics differ by only a scale transformation. The rings are crowded together toward the origin. The trajectories of several bound states,[5] which also obey this scaling law, are shown.

that at $H \ll 1/r_0$ the (a^{-1}, K) plane can be broken up into concentric rings with a radius ratio $e^{\pi/|s_0|}$, crowded together into the origin (Fig. 3). Corresponding to each ring is one bound level. Once the values of the physical quantities are found in one of the rings, the other values can be found by the scale transformation.

4. RADIAL LAW

In addition to the scaling, the physical quantities exhibit an interesting property which we will call the radial law. It turns out that the variation of these quantities with H at a fixed ξ can be calculated explicitly in terms of a few parameters.

Let us see why this is possible (here again it is useful to bear in mind the potential problem with a $1/R^2$ attraction cut off at $R \sim r_0$ and $R \sim a$; see Fig. 1). After the boundary condition at $R = R_0$ is imposed, the wave function can be written immediately for $R \ll a$; it is the wave function in an attractive $1/R^2$ field. To find the scattering amplitude, we need to examine the asymptotic behavior, i.e., to go through the region $R \sim a$. The interaction in this region (shown by the bracket in Fig. 1) can be treated as some potential barrier and can thus be assigned reflection and transmission coefficients. To determine these coefficients, it is necessary in general to solve the Schrödinger equation for $R \sim a$. If, however, we are interested in the behavior of the scattering amplitudes as a function of H for a fixed ξ, then it is not necessary to know the solution for $R \sim a$. The reflection and transmission coefficients, being dimensionless, depend on only ξ, not H. Then for a fixed value of ξ these coefficients are certain constants. Using these constants to specify the barrier, we can construct a wave function for $R \gg a$ and extract scattering amplitudes from it. The variation of these amplitudes with H is thus found explicitly; the characteristics of the interaction in the region $R \sim a$ serve as parameters.

We will go through the calculations for the elastic 0^+ scattering phase shift below the three-particle threshold, with the goal of subsequently finding the particle-pair scattering length. The calculations are simple, essentially the same as those for the potential problem.

All aspects of the problem which are peculiar to the three-particle nature of the system are incorporated in the barrier characteristics, which we treat as parameters.

The calculations are carried out in the following order.

1. The wave function at $R \sim R_0$ is found; we have already written an expression for it[3] [see Eq. (3)]. The wave function is the superposition of two waves,

$$e^{i|s_0|\ln HR} \text{ and } e^{-i|s_0|\ln HR}, \tag{4}$$

which satisfies the boundary condition at $R = R_0$. The boundary condition is satisfied by choosing the phase θ approximately. Denoting by Λ the logarithmic derivative at $R = R_0$, we have the following equation for θ:

$$\theta = -|s_0| \ln HR_0 + \operatorname{arcctg} \frac{\Lambda R_0}{|s_0|}. \tag{5}$$

2. The barrier characteristics are determined. For this purpose we examine the scattering of the first wave in (4) by this barrier. We normalize the wave to a unit flux and denote it by $\psi_1^{(+)}$. As a result of scattering by the barrier, the wave is partially reflected back into the region $R \ll a$ and partially transmitted into the region $R \gg a$. These barrier properties are described by the 2×2 scattering matrix s. The diagonal element s_{11} of this matrix determines the reflection, while s_{12} determines the transmission. Correspondingly, the matrix element s_{22} determines the reflection of a wave incident on the barrier from $R \gg a$. As usual, the s matrix is unitary and symmetric. In accordance with the discussion above, it depends on only ξ.

Then as a result of the scattering of the wave $\psi_1^{(+)}$ by the barrier we have at $R \sim R_0$ the wave function

$$\psi_1^{(+)} - s_{11}\psi_1^{(-)}, \tag{6}$$

where $\psi_1^{(-)}$ is the reflected wave, normalized to a unit flux [the second wave in (4)]. In the region $R \gg a$ the wave function is

$$-s_{12}\psi_2^{(+)}. \tag{7}$$

The diverging wave $\psi_2^{(+)}$, also normalized to a unit flux, describes a particle and a pair which are moving apart.

3. Armed with the barrier characteristics, we can construct a solution for $R \gg a$ corresponding to a superposition of the waves in (3). For this purpose, we write $(|s_0| \ln HR + \theta)$ as the sum of the wave function in (6) and the complex conjugate function:

$$\sin(|s_0| \ln HR + \theta) = c(\psi_1^{(+)} - s_{11}\psi_1^{(-)}) + c^*(\psi_1^{(-)} - s_{11}^*\psi_1^{(+)}), \tag{8}$$

where c is a complex coefficient. According to (7), at $R \gg a$ a linear combination of this type becomes

$$-cs_{12}\psi_2^{(+)} - c^*s_{12}^*\psi_2^{(-)}. \tag{9}$$

This is the solution of our problem for $R \gg a$. The ratio of the coefficients of the converging and diverging waves in (9) determines the phase shift of the particle-pair scattering:

$$e^{2i\delta} = -\frac{s_{12}}{s_{12}^*}\frac{c}{c^*}.$$

An expression for the ratio c/c^* here can be found from (8) by equating the ratio of the coefficients of the waves $e^{\pm i|s_0| \ln HR}$ on the right and left sides. As a result we find the following equation for δ:

$$e^{2i\delta} = e^{2i\theta}\frac{s_{12}}{s_{12}^*}\frac{1 - s_{11}^* e^{-2i\theta}}{1 - s_{11}e^{2i\theta}}. \tag{10}$$

Equation (10) [with (5)] is the result we have been seeking. Examining this equation, we first note that all the variation with H is incorporated in the phase θ, so that this variation has been found explicitly. Second, the expression for $e^{2i\delta}$ has poles, which determine the positions of bound states, as usual. The condition on the poles,

$$s_{11}e^{2i\theta} = 1, \tag{11}$$

has a simple meaning. The wave is first reflected from the barrier with a coefficient s_{11} and then from the region $R < R_0$ with a coefficient $e^{2i\theta}$. There is coupling when the double reflection restores the original wave. After Eq. (5) for θ is substituted into (11), the equation found in Ref. 5 for the spectrum of weakly bound three-particle states is found, as expected.[4] Finally, Eq. (10) illustrates a scaling law. As H is increased by a factor $e^{\pi/|s_0|}$, the phase θ decreases by π, and the scattering phase shift δ also decreases by π. This is as it should be, since the phase shift δ, being dimensionless, should remain the same (within $N\pi$) after the scale transformation. Its decrease by π reflects the physical fact that when H is increased by a factor $e^{\pi/|s_0|}$ the area occupied by the one three-particle bound state becomes smaller (Fig. 3).

We should also emphasize that Eq. (10) is universal: The matrix elements s_{ik} do not depend on the particular nature of the forces acting between the particles, since these matrix elements describe the wave transformation by the long-range interaction in the region $R \sim a$. Everything which depends on the nature of the forces is incorporated in the single constant Λ.

Equation (10) is analogous in meaning to the familiar low-energy equation for the phase shift for two-particle resonant scattering[8]:

$$e^{2i\delta} = \frac{1 - ika}{1 + ika}. \tag{12}$$

Both these equations are universal; the variation with the forces is concentrated in a single parameter. The more complicated structure of (10) is a consequence of the long-range interaction.

5. CALCULATION OF THE THREE-PARTICLE CHARACTERISTICS

We will now calculate the particle-pair scattering length. All the quantities in (10) must be expanded around the threshold (the sloping line in Fig. 2 with the hatching; $\xi = -\pi/4$). For this purpose we rewrite (10) in a slightly different form, using the unitarity of the s matrix:

$$e^{2i\delta} = \frac{s_{12}e^{i\theta} + s_{22}s_{12}^* e^{-i\theta}}{s_{12}^* e^{-i\theta} + s_{22}^* s_{12} e^{i\theta}}. \tag{13}$$

The matrix element s_{22} is unity at the threshold (see below), so that we have $e^{2i\delta} = 1$, as assumed. Near the threshold the variation of the phase shift δ with the momentum is governed by the scattering length A:

$$e^{2i\delta}=1-2i\varkappa A,$$

where $\varkappa$ is the relative momentum of the particle and the pair. Expanding (13) in powers of the magnitude of the deviation of s_{22} from unity, we find an equation for A:

$$A=-\frac{1}{2i\varkappa}\left[i\,\mathrm{Im}\,\delta s_{22}+\frac{s_{12}e^{-i\theta}-s_{12}e^{i\theta}}{s^{*}_{12}e^{-i\theta}+s_{12}e^{i\theta}}\mathrm{Re}\,\delta s_{22}\right], \quad (14)$$

where $\mathrm{Re}\delta s_{22}$ and $\mathrm{Im}\delta s_{22}$ are the real and imaginary parts of δs_{22}.

It is a simple matter to determine the behavior of the matrix elements s_{22} and s_{12}. Starting with s_{22}, we recall that this matrix element describes the reflection of the wave incident on the barrier from the outer region, $R\gg a$. In the limit $\varkappa\to 0$, there is a low-energy scattering which is accompanied by a process which is "inelastic" from the standpoint of channel 2: penetration into the region $R\ll a$. In this situation, the scattering length is complex,[8] with the real part determining the reflection and the imaginary part determining the switch to the "inelastic" channel. The threshold behavior of s_{22} is thus

$$s_{22}=1-2i\varkappa B=1-2i\varkappa B_1-2\varkappa B_2,$$

where $B=B_1-iB_2$ is a complex length. The only characteristic dimension of the barrier is a, so that the length B is proportional to a:

$$B_1=b_1a,\quad B_2=b_2a.$$

The proportionality coefficients b_1 and b_2 are dimensionless, and $b_2>0$, since the penetration into the region $R\ll a$ reduces the reflected flux.

With regard to the threshold behavior of the phase of the matrix element s_{12} [the modulus of this matrix element cancels out in (14)], we note that in the limit $\varkappa\to 0$ this phase is some constant. The wave number of the transmitted wave in the region $R\sim a$ is not zero even in the case $\varkappa=0$. As the wave propagates through this region it thus acquires some nonvanishing phase shift as $\varkappa=0$. We denote this phase shift by Δ_{12}^0: $s_{12}\sim e^{i\Delta_{12}^0}$.

Using these properties of the matrix elements, we find the particle-pair scattering length from (14):

$$A=B_1-B_2\,\mathrm{tg}\,(\theta+\Delta_{12}^0). \quad (15)$$

Let us examine the physical content of Eq. (15). The scattering length A is formed, in the first place, as the result of reflection from the barrier. The quantity B_1, which is a measure of the reflection, constitutes the first term in (15). In addition there is a correction for the penetration into the region $R\ll a$. This correction is proportional to B_2, which is a measure of the penetration through the barrier. Furthermore, since there are bound states in this region, a plot of the scattering length A as a function of a shows that A goes to infinity whenever some bound state goes into the continuum. This is the meaning of the factor $\tan(\theta+\Delta_{12}^0)$.

Writing out the explicit expressions for B_1, B_2, and θ [in Eq. (5) for θ we must set $H=a^{-1}\sqrt{2}$ since $\xi=-\pi/4$], we finally find

$$A=a\left[b_1+b_2\,\mathrm{tg}\left(|s_0|\ln a^{-1}R_0\sqrt{2}-\mathrm{arcctg}\frac{\Lambda R_0}{|s_0|}-\Delta_{12}^0\right)\right]. \quad (16)$$

The variation with a has thus been found explicitly in terms of a few parameters. The parameters b_1, b_2, and Δ_{12}^0 are measures of the long-range interaction and are the same for all two-particle forces. The fourth parameter, Λ, is a measure of the interaction among the three particles at short ranges $R\sim r_0$. As expected, the scattering length A increases with increasing a, since it must be equal in order of magnitude to the size of the pair, a. Superimposed on this smooth geometric variation, however, are characteristic oscillations due to the appearance of new bound states with increasing a. On the interval between two values of a differing by a factor $e^{\pi/|s_0|}$ the scattering length A must pass through infinity and zero; at the ends of this interval, the values of A also differ by a factor $e^{\pi/|s_0|}$, in accordance with the scaling law. The pattern is repeated in the next interval, and so on.

The profile of the function $A(a^{-1})$ is actually governed by a single parameter: the ratio b_1/b_2. The changes in the other parameters simply change the scale. A positive value of b_1 ensures that A will pass through infinity in the correct way as the coupling constant g increases (as a^{-1} increases); specifically, A goes from positive to negative values, as it should when the level goes into the continuum.

In analogous calculations, we find several other characteristics.

1. The cross section for 0^+ scattering at the threshold for disintegration of the three-particle system ($E=0$; the positive part of the horizontal axis in Fig. 2). At the threshold, the kinetic energy of the relative motion of the particle and the pair is equal to the pair binding energy, a^{-2}. Then we can write the cross section as

$$\sigma=3\pi a^2\sin^2\delta, \quad (17)$$

where δ is given by (10) with the threshold values of the s-matrix parameters with $H=a^{-1}$. The structure of Eqs. (17) and (16) is similar in the sense that there is a smooth geometric variation, $\sigma\sim a^2$, on which the oscillations required by the scaling law are superimposed. The cross section necessarily goes through the values of zero and $3\pi a^2$ over an a interval of a factor of $e^{\pi/|s_0|}$, since the scattering phase shift changes by π over this interval. We thus have a sort of Ramsauer–Townsend effect: The cross section for 0^+ scattering goes through zero $(|s_0|/\pi)\ln(a/r_0)$ times as a increases, reaching the geometric value on the intervals between the zeros.

The cross sections for the other partial waves do not have these oscillations; they are simply proportional to a^2.

2. The amplitude for three-particle collisions[5)] at zero energy ($E=0$; the negative part of the horizontal axis in Fig. 2). This is a low-energy characteristic analogous to the scattering length. It has the dimensions of a^4, since the phase space of the three particles contains three additional linear degrees of freedom. The equation for this amplitude is similar to (16):

$$M=a^4\left[d_1+d_2\,\mathrm{tg}\left(|s_0|\ln|a^{-1}|R_0-\mathrm{arcctg}\frac{\Lambda R_0}{|s_0|}-\Delta_{13}\right)\right],$$

where d_1 and d_2 have meanings analogous to b_1 and b_2 ($d_2>0$), and Δ_{13}^0 is the threshold phase shift for transmission through the region $R\sim a$ of a wave incident from the channel of three free particles. There is little to add to the comments regarding Eq. (16). The amplitude M becomes infinite when a bound state forms from the three-particle continuum. The law describing the passage through infinity as g is increased ($|a^{-1}|$ is reduced) is opposite to that for A: The amplitude goes from negative to positive values in this case. This difference is necessary, since in one case levels are appearing as g increases, while in the other case levels are disappearing.

3. The cross section for elastic and inelastic 0^+ scattering of a particle by a pair above the three-particle threshold. The equation for the phase shift in (10) is generalized as follows:

$$e^{2i\delta}=e^{2i\theta}(-D)\frac{1-s_{22}D^{-1}e^{-2i\theta}}{1-s_{11}e^{2i\theta}}.$$

The ratio s_{12}/s_{12}^* is replaced by $-D$, where D is the determinant of the 2×2 s matrix, which is again symmetric but in this case not unitary, because of the inelastic channel. For the same reason, the phase shift δ is now complex, with its imaginary part governing the inelastic scattering. The qualitative behavior of the cross sections as functions of H is the same as for the quantities discussed above: Superimposed on the smooth variation $\sigma\sim H^{-2}$ are oscillations with a logarithmic period $\pi/|s_0|$. The dips in the elastic cross section are filled in by the inelastic process.

6. COMPARISON OF THE PROPERTIES OF THE TWO-PARTICLE AND THREE-PARTICLE SYSTEMS

Let us summarize the results. In the region defined by conditions (1) the properties of the three-particle system are completely standardized. The nature of the level spectrum and the behavior of the scattering amplitudes are the same for all two-particle forces. The situation is thus analogous to the two-particle case.

To demonstrate the analogy more clearly, we will discuss in parallel the properties of the systems of two and three resonantly interacting particles.

1. *Number of parameters*. For two particles, the low-energy properties are governed by a single parameter, a. This parameter is a measure of the interaction between the two particles at distances $r\sim r_0$. In the case of three particles, the low-energy properties are governed by the single parameter Λ, which is a measure of the interaction among the three particles at distances $R\sim r_0$. The interaction of two particles in a three-particle system is governed by the parameter a, as in a two-particle system.[6]

To determine the properties of the three-particle system it is necessary to specify the value of the parameter Λ. This can be done directly or indirectly, by specifying one physical property (the level binding energy, the particle-pair scattering length, etc.) for some value of a. For definiteness we will speak in terms of the particle-pair scattering length.

In addition to Λ, the equations for the level spectrum, the scattering length, and similar properties contain the joining radius R_0. Clearly, the physical result must be independent of the arbitrary choice of R_0 [the only restrictions on this choice are the conditions (2)]. Then for a fixed force there will be a particular value of Λ corresponding to each choice of R_0, and any physical quantity should contain some combination of Λ and R_0 which is invariant with respect to the choice of R_0. In the equations above and in Ref. 5, this combination is the distance $R_0\exp\{-(1/|s_0|)\mathrm{arcctg}(\Lambda R_0/|s_0|)\}$, which gives us the position of the zero in the wave function at $R\sim R_0$, as follows from (3) and (5).

2. *Physical properties*. After the parameters have been specified, both the low-energy theories are completely determined for the energies and scattering lengths satisfying the conditions (1). The properties of the three-particle system are richer because of both the larger number of channels and the long-range interaction. In particular, the single level of the two-particle problem is replaced by a series of levels; the monotonic variation of the cross sections is replaced by oscillations; etc.

3. *Variation with the range of the force*. For the case of two-particles, we can set $r_0=0$ in the equations of the low-energy theory. In other words, the values of the physical quantities are independent of r_0. Here r_0 governs only the range of applicability of the theory: the conditions (1).

The situation is the same for the three-particle states other than the symmetric 0^+ state. As mentioned above, here again we can immediately set $r_0=0$. For the 0^+ state the situation seems different at first glance, since, for example, the number of levels is an explicit function of r_0: $N\sim\ln(|a|/r_0)$. However, after the particle-pair scattering length (for example) is specified, the physical quantities are determined, regardless of the particular value of r_0.

To pursue this question, we consider an example which reflects the known fact that the three-particle system collapses in the limit $r_0\to0$ (Refs. 9 and 10). We assume that two particular forms of forces have very different ranges ($r_{01}\gg r_{02}$) but the same particle-pair scattering length. We choose a joining distance R_0 which is the same for the two cases and which is larger than both r_{01} and r_{02}. Outside R_0, the long-range interaction is the same in the two cases. The logarithmic derivative Λ is also the same at R_0 in the two cases, since the particle-pair scattering length is the same. Then the low-energy properties should be identical in the two cases for values of a and K satisfying conditions (2). A difference arises only when distances between r_{01} and r_{02} come into play, i.e., outside the range of applicability in (1) for the firstcase but still inside the range of applicability for the second:

$$r_{02}\ll a\ll r_{01},\quad 1/r_{01}\ll K\ll 1/r_{02}.$$

For these values of a and K the features of the low-energy theory discussed above disappear for the case with the range r_{01}, bu they are retained for the case with r_{02}. In particular, there will be an additional $\delta N = (|s_0|/\pi)\ln(r_{01}/r_{02})$ bound states, with radii in the intermediate interval. Then, strictly speaking, the designation of the levels in the region defined in (2) (where the properties are the same) is different: The index of the level having the same energy is different by an amount δN (the scattering phase shifts differ by the same amount, in units of π). This difference in designation is the only difference between the two cases, so that even in the limit $r_{02} \to 0$ there is no effect on the behavior of the physical quantities in the region defined in (2), except that the index assigned to the levels will be infinite.

The reason for this situation is, again, that the $1/R^2$ three-particle interaction does not contain any parameter with the dimensions of length with which we could associate r_0. In the ordinary potential problem with a $1/R^2$ force we would have the same situation with regard to the variation with r_0.

In summary, for the 0^+ state (aside from this question of the numbers assigned to the levels) the physical quantities are independent[7)] of r_0.

What happens on the (a^{-1}, K) plane in the limit $r_0 \to 0$? Before Λ is specified, the radial scale of variation of the physical quantities is not fixed [a change in Λ is equivalent to a change in the scale along the variable H, as can be seen from (5)]. For example, the level trajectories, without changing in profile, may move in the radial direction. The specification of Λ implies fixing the scale. The level trajectories "freeze" in place. The only effect of a decrease in r_0 is that the boundaries of this frozen pattern at $H \sim 1/r_0$, move apart. In the case $r_0 = 0$, this scene covers the entire (a^{-1}, K) plane.

The situation is different if we reduce r_0 without fixing the particle-pair scattering length.[5] In this case the levels will move one by one to higher binding energies. The entire pattern on the (a^{-1}, K) plane swells radially without bound. The particle–pair scattering length does not have a definite limit; for any specified value of a this length oscillates as a function of r_0 with a logarithmic period $\pi/|s_0|$ [see Eq. (16), where it is necessary to take the limit $R_0 \to 0$, holding $\Lambda R_0 = \text{const}$]. The same is true of the other physical quantities.

4. *Solution method*. For the two-particle case, the low-energy problem can be solved analytically [Eq. (12) and similar equations].

In the three-particle case, the problem reduces to the solution of either a one-dimensional integral equation for a zero force range[13,3] or a two-dimensional free Schrödinger equation with boundary conditions.[5] For all states except the symmetric 0^+ state the solution decays at small distances (at large momenta). For the 0^+ state, this solution oscillates. The oscillation phase is determined from the boundary condition at $R = R_0$. Solving the problem in the momentum representation, we can replace R_0 by the large momentum K_0 and use an analogous procedure. An equivalent approach is to truncate the kernel of the integral equation at the momentum K_0 (Ref. 14). A change in K_0 corresponds to a different choice of R_0, while a change in the truncation method corresponds to a different choice of the parameter Λ. According to Subsection 3 of this section of the paper, if R_0 is much smaller than the characteristic distances of the problem (correspondingly, if K_0 is much larger than the characteristic momenta), then neither a change in R_0 and K_0 nor a change in the truncation method (provided that, say, the particle-pair scattering length is left invariant) will affect the results. In particular, we can take the limit $R_0 \to 0$, $K_0 \to \infty$ (Ref. 11).

7. CONCLUSION

These results have been found through arguments based on the concept of a long-range interaction in a system of three resonantly interacting particles. As we have shown, this physical concept is very helpful for reaching an understanding of the diversity of properties of the three-particle system. This concept may prove useful for a qualitative study of the properties of the three-nucleon system at low energies ($\lesssim 20$ MeV). Although many experimental and theoretical results have been found for this case, we lack the qualitative physical arguments for combining these results into a common framework. We will publish separately an analysis of the three-nucleon system based on the concept of the long-range interaction.

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1) For $E < 0$, this definition differs in sign from that used in Ref. 5 and is more convenient for a simultaneous study of the discrete and continuous spectra.

2) We are putting H, rather than K (as in Ref. 5), in the logarithm. This approach shows how the wave function varies with the variables in which we are interested. The replacement of K (more precisely, $|K|$), by H in accordance with footnote 1 corresponds to a redefinition of the phase, specifically, the addition of a term $|S_0|\ln|\sin\xi|$, which is unimportant for our purposes.

3) Actually, the wave function contains an angular part in addition to the radial part, given by Eq. (3). This angular part depends on dimensionless combinations of the three-particle distances.[5] It is unimportant in these calculations and thus is omitted.

4) The matrix element s_{11} must be written $s_{11} = e^{-2i\Delta}$, and we need to take into account footnote 2 regarding the phase Δ.

5) By "three-particle collisions" here we mean processes in which three particles come together within a volume $\sim a^3$.

6) Thomas[9] was the first to point out the need to introduce a three-particle parameter, when he demonstrated that the binding energy of three particles could be arbitrary for a given two-particle binding energy.

7) This assertion was actually made by Danilov,[11] who used the theory of a zero force range in calculations for the three-nucleon S state. The assertion was later questioned,[12] and the situation is actually still not clear.

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Nonscaling corrections to the asymptotic amplitude for large-angle elastic scattering of particles with spin

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The Logunov-Tavkhelidze quasipotential approach is used to study the corrections of first and second order in $1/(S)^{1/2}$ to the amplitudes for elastic meson-nucleon and nucleon-nucleon scattering at large angles. The results are compared with experiment.

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1. INTRODUCTION

The high-energy behavior of the differential cross sections for the large-angle scattering of hadrons is one of the interesting problems in high-energy physics. The assumption that the particles are composite in nature, combined with a scaling principle, leads to important conclusions regarding the asymptotic behavior of the differential cross sections for two-particle scattering[1,2]:

$$\frac{d\sigma}{dt}=\frac{1}{s^N}f\left(\frac{t}{s}\right);\ \frac{t}{s}\ \text{fixed}. \quad (1.1)$$

These conclusions correctly describe the basic aspects of the corresponding experimental data (see, for example, Ref. 3).

The Logunov-Tavkhelidze quasipotential approach is used in Refs. 4–7 to study this question. That work shows that the phenomenological quasipotentials specified by integral representation of the type

$$\hat{g}(s,\Delta)=g(s)\int_0^\infty dx\hat{\rho}(s,x)e^{-x\Delta^2},\quad t=-\Delta^2, \quad (1.2)$$

lead to a power-law asymptotic behavior of the differential cross sections like that in (1.1) for the case in which there is a weak limit for the function $\hat{\rho}(s,x)$:

$$\lim_{s\to\infty} s^M\hat{\rho}(s,x=\eta/s)=\hat{\psi}(\eta),\quad 0<\eta<\infty,\quad M>0. \quad (1.3)$$

In those energy ranges for which experimental data are available on large-angle elastic scattering [$s_{max}\sim 10$–50 (GeV)^2 for various reactions], the corrections to the leading asymptotic term are extremely important. This problem was studied in Ref. 8 for the scattering of two spinless particles of identical mass, and it was shown there that the corrections can be large, even as $s\sim 50$ (GeV)^2.

In the present paper we extend the method developed in Ref. 8 to high-energy, large-angle meson–nucleon and nucleon–nucleon scattering. We will lean heavily on the requirement of γ_5-invariance of the interaction at high energies at a large momentum transfer.[9]

In Section 2 we describe the general method for calculating the corrections of first and second order in $1/\sqrt{s}$ to the leading asymptotic term in the scattering amplitude. Specific calculations are carried out for meson–nucleon and nucleon–nucleon scattering in Sections 3 and 4, respectively. The equations are compared with experimental data and the results discussed in Sec. 5.

2. DESCRIPTION OF LARGE-ANGLE SCATTERING FOR THE CASE OF ANALYTIC QUASIPOTENTIALS

Let us examine the quasipotential equation describing the interaction of particles with spins, which we write in the general form[10,11]

$$\hat{G}(s,\mathbf{p},\mathbf{k})=\hat{g}(s,\mathbf{p},\mathbf{k})+\int d\mathbf{q}\hat{g}(s,\mathbf{p},\mathbf{q})\frac{\hat{A}(s,\mathbf{q})}{E^2(\mathbf{q})-E^2-i0}\hat{G}(s,\mathbf{q},\mathbf{k}), \quad (2.1)$$

where $\mathbf{p}$ and $\mathbf{k}$ are the momenta of the particle before and after the collision, $E=\sqrt{s}=\sqrt{m_1^2+\mathbf{p}^2}+\sqrt{m_2^2+\mathbf{p}^2}$ is the total energy of the particles in the c.m. frame, $E(\mathbf{q})=\sqrt{m_1^2+\mathbf{q}^2}+\sqrt{m_2^2+\mathbf{q}^2}$, m_1 and m_2 are the masses of the

