Loss Aversion

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Abstract: Loss aversion is traditionally defined in the context of lotteries over monetary payoffs. This paper extends the notion of loss aversion to a more general setup where outcomes (consequences) may not be measurable in monetary terms and people may have fuzzy preferences over lotteries, i.e. they may choose in a probabilistic manner. The implications of loss aversion are discussed for expected utility theory and rank-dependent utility theory as well as for popular models of probabilistic choice such as the constant error/tremble model and a strong utility model (that includes the Fechner model of random errors and Luce choice model as special cases).

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JEL classification codes: D00, D80, D81

Loss Aversion

Loss aversion is one of the most important concepts in behavioral economics (Camerer, 2008). It is consistent with a wide range of empirical findings such as the endowment effect (Thaler, 1980; Kahneman et al., 1990), status quo bias (Samuelson and Zeckhauser, 1988), equity premium puzzle (Benartzi and Thaler, 1995), labor supply of cabdrivers (Camerer et al. 1997), disposition effects in condominium sales (Genesove and Mayer, 2001) and animal behavior (Chen et al. 2006) to name a few.

Loss aversion is traditionally defined in the context of lotteries over monetary payoffs (Kahneman and Tversky, 1979; Köbberling and Wakker, 2005; Schmidt and Zank, 2005). However, people often incur losses that may not be measurable in monetary terms (e.g. loss of a close friend or a relative, loss of faith, reputation or prestige, loss of a sports title, loss of animal species etc). This paper extends the notion of loss aversion to decision problems where outcomes (consequences) may not be measurable in monetary terms.

Numerous experimental studies demonstrate that people generally have fuzzy preferences over lotteries, i.e. they choose in a probabilistic manner (e.g. Camerer, 1989; Hey and Orme, 1994; Loomes and Sugden, 1998). Therefore, this paper also extends the notion of loss aversion to allow for the possibility of fuzzy preferences.

The paper is organized as follows. Section 1 defines comparative loss aversion in the context of an arbitrary outcome set. Section 2 considers the implications of the proposed definitions of comparative loss aversion for expected utility theory. Section 3 does the same for rank-dependent utility theory. Section 4 extends the notion of loss aversion to a more general setup where people have fuzzy preferences over lotteries. Section 5 discusses probabilistic loss aversion in the context of different models of probabilistic choice. Section 6 defines absolute loss aversion. Section 7 concludes.

1. Comparative Loss Aversion

Let X denote a finite set of outcomes (consequences) that contains at least two elements. We will treat X as an arbitrary abstract set, which is not necessarily a subset of Euclidean space $\mathbb{R}^n$. Let $X_{-}\subset X$ be a nonempty proper subset of X. The elements $x_{-}\in X_{-}$ are called losses and they can be, for example, "loss of $100", "loss of a key chain", "loss of faith", "loss of virginity" etc. Let $X_{+} \equiv X \setminus X_{-}$ denote the complement of X_{-}. The elements $x_{+}\in X_{+}$ are called gains and they can be, for example, "gain of $200", "gain in experience", "weight gain" etc. If an outcome "loss of A" is in X_{-} this does not imply that a symmetric outcome "gain of A" necessarily belongs to X_{+}.

A lottery L: $X \rightarrow [0,1]$ is a probability distribution on X, i.e. it delivers an outcome $x\in X$ with a probability $L(x) \in [0,1]$ and $\sum_{x\in X} L(x)=1$. The set of all lotteries is denoted by $\mathcal{L}$. Let $L_{+} : X \rightarrow [0,1]$ denote a loss-free lottery that yields only gains with a positive probability i.e. $\sum_{x_{+}\in X_{+}} L_{+}(x_{+})=1$ and $L_{+}(x_{-})=0$ for any $x_{-}\in X_{-}$. Let $\mathcal{L}_{+} \subset \mathcal{L}$ be the set of all such loss-free lotteries. Finally, let $L_{-} : X \rightarrow [0,1]$ denote a gain-free lottery that yields only losses with a positive probability i.e. $\sum_{x_{-}\in X_{-}} L_{-}(x_{-})=1$ and $L_{-}(x_{+})=0$ for any $x_{+}\in X_{+}$. Let $\mathcal{L}_{-} \subset \mathcal{L}$ be the set of all such gain-free lotteries.

In this and the next two sections we consider a "traditional" decision maker who has a unique binary preference relation $\succsim$ on $\mathcal{L}$. As customary, we will use the sign $\succ$ to denote the asymmetric component of $\succsim$, and the sign $\sim$ to denote the symmetric component of $\succsim$. We will consider two individuals: an individual ♀ characterized by a preference relation $\succsim_{♀}$ and an individual ♂ characterized by a preference relation $\succsim_{♂}$.

We begin with a definition of comparative loss aversion that parallels Yaari's definition of comparative risk aversion (Yaari, 1969).

Definition 1 An individual ♀ is *more loss averse than* an individual ♂ if

a) $L_+ \succ_♂ L$ implies $L_+ \succ_♀ L$ for all $L_+ \in \mathcal{L}_+$ and all $L \in \mathcal{L}$;

b) $L_+ \sim_♂ L$ implies $L_+ \succsim_♀ L$ for all $L_+ \in \mathcal{L}_+$ and all $L \in \mathcal{L}$;

c) there exist $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ such that $L_+ \sim_♂ L$ and $L_+ \succ_♀ L$.

According to Definition 1, a more loss averse individual strictly prefers a loss-free lottery over another lottery whenever a less loss averse individual does so as well. In addition, a more loss averse individual weakly prefers a loss-free lottery over another lottery whenever a less loss averse individual is exactly indifferent between the two. This definition of the more-loss-averse-than relation between individuals is quite general. Specifically, Definition 1 does not require that lottery outcomes are measurable in real numbers. It also does not require that individual preferences are represented by a specific decision theory (e.g. prospect theory). In particular, comparative loss aversion is defined in terms of observable preferences and not as a property of an unobservable function (e.g. a value function in prospect theory) that represents these preferences.

Definition 1 captures a very simple idea—if a less loss averse individual likes a certain loss-free lottery then a more loss averse individual should moreover do so. Alternatively, we can define comparative loss aversion based on a logical negation of the above statement—if a less loss averse individual does not like a certain *gain*-free lottery then a more loss averse individual should moreover do so. To distinguish this alternative concept of comparative loss aversion from Definition 1 above, we refer to this second concept of comparative loss aversion as "gain proneness" rather than "loss aversion". Thus, in the remainder of this paper, we write that an individual ♀ is *more loss averse than* an individual ♂ if we refer to Definition 1 above and that an individual ♀ is *more gain prone than* an individual ♂ if we refer to Definition 2 below.

Definition 2 An individual ♀ is *more gain prone than* an individual ♂ if

a) $L \succ_{♂} L_{-}$ implies $L \succ_{♀} L_{-}$ for all $L \in \mathcal{L}$ and all $L_{-} \in \mathcal{L}_{-}$;

b) $L \sim_{♂} L_{-}$ implies $L \succsim_{♀} L_{-}$ for all $L \in \mathcal{L}$ and all $L_{-} \in \mathcal{L}_{-}$;

c) there exist $L \in \mathcal{L}$ and $L_{-} \in \mathcal{L}_{-}$ such that $L \sim_{♂} L_{-}$ and $L \succ_{♀} L_{-}$.

Definition 1 is not equivalent to Definition 2. If an individual ♀ is more loss averse than an individual ♂ this does not imply that ♀ is also more gain prone than ♂ or vice versa. Finally, we define strong comparative loss aversion as follows.

Definition 3 An individual ♀ is *strongly more loss averse than* an individual ♂ if the individual ♀ is both more loss averse and more gain prone than the individual ♂.

If an individual ♀ is more *loss* averse (or more gain prone) than an individual ♂, this does not imply that ♀ is also more *risk* averse than ♂. Specifically, it is possible that a less loss averse (or gain prone) individual ♂ strictly prefers a sure loss of $x_{-} \in X_{-}$ over a lottery $L \in \mathcal{L}$ and at the same time a more loss averse (or gain prone) individual ♀ strictly prefers L over a degenerate lottery that yields x_{-} for sure. Hence, the individual ♀ is not always more risk averse than the individual ♂ (e.g. Blavatskyy, 2008b).

However, Definition 1 implies that if a less loss averse individual ♂ strictly prefers a sure gain of $x_{+} \in X_{+}$ over a lottery $L \in \mathcal{L}$ then a more loss averse individual ♀ does so as well and if the individual ♂ is exactly indifferent between the two then the individual ♀ weakly prefers the sure gain. In other words, the individual ♀ is more risk averse than the individual ♂ in the domain of gains. Definition 2 implies that if a less gain prone individual strictly prefers a lottery $L \in \mathcal{L}$ over a sure loss of $x_{-} \in X_{-}$ then a more gain prone individual does so as well. Hence, a more gain prone individual is a more risk seeking individual in the domain of losses. On the other hand, a more risk averse individual is not necessarily a more loss averse (or less gain prone) individual as well.

Proposition 1 If an individual ♀ is more loss averse than an individual ♂, or vice versa, then

a) $L_+ \succ_♀ M_+$ if and only if $L_+ \succ_♂ M_+$ for all $L_+, M_+ \in \mathcal{L}_+$;

b) $L_+ \sim_♀ M_+$ if and only if $L_+ \sim_♂ M_+$ for all $L_+, M_+ \in \mathcal{L}_+$.

Proof is presented in the Appendix.

Proposition 1 is an intuitive implication of Definition 1. We can unambiguously rank two individuals in terms of their loss preferences only if they have identical preferences over loss-free alternatives (gain lotteries). If the two individuals do not have the same preferences in choice without any losses, one of them may choose a specific loss-free lottery because it is her most preferred alternative and not because she is averse to losses. Thus, to have a meaningful concept of comparative loss aversion, we need to consider individuals with identical preferences over the set of loss-free lotteries.

An analogous result holds for gain proneness. Since Definition 2 is effectively a mirror image of Definition 1, we prove only the results for loss aversion and state the corresponding results for gain proneness as corollaries.

Corollary 1 If an individual ♀ is more gain prone than an individual ♂, or vice versa, then

a) $L_- \succ_♀ M_-$ if and only if $L_- \succ_♂ M_-$ for all $L_-, M_- \in \mathcal{L}_-$;

b) $L_- \sim_♀ M_-$ if and only if $L_- \sim_♂ M_-$ for all $L_-, M_- \in \mathcal{L}_-$.

Corollary 1 captures the same simple intuition for gain proneness as Proposition 1 does for loss aversion. We can unambiguously rank two individuals in terms of their gain proneness only if they have identical preferences over gain-free alternatives (loss lotteries). Otherwise, one individual may dislike a particular gain-free lottery, because it is her least preferred alternative and not because she is prone to gains.

2. Loss Aversion in Expected Utility Theory

This section considers comparative loss aversion in the context of expected utility theory (von Neumann and Morgenstern, 1944). If preferences admit expected utility representation then there exists an utility function $u:X\rightarrow\mathbb{R}$ that is unique up to a positive linear transformation, such that

(1) $$L \succsim M \text{ if and only if } \sum_{x\in X} L(x)u(x) \geq \sum_{x\in X} M(x)u(x),$$

for any two lotteries $L, M \in \mathcal{L}$.

According to formula (1), a lottery L is weakly preferred over a lottery M if and only if the expected utility of L is greater than or equal to the expected utility of M. The following result follows immediately from Proposition 1.

Corollary 2 If an expected utility maximizer ♀ with utility function $u_{♀}:X\rightarrow\mathbb{R}$ is more loss averse than an expected utility maximizer ♂ with utility function $u_{♂}:X\rightarrow\mathbb{R}$, then there exist $a>0$ and $b\in\mathbb{R}$ such that $u_{♀}(x_+) = au_{♂}(x_+) + b$ for all $x_+\in X_+$.

Corollary 2 simply states that whenever two individuals can be ranked in terms of loss preferences, they must have the same utility function in the domain of gains, up to a positive linear transformation.

Proposition 2 An expected utility maximizer ♀ with utility function $u_{♀}:X\rightarrow\mathbb{R}$ is more loss averse than an expected utility maximizer ♂ with utility function $u_{♂}:X\rightarrow\mathbb{R}$ if and only if there exist $a>0$ and $b\in\mathbb{R}$ such that

a) $u_{♀}(x_+) = au_{♂}(x_+) + b$ for all $x_+\in X_+$;

b) $u_{♀}(x_-) \leq au_{♂}(x_-) + b$ for all $x_-\in X_-$;

c) there exists a loss $x_-\in X_-$ such that $u_{♀}(x_-) < au_{♂}(x_-) + b$.

Proof is presented in the Appendix.

Proposition 2 states that an individual ♀ is more loss averse than an individual ♂ if and only if we can normalize the utility function of the individual ♂ for two arbitrary gains so that ♂'s normalized utility function coincides with ♀'s utility function in the domain of gains and ♂'s normalized utility of any loss $x_-\in X_-$ is greater than or equal to ♀'s utility of x_- (and it is strictly greater for at least one loss $x_-\in X_-$).

Figure 1 illustrates Proposition 2 when X_+ is the set of positive real numbers $\mathbb{R}_+$ and X_- is the set of negative real numbers $\mathbb{R}_-$.

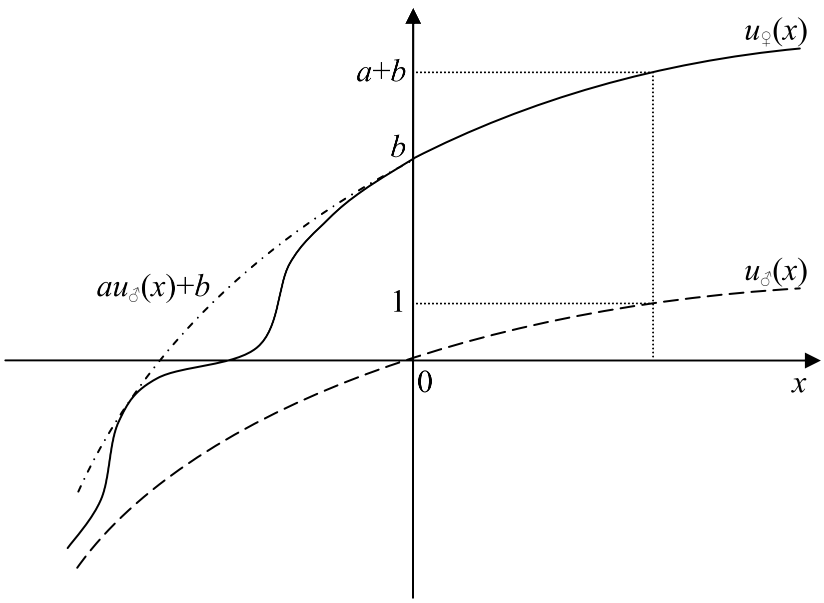

Figure 1 An expected utility maximizer ♀ with utility function $u_♀(x)$ **is more loss averse than an expected utility maximizer ♂ with utility function** $u_♂(x)$

Corollary 3 An expected utility maximizer ♀ with utility function $u_♀:X\to\mathbb{R}$ is more gain prone than an expected utility maximizer ♂ with utility function $u_♂:X\to\mathbb{R}$ if and only if there exist $a>0$ and $b\in\mathbb{R}$ such that

a) $u_♀(x_-) = au_♂(x_-) + b$ for all $x_-\in X_-$;

b) $u_♀(x_+) \geq au_♂(x_+) + b$ for all $x_+\in X_+$;

c) there exists a gain $x_+\in X_+$ such that $u_♀(x_+) > au_♂(x_+) + b$.

Corollary 3 establishes a parallel result to Proposition 2 for the case of gain proneness. In this case, an expected utility maximizer ♀ is more gain prone than an expected utility maximizer ♂ if and only if we can normalize their utility functions so that the two coincide in the domain of losses and ♀'s normalized utility function does not fall below ♂'s utility function in the domain of gains (and it is strictly above ♂'s utility function for at least one gain $x_+ \in X_+$). Thus, under expected utility theory, conditions for gain proneness are simply a mirror image of those for loss aversion.

Figure 2 illustrates Corollary 3 when X_+ is the set of positive real numbers $\mathbb{R}_+$ and X_- is the set of negative real numbers $\mathbb{R}_-$.

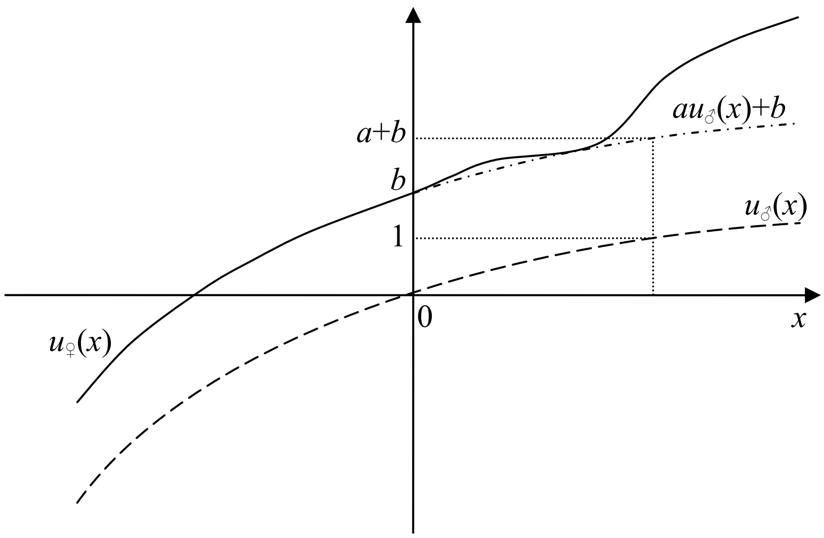

Figure 2 An expected utility maximizer ♀ with utility function $u_♀(x)$ is more gain prone than an expected utility maximizer ♂ with utility function $u_♂(x)$

By comparing necessary and sufficient conditions from Proposition 2 with those from Corollary 3 we arrive at the first impossibility result.

Corollary 4 One expected utility maximizer cannot be strongly more loss averse than another expected utility maximizer.

Corollary 4 effectively states that Definition 1 and Definition 2 are mutually exclusive under expected utility theory. In other words, if an expected utility maximizer

♀ is more loss averse than another expected utility maximizer ♂ then the individual ♀ cannot simultaneously be more gain prone than the individual ♂ and vice versa.

If preferences admit expected utility representation, we can establish a stronger relationship between risk aversion on one side and loss aversion and gain proneness on the other side. For completeness, let us first define comparative risk aversion as follows.

Definition 4 An individual ♀ is *more risk averse than* an individual ♂ if

a) $x \succ_{♂} L$ implies $x \succ_{♀} L$ for all $x \in X$ and all $L \in \mathcal{L}$,

b) $x \sim_{♂} L$ implies $x \succsim_{♀} L$ for all $x \in X$ and all $L \in \mathcal{L}$,

c) there is one outcome $x \in X$ and one lottery $L \in \mathcal{L}$ such that $x \sim_{♂} L$ and $x \succ_{♀} L$.

Definition 4 captures a simple idea—if a less risk averse individual prefers a degenerate lottery that yields one outcome $x \in X$ for sure over another lottery $L \in \mathcal{L}$ then a more risk averse individual should moreover do so.

Proposition 3 If an expected utility maximizer ♀ is more risk averse than another expected utility maximizer ♂ then:

a) the individual ♀ is also more loss averse than the individual ♂ provided that the set X_+ has no more than two elements distinct in terms of desirability,

b) the individual ♂ is more gain prone than the individual ♀ provided that the set X_- has no more than two elements distinct in terms of desirability.

Intuitively, if the set X_+ contains no more than two outcomes that are distinct in terms of desirability, then we can normalize the utility functions of two individuals so that they coincide in the domain of gains. The utility function of a more risk averse individual then does not exceed the normalized utility function of a less risk averse individual in the domain of losses i.e. a more risk averse individual is also a more loss averse individual. A similar intuition applies to the case of gain proneness.

3. Loss Aversion in Rank-Dependent Utility Theory

This section considers the concept of comparative loss aversion in the context of rank-dependent utility theory (Quiggin, 1981). In rank-dependent utility theory there exists an utility function $u{:}X\to\mathbb{R}$ that is unique up to a positive linear transformation, and a unique strictly increasing probability weighting function $w{:}[0,1]\to[0,1]$ with $w(0)=0$ and $w(1)=1$, such that

(2)
$$L \succsim M \text{ if and only if } \sum_{x\in X} u(x)\left[w\left(\sum_{\substack{y\in X\\ u(y)\geq u(x)}} L(y)\right)-w\left(\sum_{\substack{y\in X\\ u(y)>u(x)}} L(y)\right)\right]\geq$$
$$\geq \sum_{x\in X} u(x)\left[w\left(\sum_{\substack{y\in X\\ u(y)\geq u(x)}} M(y)\right)-w\left(\sum_{\substack{y\in X\\ u(y)>u(x)}} M(y)\right)\right],$$

for any lotteries $L, M \in \mathcal{L}$. The following result follows immediately from Proposition 1.

Corollary 5 If a rank-dependent utility maximizer ♀ with an utility function $u_{♀}{:}X\to\mathbb{R}$ and a probability weighting function $w_{♀}{:}[0,1]\to[0,1]$ is more loss averse than a rank-dependent utility maximizer ♂ with an utility function $u_{♂}{:}X\to\mathbb{R}$ and a probability weighting function $w_{♂}{:}[0,1]\to[0,1]$, then $w_{♀}(p) = w_{♂}(p)$ for all $p\in[0,1]$ and there exist $a>0$ and $b\in\mathbb{R}$ such that $u_{♀}(x_+) = au_{♂}(x_+) + b$ for all $x_+\in X_+$.

Recall that an unambiguous ranking of two individuals according to their loss attitudes is possible only if the two individuals share the same preferences over loss-free lotteries (Proposition 1). In the context of rank-dependent utility theory this implies the following. We can rank two rank-dependent utility maximizers according to their loss attitudes only if the two individuals have the same probability weighting function and the same utility function in the domain of gains, up to a positive linear transformation (Corollary 5).

Note that Corollary 5 implies that the two rank-dependent utility maximizers have the same ranking of gains in terms of their desirability.

Proposition 4 A rank-dependent utility maximizer ♀ with an utility function $u_♀: X \to \mathbb{R}$ and a probability weighting function $w_♀:[0,1] \to [0,1]$ is more loss averse than a rank-dependent utility maximizer ♂ with an utility function $u_♂: X \to \mathbb{R}$ and a probability weighting function $w_♂:[0,1] \to [0,1]$ if and only if there exist $a>0$ and $b \in \mathbb{R}$ such that

a) $w_♀(p) = w_♂(p)$ for all $p \in [0,1]$;

b) $u_♀(x_+) = a u_♂(x_+) + b$ for all $x_+ \in X_+$;

c) $u_♀(x_-) \leq a u_♂(x_-) + b$ for all $x_- \in X_-$;

d) there exists a loss $x_- \in X_-$ such that $u_♀(x_-) < a u_♂(x_-) + b$.

Proof is presented in the Appendix.

Note that Proposition 4 does not require that the two rank-dependent utility maximizers have the same ranking of losses in terms of their desirability.

Proposition 4 characterizes the concept of comparative loss aversion within a rank-dependent utility theory. In particular, Proposition 4 shows that comparative loss aversion is entirely captured by the curvature of the utility function and it is not related to the shape of the probability weighting function. The restrictions on the curvature of the utility function, which are required for one individual to be more loss averse than another individual, are exactly the same as in expected utility theory (cf. Proposition 2). Namely, the two individuals should have the same utility function in the domain of gains (up to a positive linear transformation) and a more loss averse individual should have an utility function that lies below the corresponding normalized utility function of a less loss averse individual in the domain of losses.

The necessary and sufficient conditions for gain proneness under rank-dependent utility theory are analogous to those given in Proposition 4 for loss aversion.

Corollary 6 A rank-dependent utility maximizer ♀ with an utility function $u_♀:X\to\mathbb{R}$ and a probability weighting function $w_♀:[0,1]\to[0,1]$ is more gain prone than a rank-dependent utility maximizer ♂ with an utility function $u_♂:X\to\mathbb{R}$ and a probability weighting function $w_♂:[0,1]\to[0,1]$ if and only if there exist $a>0$ and $b\in\mathbb{R}$ such that

a) $w_♀(p) = w_♂(p)$ for all $p\in[0,1]$;

b) $u_♀(x_-) = au_♂(x_-) + b$ for all $x_-\in X_-$;

c) $u_♀(x_+) \geq au_♂(x_+) + b$ for all $x_+\in X_+$;

d) there exists a gain $x_+\in X_+$ such that $u_♀(x_+) > au_♂(x_+) + b$.

Finally, a combination of necessary and sufficient conditions from Proposition 4 and Corollary 6 yields an impossibility result for rank-dependent utility theory.

Corollary 7 One rank-dependent utility maximizer cannot be strongly more loss averse than another rank-dependent utility maximizer.

Intuitively, if one rank-dependent utility maximizer were strongly more loss averse than another then both individuals would have the same probability weighting function, the same utility function over the domain of gains (up to a positive linear transformation) and the same utility function over the domain of losses (up to a positive linear transformation). So the two individuals can only differ to the extent how losses are valued in relation to gains. If we normalize utility functions of the two individuals so that they coincide in the domain of gains, a more loss averse individual would have a lower utility function in the domain of losses. However, this implies that if we renormalize the two utility functions so that they coincide in the domain of losses, a more loss averse individual would have a lower utility function in the domain of gains. Hence, a more loss averse individual cannot simultaneously be a more gain prone individual as well.

In the context of rank-dependent utility theory, Definitions 1 and 2 are related to the intuitive ideas of Kahneman and Tversky (1979) who pioneered the concept of loss aversion in behavioral economics. According to Kahneman and Tversky (1979), a more loss averse individual is characterized by an utility function that exhibits a greater kink at the reference point. Within the framework developed in Section 1, we can define the reference point as a unique outcome r that may be regarded both as a gain and as a loss. Technically, we can extend the set of feasible outcomes to $X \cup \{r\}$ so that the set of losses now is $X_- \cup \{r\}$ and the set of gains now is $X_+ \cup \{r\}$. It turns out that the definition proposed by Kahneman and Tversky (1979) is equivalent to a combination of greater loss aversion and lower gain proneness. Formally, this result is captured by the following Corollary 8, which follows immediately from Proposition 4 and Corollary 6.

Corollary 8 A rank-dependent utility maximizer ♀ with an utility function $u_♀:X\to\mathbb{R}$ and a probability weighting function $w_♀:[0,1]\to[0,1]$ is more loss averse *and* less gain prone than a rank-dependent utility maximizer ♂ with an utility function $u_♂:X\to\mathbb{R}$ and a probability weighting function $w_♂:[0,1]\to[0,1]$ if and only if

a) $w_♀(p) = w_♂(p)$ for all $p\in[0,1]$;

b) there exist $a>0$ and $b\in\mathbb{R}$ such that $u_♀(x_+) = au_♂(x_+) + b$ for all $x_+\in X_+$;

c) there exist $c>0$ and $d\in\mathbb{R}$ such that $u_♀(x_-) = cu_♂(x_-) + d$ for all $x_-\in X_-$;

d) $\dfrac{u_♀(x_-)-u_♀(r)}{u_♀(x_+)-u_♀(r)} < \dfrac{u_♂(x_-)-u_♂(r)}{u_♂(x_+)-u_♂(r)}$, for all $x_-\in X_-$ and $x_+\in X_+$.

According to Corollary 8, one individual is more loss averse and less gain prone than another individual if the two individuals have the same probability weighting function, the same utility function over the domain of gains (up to a positive linear transformation), the same utility function over the domain of losses (up to a positive linear transformation) but the utility function of the first individual has a greater kink at the reference point.

Figure 3 illustrates Corollary 8 when the set of gains is the set of non-negative real numbers and the set of losses is the set of non-positive real numbers (the reference point is zero).

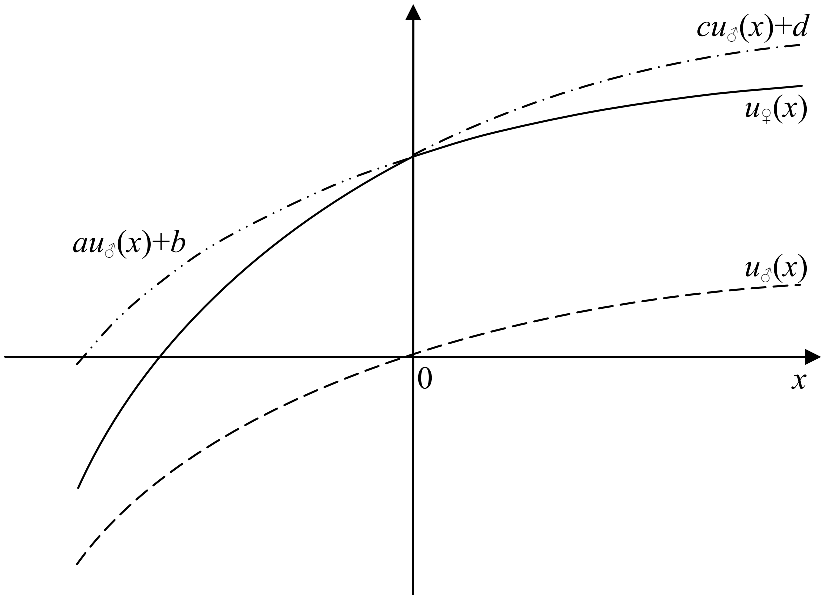

Figure 3 A rank-dependent utility maximizer ♀ with utility function $u_♀(x)$ is more loss averse and less gain prone than a rank-dependent utility maximizer ♂ with utility function $u_♂(x)$

As a final point, condition d) in Corollary 8 implies that we can use the index $I_♀(x_-, x_+, r) = -\frac{u_♀(x_-) - u_♀(r)}{u_♀(x_+) - u_♀(r)}$ as an interpersonal measure of greater loss aversion and lower gain proneness for an individual ♀. Individuals with a higher index $I(x_-, x_+, r)$ are characterized by an utility function that exhibits a greater kink at the reference point. Notice that $I(x_-, x_+, r)$ is a local index of greater loss aversion and lower gain proneness for a specific loss $x_- \in X_-$ and a specific gain $x_+ \in X_+$. Interestingly, we can consider index $I(x_-, x_+, r)$ as a discrete version of the index of loss aversion proposed by Köbberling and Wakker (2005).

4. Probabilistic Loss Aversion

Numerous experimental studies find that people do not always choose the same alternative when presented with exactly the same decision problem on two separate occasions within a short period of time (e.g. Camerer, 1989; Hey and Orme, 1994; Loomes and Sugden, 1998). In general, people often make contradictory choices if none of the lotteries transparently dominates other alternatives. In this section we will extend Definitions 1-3 to a more general setup where people may choose in a probabilistic manner.

In the remainder of this paper we assume that the primitive of choice is a binary choice probability function $P:\mathcal{L}\times\mathcal{L}\rightarrow[0,1]$, which is also known as a fuzzy preference relation (e.g. Zimmerman et al., 1984). Notation $P(L,M)$ denotes probability that an individual chooses lottery $L \in \mathcal{L}$ over lottery $M \in \mathcal{L}$ in a direct binary choice. For any L, $M \in \mathcal{L}$, $L \neq M$, probability $P(L,M)$ is observable from the relative frequency with which an individual chooses L when asked to choose repeatedly between L and M. We consider two individuals: an individual ♀ and an individual ♂ characterized by binary choice probability functions $P_{♀}(.,.)$ and $P_{♂}(.,.)$ correspondingly.

Definition 5 An individual ♀ is *probabilistically more loss averse than* an individual ♂ if $P_{♀}(L_+,L)\geq P_{♂}(L_+,L)$ for all $L_+\in\mathcal{L}_+$ and all $L\in\mathcal{L}$ and there exist at least one loss-free lottery $L_+\in\mathcal{L}_+$ and one lottery $L\in\mathcal{L}$ such that $P_{♀}(L_+,L)>P_{♂}(L_+,L)$.

Definition 5 simply states that a more loss averse individual is always at least as likely to choose a loss-free lottery over any other lottery as a less loss averse individual. Definition 5 of the more-loss-averse-than relation between individuals is very general. In particular, lottery outcomes may not be measurable in real numbers. We also do not require that fuzzy preferences over lotteries are represented by a specific model of probabilistic choice. Thus, Definition 5 applies to very distinct models of probabilistic choice, e.g. when people have multiple preference relations on $\mathcal{L}$ (Loomes and Sugden, 1995) or when people have a unique preference relation on $\mathcal{L}$ but they make random errors (Hey and Orme, 1994; Blavatskyy, 2007). Last but not least, Definition 5 is more compact than Definition 1.

Along the same lines we can extend Definitions 2 and 3 into Definitions 6 and 7.

Definition 6 An individual ♀ is *probabilistically more gain prone than* an individual ♂ if $P_{♀}(L, L_-) \geq P_{♂}(L, L_-)$ for all $L \in \mathcal{L}$ and all $L_- \in \mathcal{L}_-$ and there exist at least one lottery $L \in \mathcal{L}$ and one gain-free lottery $L_- \in \mathcal{L}_-$ such that $P_{♀}(L, L_-) > P_{♂}(L, L_-)$.

Definition 7 An individual ♀ is *probabilistically strongly more loss averse than* an individual ♂ if the individual ♀ is both probabilistically more loss averse and probabilistically more gain prone than the individual ♂.

By replacing lottery $L \in \mathcal{L}$ in the first part of Definition 5 with a loss-free lottery $M_+ \in \mathcal{L}_+$, we immediately arrive at the following result.

Corollary 9 If an individual ♀ is probabilistically more loss averse than an individual ♂, or vice versa, then $P_{♀}(L_+, M_+) = P_{♂}(L_+, M_+)$ for all $L_+, M_+ \in \mathcal{L}_+$.

According to Corollary 9, the ranking of individuals in terms of their loss attitudes is possible only if they choose in identical manner between loss-free lotteries. If this is not the case, heterogeneous loss attitudes are confounded with heterogonous tastes over loss-free lotteries and no clear comparison of individuals in terms of comparative loss aversion can be made. An analogous result holds for gain proneness.

Corollary 10 If an individual ♀ is probabilistically more gain prone than an individual ♂, or vice versa, then $P_{♀}(L_-, M_-) = P_{♂}(L_-, M_-)$ for all $L_-, M_- \in \mathcal{L}_-$.

5. Loss Aversion in Different Models of Probabilistic Choice

One of the simplest models of probabilistic choice is the constant error/tremble model. Harless and Camerer (1994) argue that people have a unique preference relation on $\mathcal{L}$ but they do not always choose their preferred lottery. With a constant probability $\tau\in[0,0.5]$ a tremble occurs and people choose a less preferred alternative (for instance, due to a lapse of concentration). Specifically, in a constant error/tremble model there exists an utility function $u:X\to\mathbb{R}$ that is unique up to a linear transformation, such that

$$(3)\qquad P(L,M)=\begin{cases}\tau, & \sum_{x\in X}L(x)u(x)<\sum_{x\in X}M(x)u(x)\\ 0.5, & \sum_{x\in X}L(x)u(x)=\sum_{x\in X}M(x)u(x)\\ 1-\tau, & \sum_{x\in X}L(x)u(x)>\sum_{x\in X}M(x)u(x)\end{cases}$$

for any two lotteries L, $M\in\mathcal{L}$ and a probability $\tau\in[0,0.5]$. The following result follows directly from the proof of Proposition 2 and Corollary 3.

Corollary 11 An individual ♀ with utility function $u_{♀}:X\to\mathbb{R}$ and the probability of a tremble $\tau_{♀}$ is probabilistically more loss averse (more gain prone) than an individual ♂ with utility function $u_{♂}:X\to\mathbb{R}$ and the probability of a tremble $\tau_{♂}$ if and only if $\tau_{♂}=\tau_{♀}$ and conditions a)-c) of Proposition 2 (Corollary 3) are satisfied.

Let us now consider probabilistic loss aversion in the context of a strong utility model (e.g. Luce and Suppes, 1965). In this model there exists an utility function $u:X\to\mathbb{R}$ that is unique up to a positive linear transformation, and a strictly increasing function $\varphi:\mathbb{R}\to[0,1]$, which is unique up to a positive dimensional constant and satisfies $\varphi(v)+\varphi(-v)=1$ for all $v\in\mathbb{R}$, such that

$$(4)\qquad P(L,M)=\varphi\left(\sum_{x\in X}L(x)u(x)-\sum_{x\in X}M(x)u(x)\right)$$

for any two lotteries L, $M\in\mathcal{L}$.

Function $\varphi(.)$ captures the sensitivity of binary choice probabilities to differences in the expected utility of the two alternatives that an individual needs to choose from. If function $\varphi(.)$ is the cumulative distribution function of a normal distribution with zero mean and constant standard deviation, model (4) becomes the Fechner model of random errors (Fechner, 1860; Hey and Orme, 1994). If function $\varphi(.)$ is the distribution function of the logistic distribution, model (4) becomes Luce choice model (Luce, 1959). Blavatskyy (2008a) provides axiomatic characterization of the choice rule (4).

Proposition 5 A strong utility maximizer ♀ characterized by a pair of functions $(u_{♀}, \varphi_{♀})$ is probabilistically more loss averse than a strong utility maximizer ♂ characterized by a pair of functions $(u_{♂}, \varphi_{♂})$ if there exist $a>0$ and $b\in\mathbb{R}$ such that

a) $u_{♀}(x_+) = au_{♂}(x_+) + b$ for all $x_+\in X_+$;

b) $u_{♀}(x_-) \leq au_{♂}(x_-) + b$ for all $x_-\in X_-$;

c) $\varphi_{♀}(av) = \varphi_{♂}(v)$ for all $v\in[-\delta, \delta]$, where $\delta = \max_{x_+\in X_+} u_{♂}\left(x_+\right) - \min_{x_+\in X_+} u_{♂}\left(x_+\right)$;

d) $\varphi_{♀}(av) \geq \varphi_{♂}(v)$ for all $v\in(\delta, \Delta]$, where $\Delta = \max_{x_+\in X_+} u_{♂}\left(x_+\right) - \min_{x_-\in X_-} u_{♂}\left(x_-\right)$;[1]

e) either there exists a loss $x_-\in X_-$ such that $u_{♀}(x_-) < au_{♂}(x_-) + b$ or there exists $v\in(\delta, \Delta]$ such that $\varphi_{♀}(av) > \varphi_{♂}(v)$ or both.

Proof is presented in the Appendix.

Proposition 5 shows that in a strong utility model loss aversion is related both to the curvature of the utility function $u(.)$ and the shape of the sensitivity function $\varphi(.)$. On the one hand, an individual ♀ can be more loss averse than an individual ♂ if they have the same utility function in the domain of gains (up to a positive linear transformation) but ♀'s utility function lies below ♂'s normalized utility function in the domain of

[1] Note that condition d) is equivalent to $\varphi_{♀}(av) \leq \varphi_{♂}(v)$ for all $v\in(-\Delta,-\delta]$ due to the skew-symmetric property of the sensitivity function $\varphi(.)$.

losses. On the other hand, an individual ♀ can be more loss averse than an individual ♂ if they have the same sensitivity function in the neighborhood of zero (up to a positive dimensional constant) but individual ♀ is more sensitive to large differences in utility.

Interestingly, a strong utility model allows individual ranking in terms of comparative *loss* aversion but not in terms of comparative *risk* aversion. Wilcox (2008) and Blavatskyy (2008b) show that risk aversion cannot be defined within a strong utility model. Thus, there are models where loss aversion is well defined even though risk aversion is not.

6. Absolute Loss Aversion

So far we considered only comparative loss aversion. To measure absolute loss aversion, we need to fix one binary choice probability function $P_{LN}: \mathcal{L}\times\mathcal{L} \rightarrow [0,1]$. An individual is called *loss neutral* if she has the binary choice probability function $P_{LN}(.,.)$. An individual is called *loss averse* if she is more loss averse (according to Definition 5) than the loss neutral individual. Similarly, an individual is called *loss seeking* or *loss loving* if the loss neutral individual is more loss averse than this individual.

Notice that the concept of absolute loss aversion depends on an ad hoc selection of a loss neutral binary choice probability function $P_{LN}(.,.)$.[2] This is similar to our temperature measurement that requires an arbitrary selection of zero temperature (e.g. the triple point of water in the Celsius scale or absolute zero in the Kelvin scale). Similarly, our time measurement also requires an ad hoc selection of an epochal date (e.g. the incarnation of Jesus in the Gregorian calendar, the creation of the world in the Hebrew calendar or the immigration of Muhammad in the Islamic calendar).

[2] The definition of risk aversion also requires a priori "normalization" of risk neutral preferences (e.g. Epstein, 1999; Blavatskyy, 2008a). Similarly, in order to define uncertainty aversion we need an arbitrary definition of uncertainty neutrality (e.g. Epstein, 1999).

In a special case when lotteries have only monetary outcomes and people have deterministic preferences, Kahneman and Tversky (1979) arbitrary selected a loss neutral preference relation so that a loss neutral individual is exactly indifferent between accepting and rejecting a symmetric bet that yields a 50%-50% chance of either a loss of $-x$ or a gain of x, for all $x \in \mathbb{R}_+$. In other words, loss aversion is defined as aversion to symmetric 50%-50% lotteries. Several later studies also adopted this convention (e.g. Schmidt and Zank, 2005). However, it is not clear how this natural "normalization" can be extended to a more general case when outcomes are not measurable in real numbers.

7. Conclusion

Loss aversion is a fundamental concept in behavioral economics. However, it is traditionally defined only in the context of lotteries over monetary payoffs. This paper extends the definition of loss aversion to a more general setup where outcomes are not necessarily measurable in real numbers and people do not necessarily have a unique preference relation over lotteries, i.e. they may choose in a probabilistic manner.

This paper proposes two alternative definitions of comparative loss aversion. A more loss averse individual prefers a loss-free lottery (that yields only gains with a positive probability) over another lottery whenever a less loss averse individual does so as well. A more gain prone individual prefers an arbitrary lottery over a gain-free lottery whenever a less gain prone individual does so as well. More generally, an individual ♀ is probabilistically more loss averse than an individual ♂ if in any decision problem ♀ chooses a loss-free lottery at least as frequently as does ♂. Similarly, one individual is probabilistically more gain prone than another individual if in any decision problem she does not choose a gain-free lottery more often than the other individual.

This paper shows that the above definitions of comparative loss aversion have very intuitive implications for well-known decision theories such as expected utility theory and rank-dependent utility theory as well as for popular models of probabilistic choice such as the constant error/tremble model, Fechner model of random errors and Luce choice model. In particular, in these models loss aversion and gain proneness are related to the curvature of the utility function. If two individuals can be ranked in terms of their loss preferences (gain proneness), then they have the same utility function in the domain of gains (losses) up to a positive linear transformation but the utility function of a more loss averse (gain prone) individual lies below (above) the normalized utility function of a less loss averse (gain prone) individual in the domain of losses (gains).

In a strong utility model, loss aversion may be also driven by the curvature of the sensitivity function—a more loss averse individual may be more sensitive to large differences in expected utility of the two lotteries that are compared. Interestingly, comparative loss aversion is well-defined in a strong utility model, even though comparative risk aversion is not. This highlights an important point that stronger loss aversion does not necessarily imply stronger risk aversion, or vice versa.

However, a more loss averse individual is always a more risk averse individual in the domain of gains. At the same time, a less gain prone individual is always a more risk averse individual in the domain of losses. Thus, if one individual is simultaneously more loss averse and less gain prone (a condition that is equivalent to a greater kink at the reference point under rank-dependent utility theory) then this individual is also a more risk averse individual. In other words, traditional definitions of comparative loss aversion (greater kink at the reference point) necessarily imply risk aversion as well. Finally, the paper also shows that under expected utility theory a more risk averse individual is also a more loss averse (less gain prone) individual provided that the set of gains (losses) contains no more than two elements that are distinct in terms of desirability.

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Appendix

Proof of Proposition 1.

Consider an individual ♀ who is more loss averse than an individual ♂.

a) According to Definition 1, if $L_+ \succ_{♂} M_+$ then $L_+ \succ_{♀} M_+$ for all $L_+, M_+ \in \mathcal{L}_+$.

Let us now assume that there exist two lotteries $L_+, M_+ \in \mathcal{L}_+$ such that $L_+ \succ_{♀} M_+$ but $M_+ \succsim_{♂} L_+$. If $M_+ \succ_{♂} L_+$ then Definition 1 implies that $M_+ \succ_{♀} L_+$. However, this contradicts to our assumption that $L_+ \succ_{♀} M_+$. If $M_+ \sim_{♂} L_+$ then Definition 1 implies that $M_+ \succsim_{♀} L_+$. Again, this contradicts to our assumption that $L_+ \succ_{♀} M_+$. Therefore, it must be the case that $L_+ \succ_{♀} M_+$ if and only if $L_+ \succ_{♂} M_+$ for all $L_+, M_+ \in \mathcal{L}_+$.

b) According to Definition 1, if $L_+ \sim_{♂} M_+$ then $L_+ \succsim_{♀} M_+$ for all $L_+, M_+ \in \mathcal{L}_+$. Moreover, if $M_+ \sim_{♂} L_+$ then $M_+ \succsim_{♀} L_+$ for all $L_+, M_+ \in \mathcal{L}_+$. Hence, if $L_+ \sim_{♂} M_+$ then it must be the case that $L_+ \sim_{♀} M_+$ for all $L_+, M_+ \in \mathcal{L}_+$.

Let us now assume that there exist two lotteries $L_+, M_+ \in \mathcal{L}_+$ such that $L_+ \sim_{♀} M_+$ but $M_+ \nsim_{♂} L_+$. If $M_+ \succ_{♂} L_+$ then Definition 1 implies that $M_+ \succ_{♀} L_+$. However, this contradicts to our assumption that $L_+ \sim_{♀} M_+$. Similarly, if $L_+ \succ_{♂} M_+$ then $L_+ \succ_{♀} M_+$ due to Definition 1 and the analogous contradiction arises. Therefore, $L_+ \sim_{♀} M_+$ if and only if $L_+ \sim_{♂} M_+$ for all $L_+, M_+ \in \mathcal{L}_+$.

Similarly, we can prove that Proposition 1 holds when an individual ♂ is more loss averse than an individual ♀. *Q.E.D*.

Proof of Proposition 2.

We will first prove that if conditions a)-c) hold then an individual ♀ is more loss averse than an individual ♂. Consider two arbitrary lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$. We will first prove that $L_+ \succ_{♂} L$ implies $L_+ \succ_{♀} L$. Condition (1) implies that $L_+ \succ_{♂} L$ if and only if

$$\sum_{x_+ \in X_+} u_{♂}(x_+) L_+(x_+) > \sum_{x \in X} u_{♂}(x) L(x). \tag{5}$$

We can rearrange (5) into

(6) $$\sum_{x_+\in X_+} u_♂(x_+)\left[L_+(x_+)-L(x_+)\right] > \sum_{x_-\in X_-} u_♂(x_-)L(x_-).$$

Furthermore, we can multiply both sides of (6) on a positive constant a>0 and add $b\sum_{x_+\in X_+}\left[L_+(x_+)-L(x_+)\right]=b\sum_{x_-\in X_-}L(x_-)$ to both sides of (6), $b\in\mathbb{R}$. This results in

(7) $$\sum_{x_+\in X_+}\left[au_♂(x_+)+b\right]\left[L_+(x_+)-L(x_+)\right] > \sum_{x_-\in X_-}\left[au_♂(x_-)+b\right]L(x_-).$$

If part a) of Proposition 2 holds then there exist a>0 and $b\in\mathbb{R}$ such that $u_♀(x_+) = au_♂(x_+) + b$ for all $x_+\in X_+$. Hence, we can rewrite (7) as

(8) $$\sum_{x_+\in X_+} u_♀(x_+)\left[L_+(x_+)-L(x_+)\right] > \sum_{x_-\in X_-}\left[au_♂(x_-)+b\right]L(x_-).$$

If part b) of Proposition 2 holds then $u_♀(x_-) \leq au_♂(x_-) + b$ for all $x_-\in X_-$. Thus, we can rewrite (8) as

(9) $$\sum_{x_+\in X_+} u_♀(x_+)\left[L_+(x_+)-L(x_+)\right] > \sum_{x_-\in X_-} u_♀(x_-)L(x_-).$$

Finally, we can rearrange (9) into

(10) $$\sum_{x_+\in X_+} u_♀(x_+)L_+(x_+) > \sum_{x\in X} u_♀(x)L(x),$$

which holds if and only if $L_+ \succ_♀ L$ due to (1). Hence, $L_+ \succ_♂ L$ implies $L_+ \succ_♀ L$ for all $L_+\in\mathcal{L}_+$ and all $L\in\mathcal{L}$ if parts a) and b) of Proposition 2 hold.

To prove that $L_+ \sim_♂ L$ implies $L_+ \succsim_♀ L$ for all $L_+\in\mathcal{L}_+$ and all $L\in\mathcal{L}$ we just need to replace the sign ">" with the sign "=" in (5)-(8) and with the sign "≥" in (9)-(10).

Let us now prove that there exist $L_+\in\mathcal{L}_+$ and $L\in\mathcal{L}$ such that $L_+ \sim_♂ L$ and $L_+ \succ_♀ L$. If part c) of Proposition 2 holds then there exists $x_-\in X_-$ such that $u_♀(x_-) < au_♂(x_-) + b$. Let $y,z\in X_+$ be two gains such that $u_♂(x_-)<u_♂(y)<u_♂(z)$. Let L_+ be a lottery that yields y for sure and let L be a lottery that yields x_- with probability p and z with probability 1-p. Obviously, there exists a probability p such that

(11) $$u_♂(y) = pu_♂(x_-) + (1-p)u_♂(z).$$

If (11) holds then condition (1) implies that $L_+ \sim_♂ L$. However, if we multiply both sides of (11) on a>0 and add $b\in\mathbb{R}$ to both sides of (11) we obtain

(12) $$u_♀(y) = p(au_♂(x_-) + b) + (1\text{-}p)u_♀(z),$$

where we used the fact that $u_♀(x_+) = au_♂(x_+) + b$ for all $x_+ \in X_+$ due to part a) of Proposition 2. Since $u_♀(x_-) < au_♂(x_-) + b$, then it must be the case that $u_♀(y) > pu_♀(x_-) + (1\text{-}p)u_♀(z)$. Hence, $L_+ \succ_♀ L$ due to condition (1). In other words, we constructed two lotteries $L_+ \in \mathscr{L}_+$ and $L \in \mathscr{L}$ such that $L_+ \sim_♂ L$ but $L_+ \succ_♀ L$.

To summarize, if parts a)-c) of Proposition 2 hold then conditions a)-c) of Definition 1 are satisfied i.e. an individual ♀ is more loss averse than an individual ♂. Let us now prove the necessity of parts a)-c) of Proposition 2. If an individual ♀ is more loss averse than an individual ♂ then part a) of Proposition 2 holds due to Corollary 2.

Suppose that an individual ♀ is more loss averse than an individual ♂ but there is a loss $x_- \in X_-$ such that $u_♀(x_-) > au_♂(x_-) + b$. In such case, for the two lotteries $L_+ \in \mathscr{L}_+$ and $L \in \mathscr{L}$ that we constructed above we must have $L_+ \sim_♂ L$ but $L \succ_♀ L_+$. However, this contradicts to condition b) in Definition 1 i.e. in this case an individual ♀ is not more loss averse than an individual ♂. Thus, part b) of Proposition 2 must hold for any $x_- \in X_-$.

Finally, if part c) of Proposition 2 does not hold, i.e. $u_♀(x_-) = au_♂(x_-) + b$ for all $x_- \in X_-$, then $L_+ \sim_♂ L$ implies $L_+ \sim_♀ L$ for all $L_+ \in \mathscr{L}_+$ and $L \in \mathscr{L}$ due to (1) and condition c) of Definition 1 cannot be satisfied. *Q.E.D.*

Proof of Proposition 3.

Blavatskyy (2008b) proves the following result.

An expected utility maximizer ♀ with utility function $u_♀: X \to \mathbb{R}$ is more risk averse than an expected utility maximizer ♂ with utility function $u_♂: X \to \mathbb{R}$ if and only if

(13) $$\frac{u_♀(y)-u_♀(x)}{u_♀(z)-u_♀(y)} \geq \frac{u_♂(y)-u_♂(x)}{u_♂(z)-u_♂(y)},$$

for any $x, y, z \in X$ such that $u_♀(x) < u_♀(y) < u_♀(z)$ and there exists at least one triple of outcomes $\{x, y, z\} \subset X$ for which inequality (13) holds with strict inequality.

Let $z \in X$ be the most preferred outcome for the individual ♀. In case there are several such outcomes, we simply let z to be one of them. Let $y \in X$ be (one of) the second most preferred outcome(s) for the individual ♀. Note that Definition 4 implies that outcomes z and y are correspondingly the best and the second best outcome for the individual ♂ as well.

Let $a = \frac{u_♀(z) - u_♀(y)}{u_♂(z) - u_♂(y)}$ and $b = u_♀(y) - a u_♂(y)$. Notice that $u_♀(z) = a u_♂(z) + b$ and $u_♀(y) = a u_♂(y) + b$. If the set X_+ has no more than two elements that are distinct in terms of desirability, it must be the case that $u_♀(x_+) = a u_♂(x_+) + b$ for all $x_+ \in X_+$. Thus, condition a) of Proposition 2 is satisfied.

Condition (13) implies that $u_♀(x) \leq a u_♂(x) + b$ for all outcomes $x \in X$. Since $X_- \subset X$, this implies that condition b) of Proposition 2 is satisfied. Finally, we know that there is at least one outcome $x \in X$, which is less desirable than the second-best outcome, such that $u_♀(x) < a u_♂(x) + b$. If the set X_+ has no more than two elements that are distinct in terms of desirability, then such outcome x must belong to the subset of losses X_-. Hence, condition c) of Proposition 2 is satisfied as well.

To summarize, we found two numbers $a>0$ and $b \in \mathbb{R}$ such that all conditions of Proposition 2 are satisfied provided that the set X_+ has no more than two elements that are distinct in terms of desirability. In other words, the individual ♀ is more loss averse than the individual ♂.

Similarly, let $x \in X$ be (one of) the least preferred outcome(s) and let $y \in X$ be (one of) the second least preferred outcome(s) for the individual ♀. Definition 4 implies that outcomes x and y are correspondingly the worst and the second worst outcome for the individual ♂ as well.

Let $c = \frac{u_♂(y) - u_♂(x)}{u_♀(y) - u_♀(x)}$ and $d = u_♂(y) - a u_♀(y)$. Using these two numbers we can renormalize the utility function of the individual ♀ so that $u_♂(x) = c u_♀(x) + d$ and $u_♂(y) = c u_♀(y) + d$. If the set X_- has no more than two elements that are distinct in terms of desirability, it must be the case that $u_♂(x_-) = c u_♀(x_-) + d$ for all $x_- \in X_-$.

Condition (13) implies that $u_{♂}(z) \geq cu_{♀}(z) + d$ for all outcomes $z \in X$ and there is at least one outcome $z \in X$ for which this inequality holds with strict inequality. Hence, we found two numbers $c>0$ and $d \in \mathbb{R}$ such that:

a) $u_{♂}(x_-) = cu_{♀}(x_-) + d$ for all $x_- \in X_-$,

b) $u_{♂}(x_+) \geq cu_{♀}(x_+) + d$ for all $x_+ \in X_+$,

c) there exists a gain $x_+ \in X_+$ such that $u_{♂}(x_+) \geq cu_{♀}(x_+) + d$.

According to Corollary 3, the individual ♂ is then more gain prone than the individual ♀. *Q.E.D.*

Proof of Proposition 4.

We will first prove the sufficiency of conditions a)-d) in Proposition 4. Consider two arbitrary lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$. We will first prove that part a) of Definition 1 must hold i.e. $L_+ \succ_{♂} L$ implies $L_+ \succ_{♀} L$.

Condition (2) implies that $L_+ \succ_{♂} L$ if and only if

$$\begin{aligned} (14)\quad & \sum_{x_+ \in X_+} u_{♂}(x_+)\left[w_{♂}\left(\sum_{\substack{y_+ \in X_+ \\ u_{♂}(y_+) \geq u_{♂}(x_+)}} L_+(y_+) \right) - w_{♂}\left(\sum_{\substack{y_+ \in X_+ \\ u_{♂}(y_+) > u_{♂}(x_+)}} L_+(y_+) \right) \right] > \\ & > \sum_{x \in X} u_{♂}(x)\left[w_{♂}\left(\sum_{\substack{y \in X \\ u_{♂}(y) \geq u_{♂}(x)}} L(y) \right) - w_{♂}\left(\sum_{\substack{y \in X \\ u_{♂}(y) > u_{♂}(x)}} L(y) \right) \right]. \end{aligned}$$

For any $a>0$ and $b \in \mathbb{R}$ we can rewrite condition (14) as follows

$$\begin{aligned} & \sum_{x_+ \in X_+} \left[au_{♂}(x_+) + b \right]\left[w_{♂}\left(\sum_{\substack{y_+ \in X_+ \\ u_{♂}(y_+) \geq u_{♂}(x_+)}} L_+(y_+) \right) - w_{♂}\left(\sum_{\substack{y_+ \in X_+ \\ u_{♂}(y_+) > u_{♂}(x_+)}} L_+(y_+) \right) \right] > \\ (15)\quad & > \sum_{x_+ \in X_+} \left[au_{♂}(x_+) + b \right]\left[w_{♂}\left(\sum_{\substack{y_+ \in X_+ \\ u_{♂}(y_+) \geq u_{♂}(x_+)}} L(y_+) \right) - w_{♂}\left(\sum_{\substack{y_+ \in X_+ \\ u_{♂}(y_+) > u_{♂}(x_+)}} L(y_+) \right) \right] + \\ & + \sum_{x_- \in X_-} \left[au_{♂}(x_-) + b \right]\left[w_{♂}\left(\sum_{\substack{y \in X \\ u_{♂}(y) \geq u_{♂}(x_-)}} L(y) \right) - w_{♂}\left(\sum_{\substack{y \in X \\ u_{♂}(y) > u_{♂}(x_-)}} L(y) \right) \right]. \end{aligned}$$

If part a) of Proposition 4 holds, then the two individuals ♀ and ♂ have identical probability weighting functions. If part b) of Proposition 4 holds, there exist $a>0$ and $b \in \mathbb{R}$ such that $u_{♀}(x_+) = au_{♂}(x_+) + b$ for all $x_+ \in X_+$ and we can rewrite (15) as follows

$$\begin{aligned}&\sum_{x_+\in X_+} u_♀(x_+)\left[w_♀\left(\sum_{\substack{y_+\in X_+\\ u_♀(y_+)\ge u_♀(x_+)}} L_+(y_+)\right)-w_♀\left(\sum_{\substack{y_+\in X_+\\ u_♀(y_+)> u_♀(x_+)}} L_+(y_+)\right)\right]>\\ &>\sum_{x_+\in X_+} u_♀(x_+)\left[w_♀\left(\sum_{\substack{y_+\in X_+\\ u_♀(y_+)\ge u_♀(x_+)}} L(y_+)\right)-w_♀\left(\sum_{\substack{y_+\in X_+\\ u_♀(y_+)> u_♀(x_+)}} L(y_+)\right)\right]+\\ &+\sum_{x_-\in X_-}\left[au_♂(x_-)+b\right]\left[w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)\ge u_♂(x_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)> u_♂(x_-)}} L(y)\right)\right].\end{aligned}\tag{16}$$

Let $z_-\in X_-$ be the most desirable loss for an individual ♀ i.e. $u_♀(z_-)\ge u_♀(x_-)$ for all $x_-\in X_-$. Let $Z_-\subset X_-$ be the set of all losses that an individual ♂ finds at least as good as z_- i.e. $u_♂(x_-)\ge u_♂(z_-)$ for all $x_-\in Z_-$. If part c) of Proposition 4 holds then we can rewrite

$$\begin{aligned}&\sum_{x_-\in Z_-}\left[au_♂(x_-)+b\right]\left[w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)\ge u_♂(x_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)> u_♂(x_-)}} L(y)\right)\right]\ge\\ &\ge\sum_{x_-\in Z_-}\left[au_♂(z_-)+b\right]\left[w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)\ge u_♂(x_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)> u_♂(x_-)}} L(y)\right)\right]=\\ &=\left[au_♂(z_-)+b\right]\left[w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)\ge u_♂(z_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)> u_♀(z_-)}} L(y)\right)\right]\ge\\ &\ge u_♀(z_-)\left[w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)\ge u_♂(z_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)> u_♀(z_-)}} L(y)\right)\right]\ge\\ &\ge\sum_{x_-\in Z_-} u_♀(x_-)\left[w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)\ge u_♀(x_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)> u_♀(x_-)}} L(y)\right)\right].\end{aligned}\tag{17}$$

We can repeat the above argument for a smaller set of losses $X_-\backslash Z_-$ and so forth. Since the set X_- is finite, we then arrive at the result

$$\begin{aligned}&\sum_{x_-\in X_-}\left[au_♂(x_-)+b\right]\left[w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)\ge u_♂(x_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♂(y)> u_♂(x_-)}} L(y)\right)\right]\ge\\ &\ge\sum_{x_-\in X_-} u_♀(x_-)\left[w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)\ge u_♀(x_-)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)> u_♀(x_-)}} L(y)\right)\right].\end{aligned}\tag{18}$$

Using (18) we can rewrite (16) as follows

$$\begin{aligned}&\sum_{x_+\in X_+} u_♀(x_+)\left[w_♀\left(\sum_{\substack{y_+\in X_+\\ u_♀(y_+)\ge u_♀(x_+)}} L_+(y_+)\right)-w_♀\left(\sum_{\substack{y_+\in X_+\\ u_♀(y_+)> u_♀(x_+)}} L_+(y_+)\right)\right]>\\ &>\sum_{x\in X} u_♀(x)\left[w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)\ge u_♀(x)}} L(y)\right)-w_♀\left(\sum_{\substack{y\in X\\ u_♀(y)> u_♀(x)}} L(y)\right)\right].\end{aligned}\tag{19}$$

If (19) holds then $L_+\succ_♀ L$ due to (2). Hence, part a) of Definition 1 must hold.

To prove that part b) of Definition 1 must hold i.e. $L_+ \sim_♂ L$ implies $L_+ \succsim_♀ L$ for all $L_+ \in \mathcal{L}_+$ and all $L \in \mathcal{L}$ we just need to replace the sign ">" with the sign "=" in (14)-(16) and with the sign "≥" in (19).

Finally, let us prove that part c) of Definition 1 must hold i.e. there exist $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ such that $L_+ \sim_♂ L$ and $L_+ \succ_♀ L$. If part d) of Proposition 4 holds then there exists $x_- \in X_-$ such that $u_♀(x_-) < au_♂(x_-) + b$. Let $y,z \in X_+$ be two gains such that $u_♂(x_-) < u_♂(y) < u_♂(z)$. Let L_+ be a lottery that yields y for sure and let L be a lottery that yields x_- with probability 1-p and z with probability p. Since function $w_♂(p)$ is strictly increasing in p with $w_♂(0)=0$ and $w_♂(1)=1$, there exists a probability p such that

(20) $$u_♂(y) = (1- w_♂(p))u_♂(x_-) + w_♂(p)u_♂(z).$$

If (20) holds then $L_+ \sim_♂ L$ due to (2). If parts a) and b) of Proposition 4 hold, we can rewrite (20) as follows

(21) $$u_♀(y) = (1- w_♀(p))(au_♂(x_-) + b) + w_♀(p)u_♀(z).$$

Since $u_♀(x_-) < au_♂(x_-)+b$, then (21) implies $u_♀(y) > (1-w_♀(p))u_♀(x_-)+w_♀(p)u_♀(z)$ i.e. $L_+ \succ_♀ L$ due to (2). Thus, we constructed lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ such that $L_+ \sim_♂ L$ but $L_+ \succ_♀ L$.

Hence, if parts a)-d) of Proposition 4 hold then conditions a)-c) of Definition 1 are satisfied i.e. an individual ♀ is more loss averse than an individual ♂. Let us now prove the necessity of parts a)-d) of Proposition 4. If an individual ♀ is more loss averse than an individual ♂ then parts a) and b) of Proposition 4 hold due to Corollary 5.

Suppose that an individual ♀ is more loss averse than an individual ♂ but there is a loss $x_- \in X_-$ such that $u_♀(x_-) > au_♂(x_-) + b$. In such case, for the two lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ that we constructed above we must have $L_+ \sim_♂ L$ but $L \succ_♀ L_+$. However, this contradicts to condition b) in Definition 1 i.e. in this case an individual ♀ is not more loss averse than an individual ♂. Thus, part c) of Proposition 4 must hold for any $x_- \in X_-$.

Finally, if part d) of Proposition 4 does not hold, i.e. $u_♀(x_-) = au_♂(x_-) + b$ for all $x_- \in X_-$, then $L_+ \sim_♂ L$ implies $L_+ \sim_♀ L$ for all $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ due to (2) and condition c) of Definition 1 cannot be satisfied. *Q.E.D.*

Proof of Proposition 5.

Consider two arbitrary lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$. Let us prove that if conditions a)-d) of Proposition 5 are satisfied then $P_♀(L_+,L) \geq P_♂(L_+,L)$. Equation (4) implies that

$$P_♀(L_+,L) = \varphi_♀\left(\sum_{x_+\in X_+} L_+(x_+)u_♀(x_+) - \sum_{x\in X} L(x)u_♀(x)\right). \tag{22}$$

If condition a) of Proposition 5 holds, we can rewrite equation (22) as follows

$$P_♀(L_+,L) = \varphi_♀\left(\sum_{x_+\in X_+} \left[L_+(x_+) - L(x_+)\right]\left[au_♂(x_+)+b\right] - \sum_{x_-\in X_-} L(x_-)u_♀(x_-)\right). \tag{23}$$

If condition b) of Proposition 5 holds and given that function $\varphi_♀(.)$ is strictly increasing, we can rewrite equation (23) as follows

$$P_♀(L_+,L) \geq \varphi_♀\left(\sum_{x_+\in X_+} \left[L_+(x_+) - L(x_+)\right]\left[au_♂(x_+)+b\right] - \sum_{x_-\in X_-} L(x_-)\left[au_♂(x_-)+b\right]\right). \tag{24}$$

Inequality (24) can be rearranged into

$$P_♀(L_+,L) \geq \varphi_♀\left(a\left[\sum_{x_+\in X_+} L_+(x_+)u_♂(x_+) - \sum_{x\in X} L(x)u_♂(x)\right]\right). \tag{25}$$

If conditions c) and d) of Proposition 5 are satisfied then we can rewrite (25) as

$$P_♀(L_+,L) \geq \varphi_♂\left(\sum_{x_+\in X_+} L_+(x_+)u_♂(x_+) - \sum_{x\in X} L(x)u_♂(x)\right). \tag{26}$$

The last inequality (26) simply states that $P_♀(L_+,L) \geq P_♂(L_+,L)$ due to equation (4).

Let us now prove that if conditions a)-e) of Proposition 5 are satisfied then there exist two lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ such that $P_♀(L_+,L) > P_♂(L_+,L)$. According to condition e) of Proposition 5, at least one of the following conditions must hold: 1) there exists a loss $x_- \in X_-$ such that $u_♀(x_-) < au_♂(x_-) + b$; 2) there exists $v \in (\delta, \Delta]$ such that $\varphi_♀(av) > \varphi_♂(v)$.

If condition 1) holds, then for any lottery $L \in \mathcal{L}$ that yields such an outcome $x_- \in X_-$ with a positive probability inequalities (24)-(26) hold as strict inequalities and we have $P_♀(L_+,L) > P_♂(L_+,L)$. If condition 2) holds, then for any two lotteries $L_+ \in \mathcal{L}_+$ and $L \in \mathcal{L}$ such that $\sum_{x\in X} L_+(x)u_♂(x) - \sum_{x\in X} L(x)u_♂(x) = v$, inequality (26) holds as strict inequality and we have again $P_♀(L_+,L) > P_♂(L_+,L)$. *Q.E.D.*