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Paying the partners

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Abstract When three or more individuals with disparate talents form a business partnership, they may find it difficult to agree on how their profits will be divided. This paper explores a rule for dividing the profits that depends only on the partners' estimates of the relative contributions of other partners. No partner can affect his own share by the input that he provides. If there is a division that is consistent with the relative contributions suggested by all partners, then the division rule assigns these shares. We provide the intuition for the division rule and investigate several of its properties.

Keywords Fair division · Aggregating opinions · Dirichlet distribution

JEL Classification D33 · D72

1 Introduction

Consider a group of three or more persons who must divide an amount of money or some other divisible good among themselves, when they recognize that their claims are subjective and unequal. The canonical example is partners in a law firm or some other enterprise who must divide their profit at the end of a year. The partners understand that their contributions to the enterprise change from year to year in unpredictable ways, so a single agreement at the beginning of the partnership will not suffice to divide the profit among them in proportion

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to their contributions. While they may be able to measure an indicator of productivity such as billable hours objectively, they recognize the need to take account of other contributions such as managing the firm, bringing in cases for other partners, and providing informal counsel. The partners agree that no disinterested mediator or arbitrator would be as able as they are to discern their relative contributions.[1]

The more subjective the criteria for division are, the more difficult and time consuming it is likely to be to reach consensus, especially for large groups. The partners may therefore find it useful to employ a formal decision rule that yields the division as a function of input from all partners. What rule should they use?

It is likely that the partners will require that their rule be *strategy-proof*—that is, they will require that no partner be able to affect his own share by the input that he provides.[2] The partners would consider a rule that permitted a partner to affect his own share unacceptable because it would give a partner an incentive to provide the input that would maximize his own share, rather than his sincere beliefs about the contributions of the other partners. It is also likely that the partners will require that their rule be *objective*—that is, they will require that the calculation of the shares not depend on any partner's statement about what he deserves relative to others. While strategy-proofness requires that a partner be unable to affect his own share, objectivity requires that a partner be unable to affect the share of any other partner by reporting a different belief about his own contribution. Because people tend to be poor judges of themselves, the partners would be suspicious of a rule that used a partner's assessment of his own contribution when calculating any of the shares. Finally, it is likely that the partners will require that their rule be *consensual*—if there is a set of shares that is consistent with all of the inputs that the partners provide, they will require the rule to assign those shares. They would regard a division rule that was not consensual to be an inadequate reflection of their views.

Quite surprisingly, if three partners agree that they want a rule with these three intuitive characteristics, then they have already chosen their division rule! There is only one division rule for three partners that is strategy-proof, objective, and consensual. This rule is a function of all three partners' proposals regarding the *relative* shares of the two partners other than themselves. In the case of four or more partners, objectivity permits each pair of partners to be assessed by two or more partners, and there are an infinite number of ways of aggregating the proposals for the share ratio for each pair. However, there is a natural extension of the unique rule for three partners to four or more, once there is a settled method of aggregating the proposals for each share ratio. There can be no strategy-proof rule for two partners because it is impossible for one partner to specify what the other ought to receive without implying what he himself ought to receive. However, the rule can be adapted to two partners by employing a mediator who receives a share of the amount to be divided as his mediation fee and who effectively acts as a third partner.

In this paper, we provide the intuition for these division rules. De Clippel et al. (2007) describe the mathematics of these rules, but their focus on mathematical rigor makes it

[1] Klein et al. (1978) suggest that partnerships are formed by persons whose efforts are uniquely complementary to one another's and who are thus more vulnerable to the appropriation of their quasi-rents under purely contractual relationships. The resulting interdependence among the partners' contributions makes it impossible to determine each partner's individual contribution. Thus the partners must base the division on their subjective assessments of these contributions.

[2] Strategy-proofness has been defined traditionally as the dominance of truth-telling. In an environment in which there is no objective truth, our usage represents a reasonable extension of the meaning of strategy-proofness.

somewhat difficult to discern intuitive reasons for their results. In addition, we investigate five properties of the division rules that de Clippel et al. do not discuss, and which are likely to be of particular interest to those who wish to use the rules. First, in the case of three partners, the rule always divides less than the total amount available for division unless the partners' proposals are consistent, and the surplus needs to be wasted to eliminate all incentives for strategic behavior. In Sect. 2 of this paper we investigate how the size of the surplus varies with the degree of divergence of the share ratios from consistency, and find that the surplus is generally very small. This makes the division rule attractive even for groups with only three members. Second, in Sect. 3 we show that the properties of strategy-proofness, objectivity, and consensuality are necessary and sufficient to determine our division rule for three partners, and that the properties are independent.[3]

Third, when there are more than three partners, there are many possible ways of aggregating the proposals of two or more partners regarding the relative shares of any two partners other than themselves. We analyze several aggregation methods and assess their properties numerically in Sect. 4. Fourth, the question arises of how to accommodate incomplete proposals. In Sect. 5 we go beyond the case, discussed by de Clippel et al. (2007), of large partnerships where not everyone is able to make a proposal about the relative share of every other partner, to the case where there are no proposals for some pairs of partners. Finally, we describe an adaptation of the division rule to the case of two partners in Sect. 6. Our discussion and analyses provide essential information for any group that might wish to adopt the division rules.

On what information do partners base their assessment of how much their fellow partners should receive? There are two meaningful interpretations of the partners' assessments of the contributions of the other partners, leading to different interpretations of our division rule. First, one can assume that each partner observes a noisy signal of the true contributions of the other partners. From this point of view, a division rule is an estimator that uses noisy signals to derive estimates of the underlying correct contributions to which the partners are entitled. In this context, strategy-proofness is attractive because it reduces a partner's incentive to report anything other than the signal that he observes.

Alternatively, every partner may be essential for the partnership to operate so that it is meaningless to define a partner's contribution as a share of the amount to be divided. Rather than rewarding past productivity, the partners' objective might then be to divide the amount in a way that maximizes the probability that the partnership will continue with all partners present. In such a case, a division rule can be interpreted simply as an aggregator of the partners' opinions of how much each partner needs to receive to be willing to continue in the partnership. In this context, strategy-proofness is attractive because it reduces a partner's incentive to make a proposal that reflects anything other than his honest opinion of what is necessary to induce the other partners to remain in the partnership.

Both interpretations are meaningful because a rule that proposes a division on the basis of the partners' input is a special type of voting rule; it should therefore not be surprising that division rules can be interpreted in the same ways in which voting rules have been interpreted. Voting rules have sometimes been interpreted as estimators of the true will of the people (see Young 1988, for a discussion), but voting rules can also be viewed simply as aggregators of opinions if the notion of the "will of the people" is not palatable. Just as voting is meaningful under both interpretations, our division rule is applicable to both cases.

[3] De Clippel et al. (2007) combine our conditions of strategy-proofness and objectivity into the single condition of *impartiality*.

Mumy (1981) discusses a problem that is complementary to ours, in which each person is the only one who knows what he deserves. He develops an example based on a Joseph Conrad story in which a storm at sea causes the separate savings of many passengers to be scrambled together. The ship's captain, despairing of giving each passenger what he had, divides the scrambled savings equally. Mumy shows that it is possible to motivate all passengers to claim only what they had by using a rule that each passenger will receive what he requests minus a multiple, greater than one, of the amount by which the sum of the requests exceeds the total available.

The idea of penalizing people when the sum of the claims exceeds what is available also arises in the game known as "divide the dollar" (DD). Brams and Taylor (1994) analyze two variants of DD that incorporate player-specific entitlements. One variant employs announced, player-specific integer entitlements e_i that sum to the available resources, along with a rule that if the players' claims, b_i, sum to more than the available resources, then the claims are given priority in the order defined by $b_i - e_i$, starting with the smallest value. Brams and Taylor show that when there are more than two players, the iterative elimination of weakly dominated strategies induces an outcome in which each player claims $e_i + 1$.

Brams and Taylor's other variant of DD with player-specific entitlements comes closest to the case in which we are interested. It is similar to the previous variant, except that each player proposes a claim for every other player as well as herself, with the sum of her proposals required to equal the available resources. While Brams and Taylor prove theorems about other variants of DD, they provide only an informal discussion of this case. They note that if all players are claiming nearly everything for themselves, any one player will benefit greatly by claiming less. They write, "This logic eventually will carry the players towards the ratings they think the other players will give them. Provided players are honest in their assessments of others (there seems no good reason they should not be in the absence of collusion), players can do no better than try to reflect the others' assessments, slightly perturbed, *in their own requests*." Brams and Taylor argue that their suggestion "would be viewed as fair by the players, because these procedures benefit players whose self-ratings agree with those of others" (p. 228).

We regard it as unlikely that Brams and Taylor's suggestion would be viewed as fair. Being able to guess the evaluation that others will give to oneself is not a talent that is particularly deserving of reward. Furthermore, out of equilibrium, a player's maximum payoff is obtained not from a self-evaluation that is in line with the evaluations of others but rather from the greatest self-evaluation that can be fully paid before the money runs out; the incentive to provide a self-evaluation that is in line with the evaluations of oneself by others occurs only when all others have provided self-evaluations in line with the evaluations of others. If Brams and Taylor's proposal is operated in two or more rounds so that the players can have reasonable estimates of the evaluations that others will give them, then each player has an incentive to make a large claim for herself prior to the final round so that others will be induced to expect to have to settle for less. What is needed to avoid inappropriate incentives is a mechanism in which a player's self-evaluation never has any effect on what she receives. That is what we offer.

We are not aware of any previous analysis, besides that of de Clippel et al. (2007), of division rules that are consensual, strategy-proof, and objective. Objective division has some affinity to the literature on how to best combine the opinions of disinterested individuals (for example, the Condorcet Jury problem), which is nevertheless different from our problem because our partners are not disinterested. There is also a relation to the literature on bargaining, although bargaining theory emphasizes strategic behavior, which we rule out by requiring strategy-proofness.

Two additional attractive criteria for division rules are efficiency and fairness. A division rule is efficient if it divides the entire amount without surplus. We show in Sects. 2 and 3 that in the case of three partners, perfect efficiency is not compatible with strategy-proofness, objectivity, and consensuality. However, we also show that the loss of efficiency from requiring these three criteria is likely to be quite small. In Sect. 4 we explain why efficiency can be attained along with the other criteria when there are more than three partners.

In the context of the problem explored in this paper, fairness cannot be related to individual productivity because no one has completely reliable knowledge of anyone's productivity. Still, all three of our criteria are related to fairness in some way. Strategy-proofness removes a partner's temptation to increase his own share through an exaggerated estimate of his own productivity, which would be patently unfair. Objectivity ensures that a partner's belief about his own productivity does not affect the relative shares of his fellow partners, which might also be considered unfair. Consensuality is a variation on the idea of unanimity, adjusting unanimity for the fact that our division rule pays no attention to anyone's belief about his own productivity. Thus consensuality is related to fairness in the same way as unanimity is.

In Sect. 4 we examine the accuracy of our division rule when there is a set of objectively correct shares that the partners deserve and a specified statistical process generates each partner's stated beliefs about the relative shares of his fellow partners. In such a setting, a division rule can be called fair if it assigns to each partner his correct share. Interpreting the division rule as an estimator of the correct shares, we assess its bias as a function of the distribution of the correct shares, the size of the partnership, the variance of distribution from which the partners' honest beliefs are drawn, and the way in which the partners' beliefs are aggregated.

2 The division rule for three partners

Consider a group of three partners who must divide the amount X among themselves. They cannot divide the amount in proportion to their contributions to their enterprise, either because they find it impossible to measure the true contributions of all of them objectively with sufficient precision, or because they find it impossible to separate their individual contributions in a meaningful way. Although it would be possible—though prohibitively expensive—to estimate the contribution of any one partner by removing him from the partnership, it would not be possible to estimate the contributions of all partners in this way. Thus the partners can base their division of X only on their personal assessments of the contributions of all of them. Depending on the definition of a partner's contribution, these assessments can be interpreted either as estimates of the partners' true contributions or as opinions about how much each partner ought to receive, for example, to induce him to stay in the partnership.

Define the share of partner i as s_i. If the shares sum to 1 so that nothing is wasted, then partner 1's share can be expressed as

$$s_1 = \frac{s_1}{s_1 + s_2 + s_3} = \frac{1}{1 + s_2/s_1 + s_3/s_1} = \frac{1}{1 + R_{21} + R_{31}}, \tag{1}$$

where $R_{ij} = s_i/s_j$ is the ratio of the shares that partners i and j receive. The ratio of the shares of partners 1 and 2 can be expressed as the product of the ratio of the shares of partners 1 and 3 and the ratio of the shares of partners 3 and 2, or

$$R_{12} = R_{13} R_{32}. \tag{2}$$

Equation (1) implies that individual shares can be computed as functions of relative shares—that is, one can determine, say, partner i's share from the relative shares R_{ji} and R_{ki} that partners j and k receive in relation to him. This suggests the following approach to determining the three shares:

Assume that when asked, each partner is willing and able to make proposals regarding the shares of all partners. From proposals regarding absolute shares, one can derive proposals for ratios of shares. Define r^i_{jk} to be partner i's proposal for the ratio of partner j's share to partner k's share, for any i, j, and k. Use such proposals in place of the share ratios in (1) to determine the three shares.

Thus the shares of the three partners under this approach are

$$s_1 = \frac{1}{1 + r^2_{31} + r^3_{21}}; \qquad s_2 = \frac{1}{1 + r^1_{32} + r^3_{12}}; \qquad s_3 = \frac{1}{1 + r^1_{23} + r^2_{13}}. \tag{3}$$

This division rule is strategy-proof because the calculation of any partner's share depends solely on the proposals of the other two partners, so no partner can affect his own share, regardless of what he proposes. The division rule is objective because no partner is asked to rank himself relative to any other partner. Finally, the division rule is consensual because the ratio of any two shares equals the respective proposal if and only if the three partners propose consistent relative shares. For example, the ratio of s_3 and s_2,

$$\frac{s_3}{s_2} = \frac{\frac{1}{1+r^1_{23}+r^2_{13}}}{\frac{1}{1+r^3_{12}+r^1_{32}}} = \frac{1 + r^3_{12} + r^1_{32}}{1 + r^1_{23} + r^2_{13}}, \tag{4}$$

equals r^1_{32} if and only if $r^3_{12} = r^2_{13} r^1_{32}$. (Multiply the denominator by r^1_{32} and remember that $r^1_{23} r^1_{32} = 1$.)

The relative shares that the division rule assigns to partners j and k are always between partner i's actual proposal and the proposal $r_{jk} = r^k_{ji} r^j_{ik}$ that would make partner i's proposal consistent with the proposals of partners j and k. For example, the ratio of s_3 and s_2 in (4) exceeds r^1_{32} and is below $r^2_{31} r^3_{12}$ if and only if $r^1_{32} < r^2_{31} r^3_{12}$, and vice versa.[4] Thus no partner needs to fear that his proposal will change the relative shares either in the opposite direction or beyond what he has proposed.

By construction, the three shares sum to 1 if the three proposals are consistent. To see what happens when the three proposals are not consistent, express the sum of the three shares as

$$\begin{aligned}
\sum_i s_i &= \frac{1}{1 + r^3_{21} + r^2_{31}} + \frac{1}{1 + r^1_{32} + r^3_{12}} + \frac{1}{1 + r^2_{13} + r^1_{23}} \\
&= \frac{1}{1 + r^3_{21} + 1/r^2_{13}} + \frac{1}{1 + r^1_{32} + 1/r^3_{21}} + \frac{1}{1 + r^2_{13} + 1/r^1_{32}} \\
&= \frac{r^2_{13}}{r^2_{13} + r^3_{21} r^2_{13} + 1} + \frac{r^3_{21}}{r^3_{21} + r^1_{32} r^3_{21} + 1} + \frac{r^1_{32}}{r^1_{32} + r^2_{13} r^1_{32} + 1}.
\end{aligned} \tag{5}$$

[4] Note that $(1 + r^1_{32})/(1 + r^1_{23}) = r^1_{32}$.

When this sum of three fractions is put over a common denominator, the result is a fraction with 27 terms in the numerator and 27 terms in the denominator, with all terms positive. Of these 54 terms, there are 25 in the numerator that are matched by the same term in the denominator. Denoting the sum of these 25 terms by $\Psi > 0$ yields, as an expression for the unallocated surplus,

$$1 - \sum_i s_i = 1 - \frac{\Psi + 2r_{32}^1 r_{21}^3 r_{13}^2}{\Psi + (r_{32}^1 r_{21}^3 r_{13}^2)^2 + 1} = \frac{(r_{32}^1 r_{21}^3 r_{13}^2)^2 - 2r_{32}^1 r_{21}^3 r_{13}^2 + 1}{\Psi + (r_{32}^1 r_{21}^3 r_{13}^2)^2 + 1}$$
$$= \frac{((r_{32}^1 r_{21}^3 r_{13}^2) - 1)^2}{\Psi + (r_{32}^1 r_{21}^3 r_{13}^2)^2 + 1}. \tag{6}$$

The numerator is zero when the proposals are consistent (that is, when $r_{23}^1 = r_{21}^3 r_{13}^2 \Leftrightarrow r_{32}^1 r_{21}^3 r_{13}^2 = 1$) and positive otherwise. Thus when the proposals are not consistent, the sum of the three shares is always less than 1 and the division rule divides less than X. Because the shares never sum to more than 1, the division rule is *feasible*: it never allocates more than the amount to be divided. However, the sum of the three shares generally will be very close to 1 because the 25 terms in Ψ are all positive, so that the denominator is generally much greater than the numerator. Thus the unallocated surplus generally will be small.

The division rule has a straightforward geometric interpretation. The set of shares that sum to 1 can be represented as vectors with three non-negative components (representing the shares of partners 1, 2, and 3 respectively) that sum to 1. The possible vectors can be represented geometrically by a triangle in a three-dimensional coordinate system, with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, represented by points A, B, and C respectively, in Fig. 1. A proposal for the ratio of two shares is represented by the set of points in the triangle in which the coordinates corresponding to the two shares have the specified ratio. This is a straight line segment from one of the vertices of the triangle to a point on the opposite side. The vertex represents a 100% share for the partner not included in the ratio. The point on the opposite side represents a 0% share for that partner. For example, the points on line segment AD represent the possible shares for the three partners that are consistent with a proposal by partner 1 that partner 3 receive 80% as much as partner 2 ($r_{32}^1 = 0.8$), the points on line segment BE represent the possible shares for the three partners that are consistent with a proposal by partner 2 that partner 1 receive 80% as much as partner 3 ($r_{13}^2 = 0.8$), and the points on line segment CF represent the possible shares for the three partners that are consistent with a proposal by partner 3 that partner 2 receive three times as much as partner 1 ($r_{21}^3 = 3$).

If the proposed share ratios are consistent, then the line segments emanating from the three vertices intersect at a common point whose coordinates are the three consistent shares. If the proposed share ratios are not consistent, as in Fig. 1, then the three line segments do not meet at a common point. To determine the share of partner 1, draw a line that is parallel to BC and passes through the intersection of BE and CF (the proposals of partners 2 and 3). This line represents all share combinations that give partner 1 the same share as in the one combination for all three partners that is consistent with the proposals of partners 2 and 3. Partner 1's share is then represented by the ratio GB/AB (or HC/AC).[5] The two other dashed lines yield the shares of partners 2 and 3 by similar constructs. Because the three dashed lines do not intersect in a common point, the proposals are inconsistent and

[5] Note that the triangle in Fig. 1 is not a unit triangle. Because its vertices are at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, its sides have length $\sqrt{2}$.

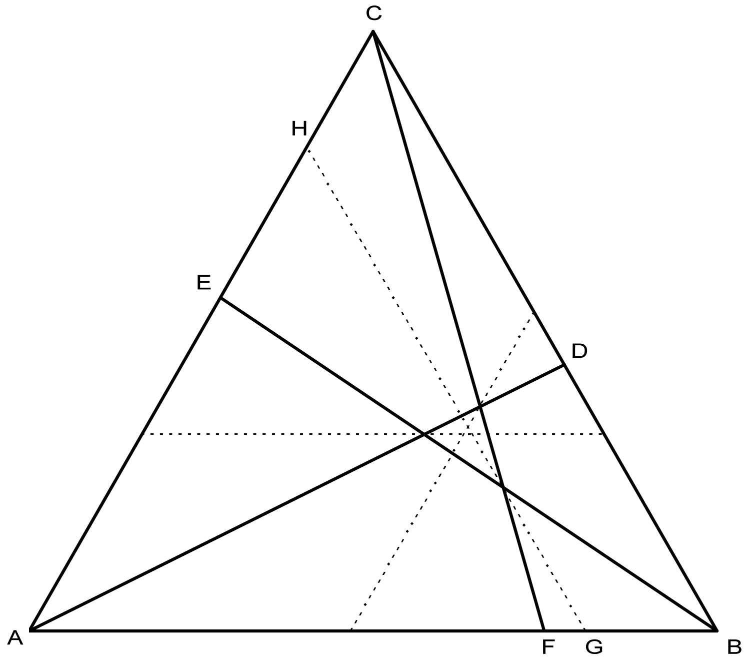

Fig. 1 A geometric representation of the division among three partners

the shares therefore sum to less than 1. Because the shares sum to less than 1, there is no point in the figure that represents the selected division. The share of the unallocated surplus is given by the ratio of the height (or the side) of the triangle formed by the three dashed lines to the height (or the side) of the original triangle.

Should the fact that the division rule divides less than X if the proposals are not consistent be a reason for concern? Figure 2 shows the surplus as a function of r^1_{32} and r^2_{13} over the range of 0.2 to 1, for $r^3_{21} = 3$. The range of 0.2 to 1 corresponds to proposals that rank one partner's share up to five times as high as another partner's share, which spans the range of proposals that we believe one is likely to encounter in applications of the division rule. The surplus is zero when $r^1_{32}r^2_{13} = 1/3$, and it is small as long as there is a reasonable degree of consistency in the proposals of the partners. Consider the same set of proposals as above (that is, $r^1_{32} = 0.8$, $r^2_{13} = 0.8$, and $r^3_{21} = 3$). These proposals imply $r^1_{32}r^3_{21}r^2_{13} = 1.92$, and the surplus (the height of the darkened rhombus in Fig. 2) is 1.29% of the amount that the partners need to divide. This surplus is very small despite the non-trivial inconsistency among the partners. The proposals of partners 1 and 2, if taken as definitive, imply that partner 2 ought to receive about one and a half times as much as partner 1 (because $1/(0.8 \cdot 0.8) = 1.56$), while partner 3 proposes a ratio twice this size. Had partner 3 proposed $r^3_{21} = 2$ instead of 3, then the surplus would have been only 0.2% of X.

The size of the surplus increases with the degree of inconsistency in the partners' proposals, and the surplus is highest, for a given value of the product of r^1_{32}, r^2_{13}, and r^3_{21}, if the three proposals are equal. Thus for any value $m = r^1_{32}r^2_{13}r^3_{21}$, the maximum surplus is

$$\max(surplus) = 1 - \frac{3}{1 + m^{1/3} + 1/m^{1/3}}, \tag{7}$$

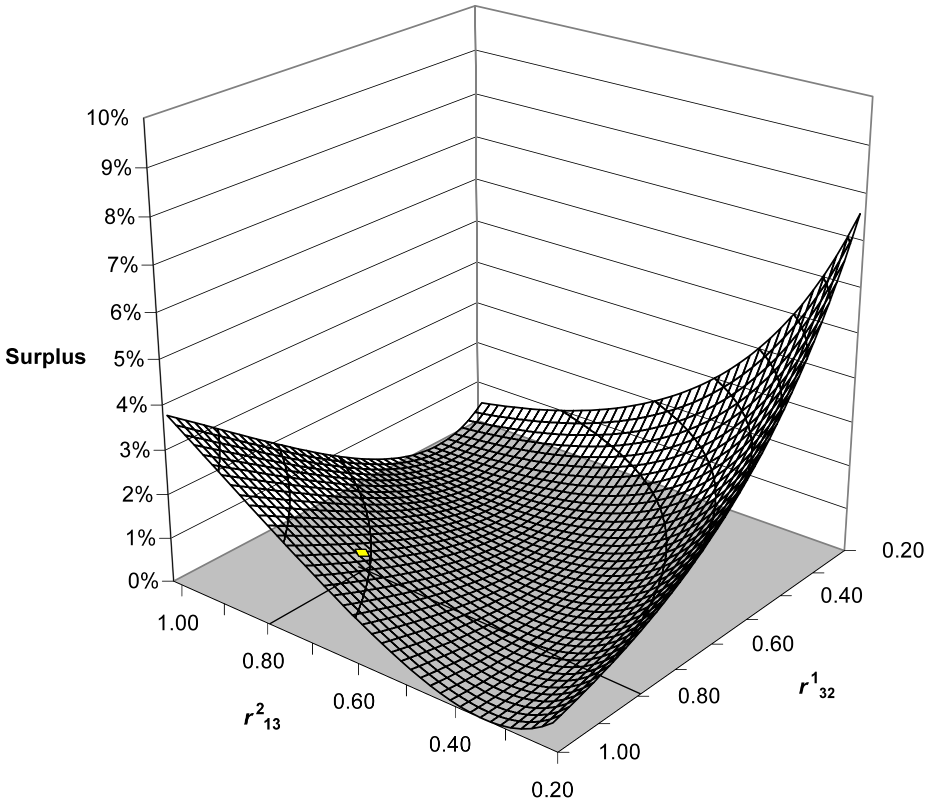

Fig. 2 Surplus as a function of proposals

while

$$f(m) = \frac{1}{27}(\ln m)^2 \tag{8}$$

provides an upper bound for the surplus. Equation (8) is a close approximation of (7) for values of m close to 1, and indicates that the maximum surplus increases quadratically with $\ln m$ for values of m close to 1.[6] However, the degree of inconsistency that is necessary to produce a substantial surplus is so high that it is very unlikely to prevail in practice. For example, if $r^1_{32} = r^2_{13} = r^3_{21} = 2$, then the proposals of partners 1 and 2 imply that partner 1 should receive 4 times as much as partner 2, while partner 3 proposes that 1 receive only half of what partner 2 receives. It is unlikely that three people with such inconsistent views of their relative contributions can maintain a viable *partnership*. And even in this extreme case, the surplus is only 14.28% of X.

The partners can dispose of the surplus in three ways. First, they can rescale the shares so that they sum to 1 and distribute the entire amount X. However, rescaling the shares opens a

[6] To see that (8) is an approximation of (7) for m close to 1, set $1 + x = m^{1/3}$, multiply both equations by 3 and compare the first three terms of the series produced by polynomial long division on (7) with the 4th order Taylor approximation for (8). The approximations are $S_7(x) = x^2 - x^3 + 2/3x^4$ and $S_8(x) = x^2 - x^3 + 11/12x^4$ respectively, which yield similar values for x close to 0.

possibility for strategic manipulation, because the size of the surplus is a function of the three proposals, and a partner's proposal now affects the share of X that he receives. Second, they can use the division rule again to divide the surplus. This also permits strategic manipulation unless, implausibly, every partner believes that a change in the size of the surplus will have no effect on the size of his own share. However, because the surplus is small as long as there is a reasonable degree of consistency in the proposals of the partners regarding relative shares, it is likely that the partners would be more concerned about maintaining goodwill in the partnership than about increasing their shares by trivial amounts. Thus the temptation to behave strategically would be fairly weak.

Finally, the partners can give away the surplus. Because the surplus is zero if the partners propose consistent share ratios, not distributing the surplus among the partners has the advantage that it provides the partners with a collective incentive to offer consistent proposals and thus be in agreement about their relative shares. However, if they donate the surplus to charity and any partner receives utility from the fact that some share of X that the other partners could have received is donated to charity, then increasing the size of the surplus may become one of his goals. Avoiding strategic behavior is therefore possible only if they dispose of the surplus in a way that does not change the utility of any partner.

Although the division rule is strategy-proof, it is nevertheless susceptible to coordinated manipulation by a coalition of two partners. If two partners agree that they will both undervalue the third partner's share, then the division rule yields a lower share for the third partner. For example, assume that the partners' sincere beliefs about the relative shares are $r_{32}^1 = 1$, $r_{13}^2 = 0.75$, and $r_{21}^3 = 0.667$, which would yield assigned shares of $s_1 = 0.333$, $s_2 = 0.286$, and $s_3 = 0.364$. If partners 1 and 2 agree to change their proposals to $r_{32}^1 = 0.75$ and $r_{13}^2 = 1$ respectively, then their shares increase to $s_1 = 0.375$ and $s_2 = 0.308$ while partner 3's share falls to $s_3 = 0.3$. In being strategy-proof for individuals but not for groups, the division rule is like the demand-revealing process.[7] However, since preference aggregation mechanisms generally are susceptible to collusive manipulation, the susceptibility of this mechanism is not surprising. In the following section, we show that it is not possible to protect partners against collusion by modifying the division rule, for example, by assigning minimum shares to which partners are entitled. So at least for groups of three partners, the division rule is for persons who can trust their partners not to collude against them. To the extent that partners have viable alternatives to remaining in the partnership, concern that a partner who was a victim of collusion would leave the partnership provides an incentive to refrain from colluding.

3 Necessity, sufficiency, and independence

Quite surprisingly, the division rule in Sect. 2 is the only division rule for three partners that is strategy-proof, objective, and consensual. The three properties are clearly necessary, since the rule possesses all three. To establish their sufficiency, consider any rule other than $s_i = 1/(1 + r_{ji}^k + r_{ki}^j)$. To be objective, it cannot use any input from partners j and i to determine their proposed relative share, so any objective rule that uses proposals for relative shares can use only r_{ji}^k and r_{ki}^j to determine partner i's share. Equation (1) indicates that the shares must correspond to $s_i = 1/(1 + r_{ji}^k + r_{ki}^j)$ when the partners' proposals for relative shares are consistent. The alternative rule must therefore assign these shares if it is to be consensual.

[7] See Tideman and Tullock (1976).

Any partner i can ensure that the proposals are consistent by altering his proposal r^i_{jk} to $r^i_{jk} = r^k_{ji} r^j_{ik}$. But if the alternative rule assigns the shares in any way other than $s_i = 1/(1 + r^k_{ji} + r^j_{ki})$ when the proposals are not consistent, then the alternative rule cannot be strategy-proof because partner i would be able to change his own share by altering his proposal. Thus the division rule $s_i = 1/(1 + r^k_{ji} + r^j_{ki})$ is the only rule that is strategy-proof, objective, and consensual. If the three partners agree that they want a rule with these three properties, then this division rule is their only option.[8]

The independence of the three properties of strategy-proofness, objectivity, and consensuality is established by the fact that for each property one can find a feasible rule that lacks that property but possess the other two. The rule "*Divide X into* 3 *equal shares*" is strategy-proof and objective, but it is not consensual because, for example, if $r^1_{23} = 3$, $r^2_{13} = 2$, and $r^3_{21} = 1.5$, then consensuality requires $s_1 = 1/3$, $s_2 = 1/2$, and $s_3 = 1/6$ rather than equal shares. Similarly, any rule that seeks to protect partners against collusion by assigning a fixed minimum share to each partner is not consensual. The rule "*Divide X into* 3 *equal shares, but use the division rule in Sect.* 2 *if the proposals are consistent*" is objective and consensual, but it is not strategy-proof because a partner who is assigned less than one-third of the profit under consistent proposals has an incentive to change his proposal and make the proposals inconsistent. Thus, any rule that assigns a minimum share but adjusts this minimum if the proposals are consistent is not strategy-proof. Finally, the rule

$$s_i = \frac{1}{1 + \max(r^j_{ji}, r^k_{ji}) + \max(r^j_{ki}, r^k_{ki})} \tag{9}$$

is strategy-proof because s_i does not depend on any report of partner i and it is consensual because consistent reports are implemented, but it is not objective because the inclusion of the terms r^j_{ji} and r^k_{ki} means that shares depend in part on how partners evaluate their own relative performances.[9]

Because the division rule in Sect. 2 is the only rule that is strategy-proof, objective, and consensual, (6) proves that these three properties together imply feasibility when there are only three partners. For groups with four or more partners, objectivity permits any pair of partners to be assessed by two or more partners, and there are an infinite number of ways of aggregating the partners' proposals. In the following section, we examine division rules for groups with four and more partners that are strategy-proof, objective, and consensual, and we explain which aggregation methods lead to feasible and attractive rules.

4 Division rules for four or more partners

For groups of more than three partners, (1) generalizes to

$$\sum_i s_i = \sum_i \frac{s_i}{\sum_j s_j} = \sum_i \frac{1}{1 + \sum_{j \neq i} R_{ji}} = 1, \tag{10}$$

[8] De Clippel et al. (2007) prove (in their Proposition 1) that this division rule is the only rule that possesses all three properties.

[9] The division rule described by (9) is feasible (that is, the sum of its shares never exceeds 1) because the proposals of the relative shares are in the denominator, and any division rule inspired by (1) that uses proposals about relative shares that are not smaller than those used by the division rule in Sect. 2 must be feasible as well.

and the following division rule is a generalization of the rule for three partners:

Have each partner supply a set of proposals for the relative shares of all partners other than himself, determine an aggregate proposal ρ_{ji} for the ratio of the shares of partners j and i from the proposals of the $n-2$ partners other than j and i (see below), and use the aggregate proposals ρ_{ji} in place of the R_{ji} to determine the shares as the n fractions in (10).

By inspection, this division rule is strategy-proof, objective, and consensual.

Because (2) holds for any three shares i, j, k, the generalized division rule divides the entire amount if the averaged ratios are consistent ($\rho_{jk} = \rho_{ji}\rho_{ik}$ for any three i, j, k), and is likely to leave a surplus otherwise.[10] However, there is a method of aggregating the partners' proposals that leads to a version of the above division rule for $n \geq 4$ partners that always distributes the entire amount without surplus:

Divide X into n parts and use the proposals from the $n-1$ partners other than partner i to determine aggregated ratios that will be used to divide the amount X_i among all n partners, using the above division rule. Partner i is the residual claimant who receives any surplus that arises from the lack of consistency in the proposals of the $n-1$ partners.

The n parts do not have to be of equal size for the rule to divide the entire amount without surplus. However, $X_i = X/n$ is a natural division that ensures that all partners are treated in the same way.

The modified division rule is still strategy-proof because partner i does not participate in the division of X_i, and it is objective because no partner ranks himself relative to any other partner. Because there are no surpluses if the n groups of $n-1$ partners all offer consistent proposals, this combination of n consensual division rules is consensual as well. This modification does not work when there are only three partners, because there is no strategy-proof, objective, and consensual division rule for subgroups with $n-1=2$ members.

There are an infinite number of such strategy-proof, objective, and consensual division rules for four or more partners because there are an infinite number of possible ways to divide X into n parts and an infinite number of possible ways to aggregate individual proposals for each ρ_{ji}. Furthermore, any rule under which groups of at least three partners divide fractions of the total among the n partners is strategy-proof, objective, and consensual, as long as each surplus goes to persons who did not contribute to the division that created it. Thus it is natural to ask whether any of these rules are better than others. De Clippel et al. (2007) prove that the sum of the shares does not exceed 1 for any set of proposed share ratios if and only if the rule for aggregating proposed share ratios satisfies $\rho_{ji}\rho_{ij} \geq 1$ for all combinations of proposals. This condition is always satisfied in the case of three partners, because the product of any proposal r^k_{ji} $(= \rho_{ji})$ and its reciprocal r^k_{ij} is 1. However, $\rho_{ji}\rho_{ij} \geq 1$ does not hold for all of the seemingly natural ways of aggregating two or more proposals. Consider the case of four partners when the two proposals for the relative shares of partners 1 and 2 are $r^3_{12} = 1.25$ and $r^4_{12} = 1.5$. Using the arithmetic mean of the two proposals as the

[10] Below we discuss the condition for the aggregation of individual proposals which guarantees that the sum of shares does not exceed 1. Aggregation rules that meet this condition lead to a surplus, unless the aggregate proposals are consistent. Aggregation rules that violate this condition can—by chance—yield shares whose sum equals 1 even if the aggregated proposals are not consistent, but the possibility of shares whose sum exceeds 1 makes those aggregation rules not generally feasible.

aggregator yields $\rho_{12} = 1.375$ and $\rho_{21} = 0.7333$, with $\rho_{12}\rho_{21} = 1.0083$. However, aggregating the proposals by their harmonic mean yields $\rho_{12} = 1.3636$ and $\rho_{21} = 0.7272$, with $\rho_{12}\rho_{21} = 0.9917$. Thus the harmonic mean is not a valid aggregator, because it can lead to shares whose sum exceeds 1.

Which aggregators ensure $\rho_{ji}\rho_{ij} \geq 1$? The two most intuitive classes of aggregators are the weighted arithmetic aggregators

$$\rho_{ji} = \sum_{k \neq i,j} \lambda_k r_{ji}^k \tag{11}$$

and the weighted geometric aggregators

$$\rho_{ji} = \prod_{k \neq i,j} \left(r_{ji}^k\right)^{\lambda_k}, \tag{12}$$

where the $\lambda_k, k \neq i, j$, are weights that sum to 1. The weighted arithmetic and geometric aggregators both have the desirable property that a partner's proposal cannot change relative shares either in the direction opposite to or beyond what he proposed.

To determine which weights lead to $\rho_{ji}\rho_{ij} \geq 1$, consider a set of $n-2$ proposals that are sorted by increasing size. De Clippel et al. (2007) prove that for such sorted sets of proposals, the weighted arithmetic and geometric aggregators satisfy $\rho_{ji}\rho_{ij} \geq 1$ for all sets of proposals if and only if the sum of the weights for the m lowest proposals does not exceed the sum of the weights for the m highest proposals, or

$$\sum_{k=1}^{m} \lambda_k \leq \sum_{k=1}^{m} \lambda_{n-k-1} \quad \text{for all } m = 1, \dots, \text{int}\left(\frac{n}{2}\right) - 1. \tag{13}$$

For example, the arithmetic mean and the geometric mean, which assign equal weights to all $n-2$ proposals, satisfy condition (13) with equality for all m. If n is odd, then the median, which assigns a weight of 1 to proposal $(n-1)/2$ and a weight of zero to all other proposals, satisfies condition (13) because the sums of the weights for the m lowest and the m highest proposals are both zero for all admissible m.[11] If n is even, then the upper median ($\lambda_{n/2} = 1$) as well as the arithmetic and the geometric averages of the upper and lower median (the "arithmetic median aggregator" and the "geometric median aggregator," with $\lambda_{n/2-1} = 0.5$ and $\lambda_{n/2} = 0.5$) satisfy condition (13), while the lower median alone ($\lambda_{n/2-1} = 1$) does not satisfy condition (13). Similarly, the largest proposal alone (the "maximum aggregator," with $\lambda_{n-2} = 1$) as well as the arithmetic and geometric averages of the smallest and the largest proposal ($\lambda_1 = 0.5$ and $\lambda_{n-2} = 0.5$) satisfy condition (13), while the smallest proposal alone (the "minimum aggregator," with $\lambda_1 = 1$) fails condition (13).

Although an infinite number of aggregators satisfy condition (13), the most natural aggregators are the arithmetic and geometric means and medians. The two means are attractive because they weigh all proposals equally, and the two medians are attractive because they are not greatly affected by single extreme proposals when there are five or more partners. We therefore limit our further analysis to these four aggregators.

While no aggregator permits a partner to increase his own share, the two mean aggregators permit any partner to reduce any other partner's share greatly by proposing highly

[11] Recall that we are aggregating $n-2$ proposals, so proposal $(n-1)/2$ and not proposal $(n+1)/2$ is the median proposal if n is odd.

inflated shares for others. Both median aggregators are highly resistant to such misbehavior when $n \geq 5$. When $n = 4$, each median aggregator is equal to the corresponding mean aggregator, since there are only two proposals to be aggregated. Between the geometric aggregator and arithmetic aggregator for $n = 4$, the geometric is less affected by an increase in the larger proposal than the arithmetic and therefore offers less opportunity to reduce a partner's share maliciously. Thus, evaluated by their degree of resistance to strategizing, the geometric median aggregator is the most attractive of the four aggregators.

If one considers it likely that all partners will state their honest beliefs about the relative shares and not behave maliciously, then it is hard to know, a priori, which of the four aggregators is best. To devise a yardstick by which to compare the aggregators, we assume that there is a set of "correct" shares $s^* = \{s_1^*, \ldots, s_n^*\}$ that the partners deserve, for example, their (imperfectly observable) marginal products as shares of the sum of their marginal products. A division rule then represents an estimator of the correct shares, and an aggregator is attractive if the implied division rule leads to an accurate division of X.

To model the partners' proposals, we assume that each partner's belief about what the n shares ought to be is a draw from an n-variate Dirichlet distribution with density function

$$f(\hat{s}; \alpha) = \frac{1}{B(\alpha)} \prod_{i=1}^{n} (\hat{s}_i)^{\alpha_i - 1}, \tag{14}$$

where $\hat{s} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n\}$ is an n-vector of proposed shares, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-vector of parameters, and the normalizing constant $B(\alpha)$ is the Beta function. The Dirichlet distribution provides an attractive framework for making coherent assumptions about a division into n shares because it is the multivariate version of a Beta distribution, which is a natural distribution to use to model a division into two shares.[12] To parameterize the Dirichlet distribution in a meaningful way, we specify its n parameters as $\alpha_i = s_i^* \cdot c$, where c is a constant. Thus we assume that all partners draw their beliefs about the relative productivities of the partners from the same multivariate distribution. While it is straightforward to use a different distribution (for example, a different c) for each partner, we found that this additional variation provided no additional insights. The mean and variance of any partner's belief about the share s_i that partner i deserves are

$$E[\hat{s}_i] = \frac{\alpha_i}{\sum_m \alpha_m} = \frac{s_i^*}{\sum_m s_m^*} = s_i^* \tag{15}$$

and

$$Var[\hat{s}_i] = \frac{\alpha_i \left(\sum_m \alpha_m - \alpha_i\right)}{\left(\sum_m \alpha_m\right)^2 \left(\sum_m \alpha_m + 1\right)} = \frac{s_i^* \left(\sum_m s_m^* - s_i^*\right)}{\left(\sum_m s_m^*\right)^2 \left(c \sum_m s_m^* + 1\right)} = \frac{s_i^*(1 - s_i^*)}{(c+1)}, \tag{16}$$

so that a change in the common constant c alters the variances of the vector of beliefs without changing the expected values. We determine partner k's proposal for the relative shares of partners j and i from his draw of his beliefs of the n shares as $r_{ji}^k = \hat{s}_j^k / \hat{s}_i^k$. For simplicity we have each partner draw a belief about his own share as well as those of others, but we obtain the same results for each partner's proposals for the relative shares of pairs of other partners as we would if each partner had not drawn a belief about his own share.

[12] There is no natural alternative to the Dirichlet distribution. Although one can construct many applicable distributions as mixtures of existing distributions, these are less intuitive. We experimented with two other distributions and obtained nearly identical results.

We explore the properties of the four aggregators while varying three dimensions of the partners' circumstances: the number of partners, the distribution of the correct shares s^*, and the variance of the distribution from which the partners' beliefs are drawn. To incorporate differences in the distribution of s^*, we undertake numerical simulations for all possible integer combinations of dividing $5n$. For example, in the case of five partners, we examine the combinations of shares $s^* = \{1, 1, 1, 1, 21\}, \{1, 1, 1, 2, 20\}, \ldots, \{5, 5, 5, 5, 5\}$. Because the draws from the Dirichlet distribution for permutations of any given distribution of s^* are qualitatively identical, we examine only one permutation of each distribution (that is, we examine $s^* = \{1, 1, 1, 2, 20\}$ but not $s^* = \{1, 1, 2, 1, 20\}$).[13] We incorporate differences in the variances by undertaking simulations for c equal to values for which the standard deviation of a share with an expected value of 0.5 is 0.02, 0.05, and 0.08.[14]

4.1 Simulation results

We report the results of our simulations in Table 1. For every combination of number of partners, distribution of shares, and standard deviation of the distribution from which the beliefs are drawn, we sampled 1000 sets of beliefs for every partner. We assess the accuracy of each aggregator through its bias—the difference between the estimated share s_i and the true share s_i^*—and its root mean square error (RMSE)—the square root of the mean of $(s_i - s_i^*)^2$. To keep the number of results manageable, we report the bias and the RMSE only for shares within the ranges of 7.5%–12.5%, 27.5%–32.5%, and 47.5%–52.5%.

We find that all four aggregators are biased, and that the bias varies nonlinearly with the share size, the number of partners, and the standard deviation of the beliefs. The bias is negative for small shares and positive for large shares. Not surprisingly, the absolute value of the bias falls as the variance in beliefs falls. All four aggregators have very similar biases for groups with four partners. As the number of partners increases, negative biases become smaller and positive biases become larger. The share size with no bias is roughly $1/n$. For large shares, the bias rises at the slowest rate for the arithmetic mean and at the fastest rate for the geometric mean. The bias of the geometric median rises at a slightly faster rate than that of the arithmetic median.

The RMSE increases with the standard deviation of the beliefs and share size, and decreases with group size. Because the averages of more precise proposals and a larger number of honest proposals lead to a better estimate of the correct share, these results are intuitive. The arithmetic aggregators have smaller RMSEs than the corresponding geometric aggregators, and the means generally have smaller RMSEs than the medians, except that the geometric median does better than the geometric mean when there are many partners, a large share, and a high variance of beliefs.

The arithmetic mean has the smallest bias and the smallest RMSE. However, the arithmetic mean is less resistant to mischievous or badly informed behavior than the two medians. The choice between median and mean aggregators therefore depends on whether the smaller bias and smaller RMSE of the mean aggregators are considered to be worth their cost in reduced resistance to mischief. The geometric aggregators have the aesthetic appeal that

[13] When aggregating, we weight each result to take account of the fact that there are $n!$ permutations if all s_i^*'s are different, but only $n!/j!$ permutations if j s_i^*'s are identical, $n!/(j!h!)$ permutations if there are two sets of j and h identical s_i^*'s, and so on.

[14] If the shares s^* are not equal, then any c leads to different variances for the n shares. We (arbitrarily) chose c in reference to the share for which $E[\hat{s}_i] = \frac{1}{2}$, which has the largest variance among the shares for a given value of c.

Table 1 Assessment of the accuracy and precision of the geometric median aggregator, the geometric mean aggregator, and the arithmetic mean aggregator

Share	Standard deviation of beliefs	Bias				RMSE			
		3 partners (wasted surplus)							
7.5%–12.5%	0.02	−0.0004				0.0095			
	0.05	−0.0028				0.0237			
	0.08	−0.0072				0.0381			
27.5%–32.5%	0.02	0.0000				0.0158			
	0.05	−0.0021				0.0399			
	0.08	−0.0038				0.0643			
47.5%–52.5%	0.02	−0.0005				0.0177			
	0.05	−0.0003				0.0450			
	0.08	−0.0030				0.0730			
Share	Standard deviation of beliefs	Bias				RMSE			
		Median/Mean				Median/Mean			
		Geom.		Arith.		Geom.		Arith.	
		4 partners (no surplus)							
7.5%–12.5%	0.02	−0.0002		−0.0002		0.0073		0.0073	
	0.05	−0.0011		−0.0011		0.0183		0.0182	
	0.08	−0.0028		−0.0026		0.0296		0.0294	
27.5%–32.5%	0.02	0.0002		0.0002		0.0117		0.0117	
	0.05	0.0007		0.0007		0.0294		0.0294	
	0.08	0.0012		0.0011		0.0474		0.0473	
47.5%–52.5%	0.02	0.0005		0.0005		0.0131		0.0131	
	0.05	0.0018		0.0018		0.0331		0.0331	
	0.08	0.0045		0.0041		0.0527		0.0525	
Share	Standard deviation of beliefs	Bias				RMSE			
		Median		Mean		Median		Mean	
		Geom.	Arith.	Geom.	Arith.	Geom.	Arith.	Geom.	Arith.
		5 partners (no surplus)							
7.5%–12.5%	0.02	−0.0002	−0.0001	−0.0003	−0.0001	0.0061	0.0061	0.0060	0.0060
	0.05	−0.0013	−0.0006	−0.0015	−0.0008	0.0153	0.0152	0.0151	0.0150
	0.08	−0.0032	−0.0013	−0.0038	−0.0017	0.0246	0.0241	0.0244	0.0238
27.5%–32.5%	0.02	0.0002	0.0001	0.0003	0.0002	0.0099	0.0099	0.0098	0.0098
	0.05	0.0014	0.0006	0.0016	0.0007	0.0250	0.0248	0.0247	0.0245
	0.08	0.0032	0.0011	0.0037	0.0012	0.0408	0.0399	0.0404	0.0394
47.5%–52.5%	0.02	0.0006	0.0002	0.0007	0.0003	0.0111	0.0111	0.0109	0.0109
	0.05	0.0039	0.0016	0.0046	0.0022	0.0285	0.0281	0.0281	0.0276
	0.08	0.0091	0.0032	0.0110	0.0042	0.0468	0.0453	0.0466	0.0447

Table 1 (*Continued*)

Share	Standard deviation of beliefs	Bias				RMSE			
		Median		Mean		Median		Mean	
		Geom.	Arith.	Geom.	Arith.	Geom.	Arith.	Geom.	Arith.
		6 partners (no surplus)							
7.5%–12.5%	0.02	−0.0002	−0.0002	−0.0002	−0.0001	0.0058	0.0058	0.0055	0.0055
	0.05	−0.0009	−0.0009	−0.0014	−0.0006	0.0145	0.0145	0.0139	0.0138
	0.08	−0.0024	−0.0024	−0.0038	−0.0015	0.0234	0.0234	0.0226	0.0219
27.5%–32.5%	0.02	0.0003	0.0003	0.0005	0.0002	0.0091	0.0091	0.0087	0.0086
	0.05	0.0017	0.0017	0.0027	0.0008	0.0229	0.0229	0.0219	0.0215
	0.08	0.0047	0.0046	0.0074	0.0016	0.0378	0.0378	0.0369	0.0351
47.5%–52.5%	0.02	0.0006	0.0006	0.0011	0.0004	0.0102	0.0102	0.0095	0.0095
	0.05	0.0046	0.0046	0.0073	0.0026	0.0262	0.0262	0.0255	0.0243
	0.08	0.0116	0.0115	0.0183	0.0052	0.0436	0.0435	0.0438	0.0391
		7 partners (no surplus)							
7.5%–12.5%	0.02	−0.0001	−0.0001	−0.0002	−0.0001	0.0052	0.0052	0.0049	0.0049
	0.05	−0.0008	−0.0007	−0.0012	−0.0004	0.0130	0.0129	0.0125	0.0123
	0.08	−0.0022	−0.0018	−0.0032	−0.0014	0.0210	0.0207	0.0205	0.0196
27.5%–32.5%	0.02	0.0005	0.0004	0.0007	0.0002	0.0082	0.0082	0.0078	0.0078
	0.05	0.0029	0.0023	0.0043	0.0011	0.0208	0.0207	0.0202	0.0195
	0.08	0.0074	0.0057	0.0109	0.0017	0.0346	0.0339	0.0345	0.0315
47.5%–52.5%	0.02	0.0010	0.0007	0.0015	0.0004	0.0092	0.0092	0.0088	0.0086
	0.05	0.0065	0.0051	0.0096	0.0026	0.0239	0.0235	0.0238	0.0216
	0.08	0.0169	0.0133	0.0249	0.0055	0.0413	0.0396	0.0437	0.0351
		8 partners (no surplus)							
7.5%–12.5%	0.02	−0.0001	−0.0001	−0.0001	0.0000	0.0048	0.0048	0.0045	0.0045
	0.05	−0.0006	−0.0006	−0.0010	−0.0003	0.0121	0.0121	0.0115	0.0113
	0.08	−0.0017	−0.0017	−0.0027	−0.0014	0.0197	0.0197	0.0190	0.0181
27.5%–32.5%	0.02	0.0005	0.0005	0.0009	0.0002	0.0076	0.0076	0.0071	0.0070
	0.05	0.0033	0.0033	0.0053	0.0012	0.0194	0.0194	0.0188	0.0177
	0.08	0.0092	0.0092	0.0148	0.0019	0.0333	0.0333	0.0340	0.0292
47.5%–52.5%	0.02	0.0012	0.0012	0.0019	0.0005	0.0087	0.0087	0.0081	0.0079
	0.05	0.0078	0.0078	0.0125	0.0031	0.0231	0.0231	0.0237	0.0200
	0.08	0.0195	0.0194	0.0317	0.0055	0.0406	0.0406	0.0460	0.0325

$\rho_{ji} = 1/\rho_{ij}$. The choice between geometric and arithmetic aggregators therefore depends on whether this aesthetic virtue is considered to be worth its cost in bias and RMSE.

So far we have assumed that the proposals of all partners ought to have equal weight. The aggregators can also be used if the proposals of some partners are to be weighted more heavily, for example, to reflect the partners' seniority. To aggregate such weighted proposals in terms of their medians, sort the proposals by size and determine the "position" of each proposal as the sum of the weights of the proposals that precede it plus half of its own weight. Next, identify the two proposals r_{ji}^h and r_{ji}^k whose positions bracket the median of the weights (0.5), and let α be the distance of r_{ji}^h's position and β be the distance of r_{ji}^k's position from the median. The weighted arithmetic median of the proposals is $\frac{\beta}{\alpha+\beta}r_{ji}^h + \frac{\alpha}{\alpha+\beta}r_{ji}^k$, while their weighted geometric median is $(r_{ji}^h)^{\beta/(\alpha+\beta)}(r_{ji}^k)^{\alpha/(\alpha+\beta)}$.

5 Large partnerships and incomplete information

It is straightforward to apply the division rule to large groups whose partners may not have sufficient information to assess the relative contributions of every partner. Even if proposals from some partners are missing, one can determine aggregate proposals ρ_{ji} as long as there is at least one proposal for every pair of partners.[15]

If combining aggregate proposals derived from vastly different numbers of individual proposals seems inappropriate, then one might want to apply the division rule in multiple stages. For partnerships that consist of several groups that are each headed by a senior partner (for example, a law firm may have offices in different cities), the senior partners can first use the division rule to divide the firm's profit among the different groups. The partners who belong to each group are likely to have sufficient information about their relative contributions, so they can use the division rule to divide the share of the firm's profits allocated to their group.

In cases in which a tiered division is not feasible and no partner is able to provide a proposal for the relative shares of some pairs of partners, it may be possible to estimate the missing ratios from the ratios that are available. For example, consider a group of five partners where none of partners 1, 2, and 3 is able to provide a relative ranking of partners 4 and 5. An estimate of the missing proposal ρ_{45} can be obtained as

$$\hat{\rho}_{45} = \frac{1}{3}\left(\rho_{43}^{12}\rho_{35}^{12} + \rho_{42}^{13}\rho_{25}^{13} + \rho_{41}^{23}\rho_{15}^{23}\right), \tag{17}$$

where the superscripts on ρ indicate the partners whose proposals, if available, are used to determine the respective aggregate proposal, so that strategy-proofness and objectivity are maintained. The estimate of ρ_{45} can also be obtained from shorter combinations like $\hat{\rho}_{45} = \frac{1}{2}(\rho_{42}^{13}\rho_{25}^{13} + \rho_{41}^{23}\rho_{15}^{23})$ or $\hat{\rho}_{45} = \rho_{43}^{12}\rho_{35}^{12}$, depending on which proposals are available.

6 Divisions between two partners

Our results have no direct implications for divisions between two partners because it is impossible for one of two partners to specify what the other ought to receive without implying what he himself ought to receive. Still, a variation on the mechanism can be used for two partners. Have the two partners appoint a "division counselor," who listens to their views about what each partner ought to receive and then makes a proposal. Each of the partners proposes a ratio for the share of the counselor (the pay for his services) to the share of the other partner. Because the division counselor is effectively treated as a third partner, the conditions for our division rule are then satisfied. Similarly, division among three partners without any surplus is possible by calling in a fourth person as a counselor and using the adjusted division rule in Sect. 4.

[15]The modified method in Sect. 4 that avoids a surplus requires that a partner's proposal be excluded from the application of the division rule for which he is the residual claimant. Thus a surplus can be avoided when some proposals are missing only if there are at least two proposals for every pair of rankings.

7 Conclusion

Although we have explored the properties of the division rules in the context of a business partnership, the rules have a wide range of possible applications. They can be used to divide a joint inheritance or any other windfall gain when the individual claims are contestable. Scientists can use the rules to divide the proceeds of a patent that stems from joint research, faculty members can use them to divide discretionary salary increases that are paid *en bloc* to their department, and creditors can use them to divide the recoverable assets after a bankruptcy.

It is worth emphasizing that the division rules do not replace discussions among the group members about their relative contributions or claims. Such discussions help the members to either obtain or recall relevant information on which to base their proposals. However, if the group members believe that they themselves are the best arbiters, they consider it too time consuming to continue their discussion until they reach consensus, and they value strategy-proofness, objectivity, and consensuality, then the division rules described here are their only choice.

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