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NON-COOPERATIVE BARGAINING OF $N \geqslant 3$ PLAYERS

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Extensions to $N \geqslant 3$ players of Rubinstein's (1982) bargaining model with fixed common discount factor are considered. If voting takes place simultaneously, then any partition of the cake is a subgame-perfect Nash equilibrium outcome.

1. Introduction

Herrero (1985) has extended Rubinstein's (1982) non-cooperative bargaining model with fixed common discount factor δ, $0 < \delta < 1$, to the case of $N \geqslant 3$ players. For two players, Rubinstein showed that all subgame-perfect Nash equilibria lead to the same partition of the cake of size 1 at time 0, namely the first player gets $1/(1+\delta)$ and the second player gets $\delta/(1+\delta)$. *For* $N \geqslant 3$, *the extensive form of the game matters.*

With *successive voting with perfect information*, two cases have to be distinguished:

(a) $\delta(N-1) \geqslant 1$. For this case, Herrero (1985) has shown that each $X \in \Delta_{N-1}$ is the payoff vector of a subgame-perfect Nash equilibrium.

(b) $\delta(N-1) < 1$. In this case, if $v(v_1, \dots, v_N)$ is the payoff vector of a subgame-perfect Nash equilibrium, then $v_i \geqslant \delta^{i-1} \cdot (1 - \delta(N-1))$ for $i = 1, \dots, N$. Therefore not every $X \in \Delta_{N-1}$ is the payoff vector of a subgame-perfect Nash equilibrium. Herrero (1985) even asserts that

$$X = \left(\frac{1-\delta}{1-\delta^N},\ \delta \cdot \frac{1-\delta}{1-\delta^N}, \dots,\ \delta^{N-1} \cdot \frac{1-\delta}{1-\delta^N} \right)$$

is the unique subgame-perfect Nash equilibrium payoff vector. But I do not fully understand her argument, which is adapted from Shaked and Sutton (1984).

It is shown below that with *simultaneous voting*, each $X \in \Delta_{N-1}$ is the payoff vector of a subgame-perfect Nash equilibrium, irrespective of δ. This result can be explained by the fact that there are less subgames under simultaneous voting than under successive voting with perfect information. The key arguments are similar to those developed by Herrero (1985) for her case $\delta(N-1) \geqslant 1$.

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2. Model

There are $N \geqslant 3$ players, numbered 1, 2,..., N; $I \equiv \{1, \ldots, N\}$. The players have to reach an agreement on the partition of a cake of size 1. In turn, each player makes a proposal as to how the cake should be divided. A proposal or offer is represented by a vector $X \in \Delta_{N-1} = \{(X_1, \ldots, X_N) \in \mathbb{R}_+^N \mid \Sigma_i X_i = 1\}$. After a player has made such an offer, each of the other players has to vote on the offer. The vote is either 'accept' or 'reject'.

Let $0 < \delta < 1$ be the fixed common discount factor. At time 0, player 1 makes the first offer $X^0 = (X_1^0, \ldots, X_N^0)$. If all the other players accept the offer X^0, then the bargaining ends with payoffs X_i^0, $i \in I$. If at least one player $i \neq 1$ rejects X^0, the bargaining continues at time 1 with an offer X^1 by player 2. If then the offer $X^1 = (X_1^1, \ldots, X_N^1)$ is unanimously accepted, the bargaining ends with payoffs δX_i^1, $i \in I$. Otherwise, player 3 makes an offer X^2 at time 2, etc. The bargaining continues as long as no unanimous agreement on an offer is reached. The offer, if any, at time $t \in \mathbb{N}_0$ is made by player $j(t)$, where $j(t)$ is determined by $j(t) - 1 \equiv t \bmod N$. If $j(t)$'s offer $X^t = (X_1^t, \ldots, X_N^t)$ is unanimously accepted at time t, then the bargaining ends with payoffs $\delta^t X_i^t$, $i \in I$. If unanimous acceptance never occurs, then each player has payoff 0 in the infinite bargaining game.

3. Result for the simultaneous voting case

To obtain a well-defined game in extensive form, it is assumed that at any time $t \in \mathbb{N}_0$, unless agreement was reached at time $t-1$, player $j(t)$ makes an offer and then the players $i \neq j(t)$ vote in the order $j(t+1)$, $j(t+2), \ldots, j(t+N-1)$. To capture the nature of simultaneous voting, it is assumed that each $j(t+k)$ with $2 \leqslant k \leqslant N-1$, when voting, is not informed of the previous votes of $j(t+l)$, $1 \leqslant l < k$ in round t. (All votes in round t become public after the last vote in round t.) Denote by Γ the resulting game in extensive form.

Theorem. *If $X = (X_1, \ldots, X_N) \in \Delta_{N-1}$ is any 'partition of the cake', then there is a subgame-perfect Nash equilibrium of the game Γ with the outcome that X is offered and unanimously accepted in round* 0.

Notation. For $h \in I$, let e^h be the hth unit vector in $\mathbb{R}^N$.

Proof of the theorem. Let us define recursively a sequence of *offer rules* R_0, R_1, ...

R_0: *Offer X.*
$R_t (t > 0)$: *Suppose that the last offer was Y.*
If R_{t-1} was followed, offer Y.
If R_{t-1} was violated, offer $e^{j(t)}$.

Further, we define voting rules E_0, $E_1, \ldots, E_t, \ldots$.

$$E_t \begin{cases} \text{Reject, if } R_t \text{ was violated.} \\ \text{Accept, if } R_t \text{ was followed.} \end{cases}$$

Following these rules whenever she has to decide, defines a strategy σ_i for any player $i \in I$. Now the claim is that $(\sigma_1, \ldots, \sigma_N)$ is a subgame-perfect Nash equilibrium of Γ with the outcome that X is offered and unanimously accepted in round 0. Clearly, $(\sigma_1, \ldots, \sigma_N)$ leads to this outcome. It remains to show that $(\sigma_1, \ldots, \sigma_N)$ is a subgame-perfect Nash equilibrium.

Let $t \geqslant 0$ and $i \in I$. Consider a subgame Σ starting in round t where i has to make the first move.

Offer case: If i has to make an offer, i.e., $j(t)=i$, then let us consider the consequences of a violation of rule R_t.

Violation of R_t by i leads to rejection and $j(t+1)$ offering $e^{j(t+1)}$ at time $t+1$. $e^{j(t+1)}$ will be accepted at some time $t' \geqslant t+1$ or there will never be acceptance. (i can always reject and offer some $\boldsymbol{Z} \neq e^{j(t+1)}$, if it is her turn, but then the other players will return to $e^{j(t+1)}$ later on.)

Anyhow, violation of R_t leads to payoff 0 for i. Therefore, following R_t is optimal for i in the offer case.

Voting case: If i has to vote, then $i=j(t+1)$ and i is the first to vote in round t. Then:

Either: R_t was violated by $j(t)$. Then there is $j \notin \{j(t), i\}$ who will reject after i. Since i is not decisive, rejection is optimal.

Or: R_t was followed by $j(t)$ who offered $\boldsymbol{Y}=(Y_1, \ldots, Y_N)$. If i accepts, the offer is accepted and i's (time 0) payoff is $\delta^t Y_i$. If i rejects, then she cannot get more than $\delta^{t+1} Y_i$. (If she deviates from R_s at some $s>t$ she will get 0. Otherwise she will get $\delta^{t+k} Y_i$ with $k \geqslant 1$.) Hence accepting is optimal.

Therefore, following E_t is optimal for i in the voting case.

Consequently, for any $i \in I$ following the rules is optimal in any subgame, if the other players follow the rules. Hence $(\sigma_1, \ldots, \sigma_N)$ is a subgame-perfect Nash equilibrium of Γ. Q.E.D.

References

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