3. EVOLUTIONARILY STABLE STRATEGIES

Game theory has been successfully applied in biology as a method for studying evolution. However, biologists approached game theory in a different way as economists have done. In the introduction to his influential book, John Maynard Smith (1982) states the differences between *evolutionary game theory* —the application of game theory to biology— and the traditional economic approach to game theory. In evolutionary game theory the payoffs of a game are identified with Darwinian fitness, and rationality is replaced with evolutionary stability. The basic object of study is a population of individuals competing with each other for survival and reproduction. This strategic competition is modeled through a game and the strategies correspond to the phenotypes whose survival is in question. The aim is to study the evolution of the different strategies in the population. In an evolutionary game, however, individuals do not consciously choose, but are programmed to play certain strategies. More successful strategies survive with higher probability and reproduce faster. A strategy is said to be *evolutionarily stable* if, once adopted by all members of the population, a small fraction of so-called mutants —i. e. individuals doing something different— will be selected against. Intuitively, if several strategies are present in a stable situation, they all must be equally successful. In a large population, a stable situation must correspond to a Nash equilibrium. In this way, evolutionary game theory provides foundations for the concept of Nash equilibrium and a different interpretation. Moreover, this interpretation is absent of the rationality assumptions of traditional game theory.

3.1. Definition and relation to NE

Consider a large population of individuals who are randomly matched to play a symmetric two-player game. Let A be the payoff matrix. Each individual is programmed to play a certain pure strategy. Let $S = \{1, \ldots, k\}$ denote the set of pure strategies. Let $\Delta = \{x \in \mathbb{R}^k_+ \mid \sum_{i \in S} x_i = 1\}$ be the set of possible population profiles, i. e. each x_i corresponds here to the fraction of individuals in the population playing strategy $i \in S$. Note that this set is formally identical to the set of mixed strategies.

Suppose that, initially, the population profile is $x \in \Delta$. The average payoff in the population is $u(x, x) = xAx$. Now suppose that a small group of *mutants* enters this population playing according to a different profile y. If we call $\epsilon \in (0, 1)$ the size of the subpopulation of mutants after normalization, then the population profile after mutation will be $\epsilon y + (1 - \epsilon)x$. After mutation, the average payoff of non-mutants *who are randomly matched to mutants* is given by $u(x, y) = xAy$, and the average payoff of non-mutants will be given by $u(x, \epsilon y + (1 - \epsilon)x) = \epsilon u(x, y) + (1 - \epsilon)u(x, x)$. Analogously, we can construct the average payoff of mutants.

Definition 1. We say that x is an *evolutionarily stable strategy* (ESS) if for all $y \neq x$, there exists some $\bar{\epsilon} \in (0, 1)$, which may depend on y, such that for all $\epsilon \in (0, \bar{\epsilon})$

$$u(x, \epsilon y + (1 - \epsilon)x) > u(y, \epsilon y + (1 - \epsilon)x). \tag{8}$$

That is, x is ESS if, after mutation, non-mutants are more successful than mutants, in which case mutants cannot invade and will eventually get extinct. We refer to $\bar{\epsilon}$ as the *invasion barrier* —the maximum rate of mutants against which x is resistant. Note that the concept of ESS, like the concept of NE, is static —once reached, it is stable. We will study evolutionary dynamics in the next chapter.

If we rewrite expression (8) as $\epsilon u(x, y) + (1 - \epsilon)u(x, x) > \epsilon u(y, y) + (1 - \epsilon)u(y, x)$, it is easy to see that (8) is equivalent to

$$u(x, x) \geq u(y, x) \quad \forall y, \tag{9}$$

$$u(x, x) = u(y, x) \Rightarrow u(x, y) > u(y, y) \quad \forall y \neq x \tag{10}$$

For x to be an ESS, condition (9) requires that non-mutants fare against themselves at least as well as mutants do. Otherwise, in a population consisting mainly of non-mutants, mutants will thrive, while non-mutants will get extinct. If both fare equally well, x can only be stable if it has additional advantages. This is what condition (10) requires. Namely that non-mutants fare against mutants better than mutants fare against themselves. Otherwise, mutants would have the same reproductive success as non-mutants after every encounter with non-mutants, and they would reproduce faster than non-mutants after every encounter with mutants.

Condition (9) corresponds to the definition of NE. It shows that if x is an ESS, then it must constitute a symmetric NE. Thus, evolutionary stability implies Nash. However, condition (10) requires that an ESS be more than a NE. Therefore, not all symmetric NE are ESS. Moreover, if x constitutes a *strict*, symmetric NE (i. e. $u(x, x) > u(y, x) \ \forall y \neq x$), then condition (10) never applies and x is immediately ESS.

3.2. Examples

Example 9. *Prisoner's dilemma.*
Recall the prisoner's dilemma (PD) game that we introduced in Example 5. In this game each player has two strategies, cooperate (C) and defect (D), where D strictly dominates C. Therefore, in equilibrium both players choose D. This NE is symmetric and strict —choosing D is always the only best reply. Thus, D is also the only ESS. This example demonstrates that evolutionary stability does not necessarily imply optimality. In the PD game both players would be better off if they both played C.

Example 10. *Coordination game.*
Recall the coordination game introduced in Example 7 with payoff matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$$

This game has three symmetric NE, namely $\{e^1, e^2, \hat{x}\}$ with $\hat{x} = (1/3, 2/3)$. The pure-strategy equilibria are strict. Therefore, they are ESS. This in another example showing that evolutionary stability does not imply optimality. Note that equilibrium e^2 is better for both players than equilibrium e^1. However, e^1 is also ESS. The third equilibrium $\hat{x}$ is not ESS. To see this, suppose that exactly 1/3 of the population is playing the first strategy. Then, the average payoff in the population is $\hat{x} \cdot A\hat{x} = 8/3$, and both pure strategies earn payoff 8/3 on average. Suppose now that a small group of mutants playing the first strategy enters the population. Both, mutants and non-mutants, who are randomly matched to non-mutants, get payoff 8/3 on average. However, mutants who are randomly matched to mutants get payoff 2, while non-mutants who are randomly matched to mutants get only payoff $\hat{x} \cdot Ae^1 = 2/3 < 2$ on average.

Example 11. *Hawk-Dove game.*
Recall the Hawk-Dove game introduced in Example 6, where two individuals compete for a resource (food, or territory) that relatively increases the fitness of the individual that gets it by ν. Individuals can behave either as hawks or as doves. Hawks fight for the resource until they get injured or until the opponent retreats. Doves do not fight for the resource and they retreat as soon as they meet a hawk. When two hawks meet, they fight until one of them gets injured, and the loser's fitness is reduced by c. It is assumed that $c > \nu > 0$. If two hawks or two doves meet, each gets the resource with probability 1/2. The payoff matrix is given by

$$A = \begin{pmatrix} \frac{\nu - c}{2} & \nu \\ 0 & \frac{\nu}{2} \end{pmatrix}$$

Let x_1 be the fraction of hawks in the population. Then the average fitness of hawks is $x_1(\frac{\nu - c}{2}) + (1 - x_1)\nu$ and that of doves is $(1 - x_1)\frac{\nu}{2}$. If the population consists mostly of doves (i. e. x_1 is close to cero), hawks typically meet doves, win the resource, and they prosper. If the population consists mostly of hawks (i. e. x_1 close to one), hawks typically meet hawks, they get frequently injured and loose fitness, while doves do not fight, do not get injured, and thrive relative to hawks. Both strategies are equally successful when the fraction of hawks in the population is exactly $\hat{x_1} = \frac{\nu}{c}$. The profile $\hat{x} = (\hat{x_1}, 1 - \hat{x_1})$ corresponds to the only symmetric NE of the game.[5] Thus, it is the only candidate to be an ESS. Since both, hawks and doves, have the same fitness when the population profile is $\hat{x}$, for $\hat{x}$ to be an ESS, we must additionally check condition (10), i. e. that for all $y \neq \hat{x},\ \hat{x} \cdot Ay > y \cdot Ay$. It is straightforward to check that for any $y = (y_1, 1 - y_1)$ with $y_1 \neq \hat{x_1}$

$$(\hat{x} - y) \cdot Ay = \frac{(v - cp)^2}{2c} > 0.$$

Example 12. Reconsider the Rock-scissors-paper game introduced in Example 8 with payoff matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

The only NE of the game is a symmetric one where each strategy is played with probability 1/3, $\hat{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Since it is a mixed-strategy equilibrium, it is not strict. We need to check condition (10) to see if $\hat{x}$ is an ESS. Suppose that the population profile is $\hat{x}$ and that a small group of mutants enters the population, all playing e^1. Non-mutants who are matched with mutants get on average $\hat{x}Ae^1 = 1$, and mutants who are randomly matched with other mutants also get payoff 1. Thus, the population profile $\hat{x}$ is not evolutionarily stable since a small group of mutants playing for example e^1 would survive.

[5] The game has also two asymmetric NE in pure strategies.

3.3. *Properties of ESS*

Proposition 1. If $x \in \Delta$ is an ESS and $C(y) \subset C(x)$ for a different $y \in \Delta$, then y cannot constitute a symmetric NE. In particular, it cannot be an ESS.

Proof. Let x be an ESS, and let $C(y) \subset C(x)$ for $y \neq x$. Since x constitutes a NE, then $u(s,x) = u(x,x)$ for all $s \in C(x)$. It follows that $u(y,x) = u(x,x)$. Condition (10) implies that $u(x,y) > u(y,y)$. Thus, y cannot constitute a NE. □

This additionally implies that a completely mixed ESS must be unique, since no pure strategy can constitute a symmetric NE, and no other mixed strategy that assigns positive probability to a strict subset of pure strategies can constitute a symmetric NE. Therefore, they can also not be ESS. An example of this is the hawk-dove game in Example 11. Note also that a game with a pure ESS cannot have a completely mixed ESS, since the former cannot be in the support of the latter. An example of this is the coordination game of Example 10.

Proposition 2. If $x \in \Delta$ is an ESS, then it constitutes a perfect NE.

Proof. Recall that, for two-player games, a NE is perfect if and only if no weakly dominated strategy is played with positive probability. Since ESS implies NE, it is enough to show that an ESS cannot assign positive weight to a weakly dominated strategy.

Suppose $x \in \Delta$ is an ESS (and thus also NE) and that it assigns positive weight to a weakly dominated pure strategy. Then x itself is weakly dominated. Let $y \in \Delta$ be a strategy that dominates x. Then for all $z \in \Delta$, $u(y,z) \geq u(x,z)$. In particular, taking $z = x$ we have $u(y,x) \geq u(x,x)$. Since x constitutes a NE, $u(y,x) = u(x,x)$. Taking $z = y$, $u(y,y) \geq u(x,y)$, which contradicts condition (10), and the fact that x is ESS. □

3.4. *Multipopulation models*

In order to analyze asymmetric contexts like the one of the entry deterrence (ED) game introduced in Examples 1–4, we would need to extend the definition of an ESS to such asymmetric situations. In the ED game entrant and incumbent do not have the same payoff matrix. We need to consider two populations, one of potential entrant firms and one of incumbent firms. In general, for an n-player game, we will have n populations.

Let $I = \{1, \dots, n\}$ the set of players. Assume there is a large population of individuals in each player position. Call S_i the set of pure strategies available to individuals of population $i \in I$, and Δ_i the set of possible population profiles within population i (formally equivalent to the set of mixed strategies). Let $\Theta = \times_{i \in I} \Delta_i$ be the set of all possible *multipopulation* profiles.

Definition 2. We say that $x \in \Theta$ is *evolutionarily stable* if for all $y \neq x$ there exists $\bar{\epsilon} \in (0,1)$, which may depend on y, such that for all $\epsilon \in (0, \bar{\epsilon})$

$$u_i(x_i, \epsilon y_{-i} + (1-\epsilon)x_{-i}) > u(y_i, \epsilon y_{-i} + (1-\epsilon)x_{-i}) \quad \text{for some } i \in I. \tag{11}$$

Proposition 3. $x \in \Theta$ is evolutionarily stable if and only if x is a strict NE.

Proof. Suppose $x \in \Theta$ is evolutionarily stable. Fix $i \in I$. For player i, let y_i be any strategy different to x_i. For all $j \neq i$ let $y_j = x_j$. Then $y \neq x$. Define $z = \epsilon y + (1-\epsilon)x$ with $\epsilon \in (0, \bar{\epsilon})$, where $\bar{\epsilon}$ is the invasion barrier for y. Obviously, $u_j(x_j, z_{-j}) = u_j(y_j, z_{-j})$ for all $j \neq i$. Since x is evolutionarily stable, by condition (11) $u_i(x_i, z_{-i}) > u_i(y_i, z_{-i})$, since $z_{-i} = x_{-i}$. Since y_i is arbitrary, it follows that $u(x_i, x_{-i}) > u(y_i, x_{-i})$ for all $y_i \neq x_i$, which means that x is a strict NE.

Suppose now that x is a strict NE. Let $y \neq x$. Then $y_i \neq x_i$ for some i, and $u_i(x_i, x_{-i}) > u_i(y_i, x_{-i})$. For ϵ small enough

$$\epsilon u_i(x_i, y_{-i}) + (1-\epsilon)u_i(x_i, x_{-i}) > \epsilon u_i(y_i, y_{-i}) + (1-\epsilon)u_i(y_i, x_{-i}).$$

which is equivalent to (11). □

Remark 2. Note that in the entry deterrence game the only evolutionarily stable profile is the one corresponding to the strict NE where the entrant enters and the incumbent yields.

4. THE REPLICATOR DYNAMICS

In the last section we defined the concept of evolutionarily stable strategy (ESS) and showed that evolutionary stability implies (perfect) Nash. However, ESS is a static concept —a strategy such that, if already adopted by all individuals in the population, will be resistant to any small fraction of mutants playing otherwise. In the present section, we explicitly introduce natural selection forces over time. In the same analytical framework as in the last section, we consider individuals from a large population who are randomly drawn in pairs to play a two-player game over and over again. Each individual is programmed to play a pure strategy —the *replicator*— which will be inherited by all her offspring. The first question is how the distribution over pure strategies in the population changes over time according to some dynamic selection process. In particular, we will focus on the basic evolutionary dynamics known as the *replicator dynamics* (RD). The second question is whether the distribution of pure strategies converges to some profile in the long run, and if so, whether this profile corresponds to a Nash equilibrium (NE). We will see that, indeed, long-run aggregate behavior corresponds to NE. This provides dynamic foundations for the concept of NE. Even if agents in a large population are not rational and do not have coordinated beliefs, in the aggregate the population behaves *as if* each individual fulfilled all the rationality assumptions.

4.1. The replicator equation

Consider the analytical framework described in Section 3.1. Let $t \in [0, \infty)$ denote continuous time. Each vector $x(t) = (x_1(t), \dots, x_k(t)),\ x(t) \in \Delta$ —called *population state*— corresponds to the shares of individuals in the population playing each pure strategy $i \in S$ at any instant t, although we will drop the time index whenever this does not lead to confusion. We will put up a model to analyze how these proportions x_i change over time.

Assume that the population is very large but finite. Let $p_i(t) \geq 0$ be the number of individuals in the population who display pure strategy i at time t. Let $p(t) = \sum_{j \in S} p_j(t) > 0$ be the total size of the population at t. Then $x_i(t) = p_i(t)/p(t) \geq 0$. Given the population profile x, recall that $u(e^i, x)$ denotes the average payoff of any pure strategy i, and that the average payoff in the population is given by $u(x, x) = \sum_{j \in S} x_j u(e^j, x)$. Here we interpret $u(e^i, x)$ as the increment in the average number of offspring per unit of time with respect to some basic birthrate $\beta \geq 0$.[6] We assume that all individuals reproduce in one unit of time and that they die after reproduction at some rate $\delta \geq 0$. Each offspring replicates the pure strategy adopted by its single parent. Therefore, the number of individuals that will display strategy i at $t+1$ will be [7]

$$p_i(t+1) = [\beta + u(e^i, x(t)) + (1 - \delta)]p_i(t)$$

The standard way to approximate such a discrete-time equation by its continuous-time counterpart is to postulate that agents reproduce (and potentially die) uniformly along any single period, so that in any time interval of length $0 < h < 1$ a fraction h of the population reproduces. That is,

$$p_i(t+h) = h \cdot [\beta + u(e^i, x(t)) + (1 - \delta)]p_i(t) + (1 - h)p_i(t)$$

Rearranging we obtain

$$\frac{p_i(t+h) - p_i(t)}{h} = [\beta + u(e^i, x(t)) - \delta]p_i(t)$$

Taking limits when $h \to 0$,

$$\dot{p}_i(t) = [\beta + u(e^i, x(t)) - \delta]p_i(t) \tag{12}$$

where the upper dot denotes time derivative $\dot{p}_i(t) = \frac{d}{dt}p_i(t)$. Note that

$$\dot{x}_i = \frac{\dot{p}_i}{p} - \frac{\dot{p}}{p}x_i \tag{13}$$

$$\dot{p}(t) = [\beta + u(x(t), x(t)) - \delta]p(t) \tag{14}$$

From equations (12)–(14) the replicator equations follow.

$$\dot{x}_i = [u(e^i, x) - u(x, x)]x_i \tag{15}$$

[6] The idea is that strategies spread at a rate that is proportional to their average payoff, where this 'proportional' is taken here equal to one.

[7] Technically, equation (12) describes the *expected* number of individuals playing strategy i at $t+1$. By the Weak Law of Large Numbers, if the population is large enough, the distribution of $p_i(t+1)$ concentrates arbitrarily close to its expectation. Hence, it is justified to suppress the expectation term.

The replicator equations say that the *growth rate* of the population share using strategy $i,\ \dot{x}_i/x_i$, equals the difference between the strategy's current payoff (fitness) and the current average payoff (fitness) in the population.

Note that the sub-populations associated with pure *best* replies to x will have the maximum growth rates. If strategy i is a pure best reply to x, then $u(e^i, x) \geq u(e^j, x)$ for all j. Note also that the share of any strategy x_i will increase faster than the share of any other strategy x_j over time if i earns higher payoff than j.

$$\frac{d}{dt}\left(\frac{x_i}{x_j}\right) = \frac{x_i}{x_j}[u(e^i, x) - u(e^j, x)] \tag{16}$$

4.2. Examples

Example 13. *Prisoner's dilemma.*
Consider the following prisoner's dilemma game, which results from adding 5 to all payoffs in Example 5.

$$A = \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}$$

Note that $u(e^1, x) = 2x_1,\ u(e^2, x) = 1 + 4x_1,\ u(x, x) = 2x_1^2 + (1 - x_1)(1 + 4x_1)$. Thus,

$$\dot{x}_1 = [u(e^1, x) - u(x, x)] \cdot x_1 = -(1 + 2x_1)(1 - x_1)x_1 \tag{17}$$

Of course, $\dot{x}_2 = -\dot{x}_1$, i. e. the change in the proportion of defectors must be equal, but of different sign, to the change in the proportion of cooperators. Observe first that $\dot{x}_i = 0,\ i = 1, 2$, if $x_1 = 0$ and if $x_1 = 1$. That is, both population states e^1 —all cooperate— and e^2 —all defect— are *stationary*. If the system starts at any of them, it will never leave them. However, observe that $\dot{x}_1 < 0$ for all $0 < x_1 < 1$. That is, starting at a state with a positive fraction of cooperators, this fraction will steadily decrease over time, while the fraction of defectors will increase. In the long run, the system converges to all individuals defecting for all initial states $x(0) = (x_1(0), x_2(0))$ with $x_1(0) > 0$. We say that e^2 is *globally asymptotically stable*. Recall that e^2 was also the only NE and ESS.

Example 14. *Coordination game.*
Consider again the coordination game of Examples 7 and 10 with payoff matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$$

Note that $u(e^1, x) = 3 - x_1,\ u(e^2, x) = 4(1 - x_1),\ u(x, x) = x_1(3 - x_1) + 4(1 - x_1)^2$. Thus,

$$\dot{x}_1 = [u(e^1, x) - u(x, x)] \cdot x_1 = (3x_1 - 1)(1 - x_1)x_1 \tag{18}$$

Again, of course, $\dot{x}_1 = -\dot{x}_2$. Note that $\dot{x}_1 = 0$ whenever $x_1 = 0,\ x_1 = 1$, or $x_1 = 1/3$. Thus, the states $\{e^1, e^2, \hat{x}\}$ with $\hat{x} = (1/3, 2/3)$ are the stationary states. Recall these were precisely the NE of the game. Thus, if the system starts in one of the NE, it will never abandon it. We saw in Example 10 that only the pure-strategy NE are ESS. Note here that, for all x such that $x_1 > 0,\ \dot{x}_1 > 0$ if and only if $x_1 > 1/3$. That is, starting at any state $x(0)$ with more than 1/3 of individuals playing the first strategy $x_1(0) > 1/3$, the system will converge to all playing the first strategy. If at least 1/3 of the population is playing the first strategy then this strategy earns payoffs higher than the average. Otherwise, the system converges to the second strategy. Both e^1 and e^2 are asymptotically stable, while $\hat{x}$ is unstable. If the system starts at $x(0) \in [0, 1/3)$, then it converges to e^2. If it starts at $x(0) \in (1/3, 1]$, it converges to e^1. The system displays *path dependence*. The long-run outcome depends on initial conditions. The set $[0, 1/3)$ is called the *basin of attraction* of the state e^2, and the set $(1/3, 1]$ is the basin of attraction of e^1.

Example 15. *Hawk-Dove game.*
Consider the hawk-dove game of Examples 6 and 11 with payoff matrix

$$A = \begin{pmatrix} \frac{\nu - c}{2} & \nu \\ 0 & \frac{\nu}{2} \end{pmatrix}$$

$u(e^1, x) = \frac{\nu - c}{2}x_1 + \nu(1 - x_1),\ u(e^2, x) = \frac{\nu}{2}(1 - x_1),\ u(x, x) = x_1(\frac{\nu - c}{2}x_1 + \nu(1 - x_1)) + \frac{\nu}{2}(1 - x_1)^2$. Thus,

$$\dot{x}_1 = [u(e^1, x) - u(x, x)] \cdot x_1 = \left(\frac{\nu}{2} - \frac{c}{2}x_1\right)(1 - x_1)x_1 \tag{19}$$

Again, $\dot{x}_1 = -\dot{x}_2$. Note that $\{e^1, e^2, \hat{x}\}$ with $\hat{x} = (\frac{\nu}{c}, 1 - \frac{\nu}{c})$ are stationary. Recall that $\hat{x}$ constitutes the only symmetric NE of this game and the only ESS. Finally, note that $\dot{x}_1 > 0$ (resp. $\dot{x}_1 < 0$) if and only if $x_1 < \frac{\nu}{c}$ (resp. $x_1 > \frac{\nu}{c}$). Thus $\hat{x}$ is globally asymptotically stable. If $x_1(0) < \frac{\nu}{c}$, then the fraction x_1 increases over time until $\hat{x}_1$. If $x_1(0) > \frac{\nu}{c}$, then the fraction x_1 decreases over time until $\hat{x}_1$. The states e^1 and e^2 are unstable.

Example 16. *Rock-Scissors-Paper game.*
Consider the rock-scissors-paper game of Examples 8 and 12 with payoff matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Note that $u(e^1, x) = x_1 + 2x_2,\ u(e^2, x) = x_2 + 2x_3,\ u(e^3, x) = x_3 + 2x_1$, and $u(x, x) = x_1(x_1 + 2x_2) + x_2(x_2 + 2x_3) + x_3(x_3 + 2x_1) = (x_1 + x_2 + x_3)^2 = 1$. Thus,

$$\dot{x}_1 = [x_1 + 2x_2 - xAx] \cdot x_1 \tag{20}$$

$$\dot{x}_2 = [x_2 + 2x_3 - xAx] \cdot x_2 \tag{21}$$

$$\dot{x}_3 = [x_3 + 2x_1 - xAx] \cdot x_3 \tag{22}$$

Note also that the function $h(x_1, x_2, x_3) = \log(x_1 x_2 x_3)$ is constant over time. That is, $\dot{h} = 0$.

$$\dot{h} = \frac{\dot{x}_1}{x_1} + \frac{\dot{x}_2}{x_2} + \frac{\dot{x}_3}{x_3} = 3 - 3xAx = 0 \tag{23}$$

Starting at any level $\gamma_0 = x_1(0)x_2(0)x_3(0)$ of the function h —the closed curve $x_1 x_2 x_3 = \gamma_0$— the system remains at level γ_0 forever —it *cycles* around $(1/3, 1/3, 1/3)$ forever. There is no asymptotically stable state.

4.3. Stationarity

As mentioned in Example 13, a *stationary state* is one where $\dot{x}_i = 0$ for all i, i. e. a state such that, once reached, is never abandoned. In the examples above we have seen that situations where all individuals choose the same strategy are always stationary even if they do not constitute a NE —a strategy that is not present cannot be replicated. We have also seen that all pure and mixed strategy NE are always stationary, since all strategies that are assigned positive probability give the same average payoff. Since an ESS is always a NE, then ESS are also stationary. Finally, a mixed strategy that is stationary must be NE, since then all pure strategies give the same payoff. Mere examination of the replicator equation (15) shows that:

Proposition 4. (a) e^i is stationary for all i.
(b) If x constitutes a symmetric NE, then it is stationary.
(c) If x is ESS, then it is stationary.
(d) If x is completely mixed and stationary, then it must be a NE.

4.4. Stability

Definition 3. A state x is *Lyapunov stable* if every neighborhood B of x contains another neighborhood B' of x such that, if $x(0) \in B'$, then $x(t) \in B$ for all $t \geq 0$.

Proposition 5. If x is Lyapunov stable, then it is stationary and constitutes a symmetric NE.

Note that in Example 16 the mixed strategy $(1/3, 1/3, 1/3)$ is Lyapunov stable. If the system starts at this point, it will never abandon it. However, if it starts close to it, it will cycle forever around and close to it.

Definition 4. A state x is *asymptotically stable* if it is Lyapunov stable and there exists a neighborhood B^* of x such that, if $x(0) \in B^*$, then $\lim_{t\to\infty} x(t) = x$. We say that a state x is *globally* asymptotically stable if B^* can be taken to be the set of all completely mixed profiles, i. e. the *interior* of the state space Δ.

In Example 16 the profile $(1/3, 1/3, 1/3)$ fails to be asymptotically stable. Independently of how close to it the system starts, it will always cycle around it but never converge to it.
We saw in Example 13 that the only ESS is globally asymptotically stable. However, in Example 14 the two ESS are asymptotically stable, but not globally, since long-run convergence depends on initial conditions. In Example 15 the only ESS, which is completely mixed, is globally asymptotically stable. In general, it can be shown that:

Proposition 6. (a) If x is an ESS, then it is asymptotically stable.
(b) If x is a completely mixed ESS, then it is globally asymptotically stable.

Recall that all ESS are perfect NE. This result can be extended to all asymptotically stable states of the RD.

Proposition 7. If x is asymptotically stable, then it constitutes a symmetric perfect NE, and it is isolated ($\{x\}$ is a component of the set of NE).