# Voluntary Contributions when the Public Good is not Necessarily Normal*

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**Abstract.** *In this paper, we argue that important examples of privately provided public goods do not satisfy the assumption of strict normality, and reconsider voluntary contribution games in a more general framework. It is shown that, in general, (i) equalizing transfers between individuals with identical tastes can increase total supply of the public good, and (ii) more of the public good can be supplied if agents move sequentially rather than simultaneously. These results are in sharp contrast to conclusions derived in the literature under the assumption of strict normality. We also provide a general condition for generic uniqueness of Stackelberg equilibrium in the sequential move game. In this context, we prove and use the fact that the neutrality result for interior Stackelberg equilibria only applies to transfers from the leader to the follower, a result with several other interesting implications.*

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# 1 Introduction

A standard assumption in the theory of private provision of public goods is strict normality of the public good(s) for all consumers at all levels of wealth. This assumption seems to be motivated mainly by technical reasons since together with strict normality of private consumption it guarantees uniqueness of Nash equilibrium in the simultaneous move game. Indeed, there is no inherent reason to assume that – on an individual level – demand for public goods should behave differently from demand for private goods, and for the latter strict normality is only one among other relevant cases to study.

Consider, for instance, donations to charitable organizations that provide help for the poor. Clearly, those who benefit directly from these donations are the poor. However, there are important *ethical considerations* that lend "help for the poor" the character of a public good. The principle that no one in our society should suffer from hunger and deprivation is certainly important to most of us. More specifically, a main motive for caring about the poor is solidarity. It is argued that average or below-average income groups have a much stronger feel of solidarity with the very poor than rich individuals. This would imply that "help for the poor," considered as a public good, becomes inferior at high incomes. There are two different but interrelated reasonings that lead to that conclusion. The first is based on the *psychological* fact that people tend to "feel closer" to individuals that share common characteristics. Income is certainly one of the most important determinants of how people shape their lives. Consequently, one should expect the impact of solidarity to be much larger among similar income groups than among individuals with very different endowments, and hence also very different other characteristics. The second reasoning that supports our claim is based on *biological* considerations. Several authors have emphasized that certain ethical perceptions that govern human behaviour today can be explained in terms of the evolution of social interaction. The desire to share wealth with the poor can on this interpretation be understood as a sort of "reciprocal altruism" based on an insurance motive: Those who are in need now receive help from those who may be in need of help tomorrow. This kind of reciprocal altruism was certainly a rational and biologically useful strategy in primitive societies where people used to live in small groups with face-to-face contact. It survives in modern society as part of an "intuitive" ethics, and thus still influences human interaction (see e.g. Rubin [1982] and the references therein). What is important in our context is that also this reciprocal altruism based on an insurance motive gives rise to inferiority of the public good "help for the poor" at high incomes: Most likely, the subjective probability that one could need help in the future is higher in average income groups than in high income groups.

Besides these considerations based on the notion of altruism, there are also purely *egoistic motives* that may imply inferiority of public goods. Consider again donations to charity. One important aspect of the good consumed through donations to charity is "absence of the threat of crime," or, put differently, "a feeling of safety in the social environment." For income that is not well above average there seems to be no possibility to substitute donations to charity in order to consume that kind of safety. High income, on the other hand, allows for such substitution. For instance, one can experience "a feeling of safety" by employing private security services, hiring bodyguards, and so on. Consequently, one should expect the demand for "absence of crime" as a *public* good to decrease for income above a certain critical level.

The purpose of the present paper is to reconsider the theory of private provision of public goods without imposing the normality assumption. Specifically, it is assumed that individual Engel curves are either strictly increasing everywhere (corresponding to strict normality), or *unimodal*, i.e. strictly increasing up to a certain critical income level and decreasing above that level.

Our analysis reveals that many results obtained under the assumption of strict normality of public good(s) fail to hold in this more general setting. First, we prove that, in contrast to the strictly normal case analyzed by Varian [1994], sequential contributions can yield a larger equilibrium amount of the public good than simultaneous contributions. This suggests that the sequential contribution setup might be relevant even beyond its descriptive adequacy in certain contexts: Namely, as the result of a *design strategy* for a central agency that, in collecting individual donations, can reveal information to prospective contributors in order to increase total supply of the public good. Secondly, we show that, in contrast to the results obtained by Bergstrom, Blume and Varian [1986] for the strictly normal case, equalizing transfers between individuals with identical tastes can increase total equilibrium supply of the public good, both in the simultaneous and the sequential move game.

Our general analysis of wealth redistributions is based upon the neutrality results of Warr [1983] for interior Nash equilibria in the simultaneous move game, and Varian [1994] for interior Stackelberg equilibria in the sequential move game. For the case of Stackelberg equilibria we provide a new proof which avoids first-order conditions and which reveals that – even in the strictly normal case – the neutrality statement in fact only applies to transfers from the leader to the follower. This result implies that the set of interior Stackelberg equilibria obtainable by redistributing wealth among individuals is bounded away from the set of boundary equilibria in which the follower does not contribute. In other words, there cannot exist interior Stackelberg equilibria in which the follower contributes an arbitrarily small positive amount. This, in turn, implies the surprising result that transfers from the follower to the leader may involve strict Pareto improvements, a point that has also been observed by Buchholz, Konrad and Lommerud [1996]. The asymmetry of transfers between the leader and the follower in interior Stackelberg equilibria also plays a crucial role in our uniqueness analysis. Contrary to what has sometimes been claimed in the literature there is a problem of multiplicity of Stackelberg equilibria even in the strictly normal case. We prove that concavity of the follower's Engel curve in its increasing part guarantees *generic uniqueness* of Stackelberg equilibrium, both in the normal and the unimodal case. The fact that Stackelberg-neutrality only holds for transfers in one direction is used there to establish that the equilibrium correspondence that assigns the set of Stackelberg equilibria to each income distribution is upper hemicontinuous.

The plan of the paper is as follows. Section 2 introduces our basic assumptions on individual preferences and demand, and discusses the resulting properties of individual reaction functions. Section 3 addresses the issue of existence and multiplicity of equilibria, both in the simultaneous and the sequential move game. Section 4 compares the equilibria in the corresponding games while Section 5 provides the analysis of wealth redistributions. A discussion of welfare effects implied by the various comparative statics results is found in the concluding Section 6.

# 2 Voluntary Contribution Games

Consider a simple economy with 2 individuals, indexed by $i = 1, 2$, and two goods. Each individual's utility is given by a strictly quasi-concave utility function $u^i(c_i, G)$, where $c_i$ denotes $i$'s consumption of a private good and $G$ the consumption of a purely public good. Each individual has an initial endowment of $m_i^e$ units of the private good. For simplicity, let the price of the private good be equal to 1. Hence, one may think of $m_i^e$ as consumer $i$'s income. The public good is produced from the private good at a cost of one unit private good per unit of public good. Throughout, it is assumed that each consumer's preferences are strictly monotone and continuous. However, in contrast to most of the literature we do not assume that the public good is "normal" at every level of wealth. Instead, we will assume that for each individual there exists some income $\hat{m}_i$ such that the "demand" for the public good decreases in $m_i$ for incomes larger than $\hat{m}_i$. Specifically, let $\tilde{G}_i(\cdot)$ denote consumer $i$'s *Engel curve* for the public good, i.e. $\tilde{G}_i(m_i)$ is the amount of the public good supplied by agent $i$ with income $m_i$ given that she is the only supplier.[1] Given a current income $m_i^e$, $\tilde{G}_i(m_i^e)$ will also be referred to as agent $i$'s *standalone contribution.* Our basic assumption is as follows.

**Assumption I (Unimodality)** *For individual $i$, either $\tilde{G}_i(\cdot)$ is strictly increasing everywhere, or there exists $\hat{m}_i > 0$ such that $\tilde{G}_i(\cdot)$ is strictly increasing in the interval $[0, \hat{m}_i)$ and non-increasing in $[\hat{m}_i, \infty)$. Furthermore, in the latter case, $\tilde{G}_i(\cdot)$ is strictly decreasing in $[\hat{m}_i, \hat{\hat{m}}_i)$ for some $\hat{\hat{m}}_i > \hat{m}_i$.*

[1] Note that since prices for both goods are held fixed throughout the paper, demand and Engel curves coincide.

Note that Assumption I includes the standard assumption of strict normality of the public good everywhere as a special case. However, for the most part of this paper we will be interested in the case of "proper" unimodality, i.e. in settings where the public good is strictly inferior at some (high) incomes. For private consumption, on the other hand, we will assume non-inferiority everywhere. Indeed, in the two-good model considered here, private consumption should be viewed as an aggregate good. Therefore, the assumption of normality of private consumption seems much less problematic than the assumption of normality (resp. strict normality) of the public good common in the literature on voluntary contributions to public goods. Note that non-inferiority of private consumption implies that $\tilde{G}_i(\cdot)$ can never increase at a rate greater than 1.

Our assumptions on individual preferences and demand are jointly satisfied in the following example.

**Example 1** Consider the following utility function. For all $c, G \geq 0$, $(c, G) \neq (0, 0)$,

$$u(c, G) = 2\ln(c + 2G) + c. \tag{2.1}$$

Obviously, the corresponding preferences are continuous and strictly monotone. Also, it is easily verified that $u$ is strictly quasi-concave. Maximization of (2.1) subject to the budget constraint

$$c + G = m$$

gives the following "demand" $\tilde{G}(m)$ for the public good.

$$\tilde{G}(m) = \begin{cases} m & \text{if} \quad m \leq 1 \\ 2 - m & \text{if} \quad 1 \leq m \leq 2 \\ 0 & \text{if} \quad 2 \leq m \end{cases} \tag{2.2}$$

Hence, in this example $\tilde{G}(\cdot)$ is strictly increasing in $[0, \hat{m})$ with $\hat{m} = 1$, strictly decreasing in $[\hat{m}, \hat{\hat{m}})$ with $\hat{\hat{m}} = 2$, and $\tilde{G}(\cdot) = 0$ if $m \geq \hat{\hat{m}}$.[2]

The public good is supplied by voluntary contributions of the consumers. For each $i$, denote by $g_i$ consumer $i$'s contribution to the public good. In the following, we will consider two different models of voluntary contributions. The first is the simultaneous-move game analyzed in Warr [1983] and Bergstrom, Blume and Varian [1986], among others. In this framework, the basic assumption is that the individuals choose their contributions simultaneously, taking the activities of the other agent(s) as given for their own decision. Consequently,

[2]We are indebted to Klaus Nehring for suggesting the form of the utility function used in this example. Observe that by (2.2) private consumption is not a strictly normal good at every level of wealth. Example 1 has, of course, not been chosen for the sake of representation of "real" demand behaviour. The preferences described by (2.1) should rather be seen as a particularly simple *reference* example illustrating some basic properties of our model – just as quasi-linear preferences serve as a useful reference example in standard consumer theory.

consumer $i$'s decision problem is

$$\max_{c_i, g_i} u^i(c_i, g_i + g_j) \tag{2.3}$$

s.t.

$$c_i + g_i = m_i^e \quad \text{and} \tag{2.4}$$

$$g_i \geq 0, \tag{2.5}$$

where $j \neq i$. A pair $(c_1^*, c_2^*)$ and $(g_1^*, g_2^*)$ satisfying (2.3) – (2.5) for $i = 1, 2$ is a Nash equilibrium of the corresponding contribution game played by the 2 individuals. In order to distinguish this equilibrium concept from the one described below, we will refer to such an equilibrium as a *Cournot-Nash* equilibrium emphazising the simultaneous nature of the game and its formal similarities to the model of Cournot competition in industrial organization.

The second model considered here is Stackelberg leadership of one individual. This model has first been examined in Varian [1994] and further analyzed in Buchholz, Konrad and Lommerud [1996] (for the case of incomplete information, see also Haslbeck [1995]). Here, the assumption is that one individual, the Stackelberg leader, can commit to a certain quantity of contribution while anticipating the optimal response of the follower. Such a possibility of unilateral commitment may be due to the timing of decisions, i.e. when contributions are made sequentially with the Stackelberg leader ("her") moving first. Suppose that individual 1 is the leader. Then, her problem is

$$\max_{c_1, g_1} u^1(c_1, g_1 + g_2^R(g_1)) \tag{2.6}$$

s.t.

$$c_1 + g_1 = m_1^e \quad \text{and} \tag{2.7}$$

$$g_1 \geq 0, \tag{2.8}$$

where $g_2^R(\cdot)$ denotes the follower's ("his") reaction function, i.e. for any given $g_1$, $g_2^R(g_1)$ is the solution of the follower's problem (2.3) – (2.5).

In order to solve problem (2.3) – (2.5) it is convenient to rewrite (2.4) by adding $g_j$ on both sides of the equation. This yields

$$c_i + G = m_i^e + g_j.$$

Using this, it is easily verified that $i$'s reaction function can be written as follows. For all $g_j \geq 0$, $j \neq i$,

$$g_i^R(g_j) := g_i^R(m_i^e, g_j) = \max\{\tilde{G}_i(m_i^e + g_j) - g_j, 0\}. \tag{2.9}$$

In (2.9), $\tilde{G}_i(\cdot)$ denotes, as before, individual $i$'s Engel curve for the public good, i.e. the unique solution to $\max_G u^i(m_i - G, G)$. Consequently, for all $g_i^R(g_j) > 0$, individual $i$'s best response to the other agent's contribution $g_j$ can be thought of as $i$'s demand for the public good if her income was $m_i^e + g_j$, minus the other agent's contribution. Clearly, given continuity of $u^i$, $g_i^R(\cdot)$ will be a continuous function. In order to describe the qualitative behaviour of $g_i^R(\cdot)$ implied by our assumptions on individual preferences, let

$$\hat{g}_j := \hat{g}_j(m_i^e) := \hat{m}_i - m_i^e,$$

where $i \neq j$ and $m_i^e$ is $i$'s actual income. If $g_i^R(\hat{g}_j)$ is positive, our assumptions on individual demand for the public good imply that $g_i^R(\cdot)$ is decreasing at a rate less than 1 in the interval $[0, \hat{g}_j)$, and at a rate greater than 1 for values of $g_j$ greater than $\hat{g}_j$ (until $g_i^R(\cdot)$ becomes zero).[3] Next, define

$$\bar{g}_j := \bar{g}_j(m_i^e) := \min\{g_j : \tilde{G}_i(m_i^e + g_j) = g_j\}.$$

Note that by continuity of $g_i^R(\cdot)$ the minimum exists, so that $\bar{g}_j$ is well defined. Given that $\tilde{G}_i(\cdot)$ can never increase at a rate greater than 1, i.e. given that private consumption is non-inferior everywhere, one obtains that $g_i^R(g_j) = 0$ if and only if $g_j \geq \bar{g}_j$. That is, at point $\bar{g}_j$ agent $j$ contributes so much that agent $i$ chooses to contribute zero. Therefore, $\bar{g}_j$ is called agent $j$'s *complete crowding out contribution*. Finally, define $m_i^0$ by

$$m_i^0 := \hat{m}_i - \tilde{G}_i(\hat{m}_i).$$

Obviously, $0 \leq m_i^0 < \hat{m}_i$. The following observation summarizes the qualitative behaviour of the reaction function.

**Fact 2.1** **Case a)** *Suppose that $0 < m_i^e \leq m_i^0$. Then, the reaction function $g_i^R(\cdot)$ is decreasing at a rate less than 1 in the interval $[0, \bar{g}_j)$ and $g_i^R(g_j) = 0$ for all $g_j \geq \bar{g}_j$ (see Figure 1a).*
**Case b)** *Suppose that $m_i^0 < m_i^e < \hat{m}_i$. Then, the reaction function $g_i^R(\cdot)$ is decreasing at a rate less than 1 (possibly at a rate 0) in the interval $[0, \hat{g}_j)$. Furthermore, the reaction function is decreasing at a rate greater than 1 in the interval $[\hat{g}_j, \bar{g}_j)$, and it is zero for all $g_j \geq \bar{g}_j$ (see Figure 1b).*
**Case c)** *Suppose that $m_i^e \geq \hat{m}_i$. Then, the reaction function $g_i^R(\cdot)$ is decreasing at a rate greater than 1 in the interval $[0, \bar{g}_j)$, and it is zero for all $g_j \geq \bar{g}_j$ (see Figures 1c and 1d).*

In order to verify this, consider first case a). First, we show that $\hat{g}_j < \bar{g}_j$ if and only if $m_i^e > m_i^0$. Indeed, $\hat{g}_j < \bar{g}_j$ if and only if $g_i^R(\hat{g}_j) > 0$. However, by (2.9) this is equivalent to $\tilde{G}_i(\hat{m}_i) - (\hat{m}_i - m_i^e) > 0$, i.e. to $m_i^e > m_i^0$. Hence, in case a), $\bar{g}_j \leq \hat{g}_j$. This implies that the reaction function becomes zero before it reaches

[3] Hence, just as $\hat{m}$ demarcates the "normal" region of the Engel curve (positive slope) from the "inferior" region (negative slope), $\hat{g}$ demarcates the "flat" region of the reaction function (slope greater than $-1$) from the "steep" region (slope smaller than $-1$). The symbol "ˆ" has been chosen as a graphical representation of the qualitative change of the corresponding function.

the region where its slope can become smaller than $-1$. Since private consumption is non-inferior everywhere, $g_i^R(\cdot)$ can nowhere be increasing. Consequently, $g_i^R(\cdot)$ looks qualitatively as shown in Fig. 1a.

Given our definitions and notation, case b) is straightforward: By assumption, $\tilde{G}_i(m_i)$ is strictly increasing at a rate less or equal to 1 in the interval $[0, \hat{m}_i)$. Hence, if $m_i^0 < m_i^e < \hat{m}_i$, then by (2.9), $g_i^R(\cdot)$ is decreasing at a rate less than 1 in the interval $[0, \hat{g}_j)$. Similarly, since $\tilde{G}_i(m_i)$ is non-increasing in the interval $[\hat{m}_i, \infty)$, $g_i^R(\cdot)$ is decreasing at a rate greater than or equal to 1 in the interval $[\hat{g}_j, \bar{g}_j)$, and it is zero for all $g_j \geq \bar{g}_j$ (see Fig. 1b).

Finally, in case c), when $m_i^e \geq \hat{m}_i$, $\tilde{G}_i(m_i^e + \cdot)$ is non-increasing everywhere. Hence, if $\tilde{G}_i(m_i^e) > 0$, $g_i^R(\cdot)$ is decreasing at a rate greater or equal to 1 until it hits the axis (Fig. 1c). On the other hand, there is also the possibility that $\tilde{G}_i(m_i^e) = 0$ in which case $g_i^R(\cdot) = 0$ everywhere (Fig. 1d).

*Figure 1 here*

# 3 Equilibria

Existence of Cournot-Nash and Stackelberg equilibria follows in our model from standard arguments. However, in contrast to the case where both, private consumption and public good, are strictly normal, there will, in general, be a multiplicity of Cournot-Nash equilibria.

## 3.1 Cournot-Nash Equilibria

In order to illustrate the possibility of multiple Cournot-Nash equilibria, consider the following example.

**Example 2** Suppose that the two individuals have identical preferences given by the utility function (2.1) of Example 1. Calculating $i$'s reaction function gives

$$g_i^R(g_j) = \begin{cases} m_i^e & \text{if} \quad g_j \leq 1 - m_i^e \\ 2 - m_i^e - 2g_j & \text{if} \quad 1 - m_i^e < g_j < 1 - \frac{m_i^e}{2} \\ 0 & \text{if} \quad 1 - \frac{m_i^e}{2} \leq g_j \end{cases}$$

where $i \neq j$. Suppose that $m_1^e = 0.9$ and $m_2^e = 1$. Then there exist exactly three Cournot-Nash equilibria with the following distribution of individual contributions: $(0.9, 0)$, $(0, 1)$, and $(0.3, 0.4)$ (see Figure 2).[4]

[4]Note that there is an issue of *stability* here. Indeed, in the above example the interior equilibrium is not stable with respect to any dynamics where agents adjust their contribution in direction to their best responses. However, the focus of our analysis is not on stability and the issue is not relevant for our main results.

*Figure 2 here*

## 3.2 Stackelberg Equilibria

In contrast to Cournot-Nash equilibrium, multiplicity of Stackelberg equilibria is also an issue when the public good is strictly normal. In fact, the following result suggests that the assumption of inferiority of the public good for high income typically *reduces* the number of Stackelberg equilibria.

**Fact 3.1** *Suppose that in a Stackelberg game with individual* 1 *as leader, the follower's Engel curve* $\tilde{G}_2(\cdot)$ *is non-increasing in the interval* $[m_2^e, \infty)$*. Then, there exist at least one and at most two Stackelberg equilibria. The two potential equilibria are* $(g_1^S, g_2^S) = (0, \tilde{G}_2(m_2^e))$ *and* $(g_1^S, g_2^S) = (\tilde{G}_1(m_1^e), 0)$*.*

**Proof** Obviously, among all $(g_1, 0)$, the leader's standalone contribution $g_1 = \tilde{G}_1(m_1^e)$ maximizes her utility. Consequently, $(\tilde{G}_1(m_1^e), 0)$ is at least a candidate for equilibrium (though not always actually an equilibrium). Next, suppose that the follower chooses his contribution from the strictly positive region of his reaction function $g_2^R(\cdot)$. We show that in this case $g_1^S = 0$ is optimal for the leader. The leader maximizes

$$u^1\left(m_1^e - g_1, g_1 + g_2^R(g_1)\right) = u^1\left(m_1^e - g_1, \tilde{G}_2(m_2^e + g_1)\right).$$

By assumption, $\tilde{G}_2(m_2^e + \cdot)$ is non-increasing. Hence, by strict monotonicity of preferences,

$$u^1\left(m_1^e, \tilde{G}_2(m_2^e)\right) > u^1\left(m_1^e - g_1, \tilde{G}_2(m_2^e + g_1)\right)$$

for all $g_1 > 0$. Consequently, $g_1^S = 0$ is optimal for the leader.

Fact 3.1 shows that there cannot exist a Stackelberg equilibrium in which both agents contribute a positive amount if the public good is inferior for the follower at his initial level of wealth. For the intuition behind this fact, let $T(g_1) := g_1 + g_2^R(g_1)$ denote total provision of the public good when the leader chooses $g_1$. The follower being in his inferior region at $m_2 = m_2^e$ implies that his reaction function is decreasing at a rate greater than, or equal to, 1 in the whole positive range, so that $T(g_1)$ is decreasing in $g_1$ for all $g_1 < \bar{g}_1$. Consequently, choosing a contribution in the interval $(0, \bar{g}_1)$ can never be optimal for the leader, since any reduction of her contribution would increase not only her private consumption but also the total quantity of the public good.

**Remark** Suppose, as in Fact 3.1, that $\tilde{G}_2(\cdot)$ is non-increasing in $[m_2^e, \infty)$. A sufficient condition for $(0, \tilde{G}_2(m_2^e))$ to be the unique Stackelberg equilibrium is that $\tilde{G}_2(m_2^e) \geq \tilde{G}_1(m_1^e)$. This follows at once from monotonicity of preferences.

By Fact 3.1, if the follower is in his "inferior region" there is only one equilibrium in which he contributes a positive amount. On the other hand, it is quite clear that if $\tilde{G}_2(\cdot)$ is strictly increasing at $m_2^e$ there may be several Stackelberg equilibria in which the follower contributes a positive amount. A sufficient condition that rules this out is *concavity* of $\tilde{G}_2(\cdot)$ in the region where it is increasing. Note that concavity of $\tilde{G}_2(\cdot)$ implies that the public good is a *necessary* good.[5]

**Assumption II (Concavity)** *Individual i's Engel curve $\tilde{G}_i(\cdot)$ is a concave function in its increasing part, i.e. on the interval $[0, \hat{m}_i)$.*

**Fact 3.2** *Suppose that the follower's Engel curve satisfies Assumptions I and II. Then there exists at least one and at most two Stackelberg equilibria. Furthermore, there exists at most one Stackelberg equilibrium $(g_1^S, g_2^S)$ in which the follower (individual 2) contributes a positive amount. In any such equilibrium, $g_1^S \leq \hat{g}_1$.*

**Proof** As in Fact 3.1 above, a possible candidate for equilibrium is $(\tilde{G}_1(m_1^e), 0)$. Next, suppose that the follower chooses his contribution from the strictly positive region of his reaction function $g_2^R(\cdot)$. By Fact 3.1, the leaders utility is strictly decreasing in the interval $(\hat{g}_1, \bar{g}_1]$ in that case. Hence, in any Stackelberg equilibrium with $g_2^S > 0$ one must have $0 \leq g_1^S \leq \hat{g}_1$. Suppose there were two equilibrium values $g_1^S \neq g_1'^S$ with $0 \leq g_1^S, g_1'^S \leq \hat{g}_1$. Let

$$u_{max}^1 := u^1\left(m_1 - g_1^S, \tilde{G}_2(m_2^e + g_1^S)\right) = u^1\left(m_1 - g_1'^S, \tilde{G}_2(m_2^e + g_1'^S)\right).$$

Consider the value $g_1^0 := (g_1^S + g_1'^S)/2$. By strict quasi-concavity of $u^1$,

$$u^1\left(m_1 - g_1^0, \frac{1}{2}[\tilde{G}_2(m_2^e + g_1^S) + \tilde{G}_2(m_2^e + g_1'^S)]\right) > u_{max}^1.$$

Since, $g_1^S, g_1'^S \leq \hat{g}_1$ we know that $0 \leq m_2^e + g_1^S, m_2^e + g_1'^S \leq \hat{m}_2$. Hence by concavity of $\tilde{G}_2(\cdot)$ in that region,

$$\frac{1}{2}\left[\tilde{G}_2(m_2^e + g_1^S) + \tilde{G}_2(m_2^e + g_1'^S)\right] \leq \tilde{G}_2(m_2^e + g_1^0).$$

Consequently, by monotonicity of $u^1$

$$u^1\left(m_1^e - g_1^0, \tilde{G}_2(m_2^e + g_1^0)\right) > u_{max}^1,$$

which is obviously not possible. Hence, a utility maximizing value $g_1^S \in [0, \hat{g}_1]$ must be unique.

To further illustrate the role of concavity of the follower's Engel curve in Fact 3.2, consider potential equilibria with $g_2^S > 0$ and $g_1^S \leq \hat{g}_1$, and let $T(g_1) = g_1 + g_2^R(g_1)$ denote total provision of the public good. Concavity of the follower's

[5] A good is necessary at a certain income level $m$, if demand goes up by a lesser proportion than income, i.e. in our case, if $d\tilde{G}(m)/dm \leq \tilde{G}(m)/m$.

Engel curve in $[0, \hat{m}_2]$ implies concavity of his reaction function in $[0, \hat{g}_1]$, and hence concavity of $T(\cdot)$ in this range. In words, if the leader increases her own contribution in the considered interval, each additional dollar of contribution has a smaller effect on total quantity of the public good than the previous one. From the leader's point of view this is equivalent to an increasing marginal cost of providing the public good. Since her marginal evaluation for the public good is decreasing there must be a unique optimal value $g_1$ in the considered range. This potential candidate for Stackelberg equilibrium has to be compared with the second candidate $g_1 = \tilde{G}_1(m_1^e)$. The one with higher utility for the leader becomes the actual Stackelberg equilibrium.

Summarizing, under Assumption II, i.e. under concavity of the follower's Engel curve in its increasing part, there are at most two Stackelberg equilibria. In fact, one can establish a much stronger result. The following theorem demonstrates that essentially the same assumption implies *generic uniqueness* of Stackelberg equilibrium, i.e. uniqueness for almost all income distributions.

**Theorem 3.1 (Generic Uniqueness of Stackelberg Equilibrium)** *In a Stackelberg game with individual 1 as leader, let the follower's Engel curve $\tilde{G}_2(\cdot)$ satisfy Assumptions I and II. In addition, assume that in the interval $[\hat{m}_2, \infty)$, $\tilde{G}_2(\cdot)$ is strictly decreasing whenever positive. Then, the set of $(m_1^e, m_2^e)$ for which Stackelberg equilibrium is non-unique is a closed set of measure zero.*

The proof of Theorem 3.1 is partly based on results on wealth redistributions presented in Section 5 below. Therefore, the proof is deferred to an appendix where we also verify necessity of the additional condition of strict monotonicity of $\tilde{G}_2(\cdot)$ in its non-increasing part.

# 4 Comparison of Cournot-Nash and Stackelberg Equilibria

This section is devoted to the comparison of Cournot-Nash and Stackelberg equilibria. In particular, we show that, contrary to the case where the public good is strictly normal, total provision in the sequential move game may be higher than in the corresponding simultaneous move game. The following preliminary result does not depend on our specific assumptions on individual demand for the public good and is essentially the same as in Varian [1994, Th. 2].

**Fact 4.1** *The amount of the public good provided by the leader in any Stackelberg equilibrium is never larger than the smallest amount she will provide in any Cournot-Nash equilibrium.*

**Proof** Let $(g_1^*, g_2^*)$ be the Cournot-Nash equilibrium with minimal $g_1^*$. By strict quasi-concavity of utility functions, and by the fact that $g_1 = g_1^*$ is a best response to $g_2 = g_2^* = g_2^R(g_1^*)$,

$$u^1(m_1^e - g_1^*, g_2^R(g_1^*) + g_1^*) > u^1(m_1^e - g_1, g_2^R(g_1^*) + g_1)$$

for all $g_1 \neq g_1^*$. Since $g_2^R(\cdot)$ is non-increasing everywhere,

$$u^1(m_1^e - g_1, g_2^R(g_1^*) + g_1) \geq u^1(m_1^e - g_1, g_2^R(g_1) + g_1)$$

for all $g_1 \geq g_1^*$. Consequently,

$$u^1(m_1^e - g_1^*, g_2^R(g_1^*) + g_1^*) > u^1(m_1^e - g_1, g_2^R(g_1) + g_1)$$

for all $g_1 > g_1^*$. Hence, for any Stackelberg equilibrium $(g_1^S, g_2^S)$, $g_1^S \leq g_1^*$.

Fact 4.1 at once implies the following two corollaries.

**Corollary 4.1** *Suppose that* $(0, \tilde{G}_2(m_2))$ *is a Cournot-Nash equilibrium in the simultaneous move game. Then* $(0, \tilde{G}_2(m_2))$ *is the unique Stackelberg equilibrium.*

**Corollary 4.2** *Suppose that there is a Cournot-Nash equilibrium* $(g_1^*, g_2^*)$ *with* $0 \leq g_1^* \leq \hat{g}_1$. *Then total provision of the public good in any Stackelberg equilibrium is not greater than* $g_1^* + g_2^*$.

**Proof** By Fact 4.1, in any Stackelberg equilibrium $(g_1^S, g_2^S)$, one must have $g_1^S \leq g_1^*$. The conclusion thus follows from the fact that $g_1 + g_2^R(g_1)$ is non-decreasing in the interval $[0, \hat{g}_1]$.

If the public good is a normal good, Corollary 4.2 implies that the total amount of the public good provided in a Stackelberg equilibrium is never larger than that provided in any Cournot-Nash equilibrium. This conclusion, however, is not valid under our present assumption that the public good may become inferior at high income levels. To illustrate the possibility that in a Stackelberg equilibrium total provision of the public good is greater than in *any* Cournot-Nash equilibrium consider the following example.

**Example 2.1** Suppose that in Example 2, incomes are given by $m_1^e = 0.5$ and $m_2^e = 1.4$, respectively. In this case, the unique Cournot-Nash equilibrium is $(0.5, 0)$ (see Fig. 3). On the other hand, it is easily verified that $\tilde{G}_2(m_2^e) = 0.6 > \tilde{G}_1(m_1^e) = 0.5$, hence by the remark to Fact 3.1, $(0, 0.6)$ is the unique Stackelberg equilibrium.

*Figure 3 here*

In Example 2.1, individual 1 is in her "strictly normal region" with her initial level of wealth. For individual 2, on the other hand, the public good is inferior at $m_2 = m_2^e$, so that his reaction function is decreasing at a rate greater than 1 in the whole positive range. In the unique Cournot-Nash equilibrium of the simultaneous move game player 1 chooses her standalone contribution and player 2 free rides, i.e. contributes zero. If the players move sequentially (with player 1 as leader) then player 1 gets to pick among two allocations which are potential candidates for Stackelberg equilibrium (cf. Fact 3.1): (i) the unique Cournot-Nash equilibrium, and (ii) the allocation in which player 1 free rides on player 2's contribution. Since both, the total amount of public good, and 1's private consumption are higher in (ii), player 1 will, of course, choose that allocation. Thus, total provision of the public good is greater in Stackelberg than in Cournot-Nash equilibrium.

The following result gives sufficient conditions for total provision of the public good being never less in Stackelberg equilibrium than in any Cournot-Nash equilibrium.

**Theorem 4.1** *Suppose that in a Stackelberg game with individual* 1 *as leader, the follower's Engel curve* $\tilde{G}_2(\cdot)$ *is non-increasing in* $[m_2^e, \infty)$*, i.e. the follower is in his "inferior" region. Furthermore, suppose that* $\tilde{G}_2(m_2^e) \geq \tilde{G}_1(m_1^e)$*. Then, total provision of the public good in the unique Stackelberg equilibrium is at least as great as in any Cournot-Nash equilibrium.*

**Proof** Under the assumptions of Theorem 4.1, $(g_1^S, g_2^S) = (0, \tilde{G}_2(m_2^e))$ is the unique Stackelberg equilibrium (cf. Fact 3.1 and the following remark). By $\tilde{G}_2(m_2^e) \geq \tilde{G}_1(m_1^e)$, it suffices to show that $\tilde{G}_2(m_2^e)$ is always at least as great as the total amount of public good provided in any Cournot-Nash equilibrium $(g_1^*, g_2^*)$ with $0 \leq g_1^* < \bar{g}_1$. In any such equilibrium, total amount of public good is

$$g_2^R(g_1^*) + g_1^* = \tilde{G}_2(m_2^e + g_1^*).$$

Since $\tilde{G}_2(\cdot)$ is non-increasing in $[m_2^e, \infty)$ the conclusion follows.

**Remark** Suppose that $\tilde{G}_2(\cdot)$ is non-increasing in $[m_2^e, \infty)$ and strictly decreasing at $m_2^e$. Furthermore, assume that $\tilde{G}_2(m_2^e) > \tilde{G}_1(m_1^e)$. Then total provision of the public good in Stackelberg equilibrium is *strictly* greater than in any Cournot-Nash equilibrium if and only if $\tilde{G}_2(m_2^e) < \bar{g}_2$. This is easily verified along the lines of Theorem 4.1 noting that $\tilde{G}_2(m_2^e) < \bar{g}_2$ holds if and only if $(0, \tilde{G}_2(m_2^e))$ (the unique Stackelberg equilibrium) is *not* a Cournot-Nash equilibrium.

It is emphasized that the assumption of the follower being in his "inferior region" *alone* is not sufficient for the conclusion of Theorem 4.1. Hence, the additional condition $\tilde{G}_2(m_2^e) \geq \tilde{G}_1(m_1^e)$ cannot be dropped. To illustrate this, consider the following example.

**Example 2.2** Suppose that in Example 2 above, individual income is given by $m_1^e = 0.8$ and $m_2^e = 1.5$, respectively. It is easily verified that $(0.8, 0)$ is the unique Cournot-Nash equilibrium, whereas $(0, 0.5)$ is the unique Stackelberg equilibrium.

On the other hand, the assumption of the follower being in his "inferior region" is also not necessary for the conclusion of Theorem 4.1. Indeed, even if $\tilde{G}_2(\cdot)$ and $\tilde{G}_1(\cdot)$ are both strictly increasing at $m_2^e$ and $m_1^e$, respectively, one can have a greater total provision of public good in Stackelberg equilibrium. To illustrate this possibility consider the following example.

**Example 3** Suppose that the follower's utility is given as in Example 2 by $u^2(c_2, G) = 2\ln(c_2 + 2G) + c_2$, whereas the leader's utility is $u^1(c_1, G) = 4\ln(c_1 + 2G) + c_1$. Furthermore, suppose that $m_1^e = 0.7$ and $m_2^e = 0.8$. It can be verified that the unique Cournot-Nash equilibrium is $(0.7, 0)$, whereas $(0.2, 0.8)$ is the unique Stackelberg equilibrium.

Example 3 illustrates that total supply of the public good can be larger in Stackelberg equilibrium than in any Cournot-Nash equilibrium even when both individuals are still in their "strictly normal region" with their initial income. Hence, the assumption of inferiority for high income can have an indirect influence on contribution behaviour even when actual income is not high.

# 5 Wealth Redistributions

We are now ready to deal with the central issue of income redistributions. The section is divided into three parts. In the first part, we state some general facts about redistribution that are independent from our specific assumptions on individual demand. These facts are closely related to the neutrality results for interior equilibria due to Warr [1983] in the case of Cournot-Nash equilibrium, and Varian [1994] in the case of Stackelberg equilibria. Perhaps surprisingly, the neutrality result for interior Stackelberg equilibria only applies to transfers from the leader to the follower. The second part is devoted to *local* redistributions, i.e. sufficiently small transfers. In particular, it is observed that arbitrarily small transfers from the follower to the leader can discontinuously increase aggregate supply of the public good. The third part deals with global redistributions under the assumption of identical preferences. We demonstrate that there always exists an *optimal* distribution of income in the sense that for no other distribution total provision of the public good can be greater. In many cases, such an optimal distribution involves equalizing transfers, a result which is in sharp contrast to the findings of Bergstrom, Blume and Varian [1986].

## 5.1 General Facts about Redistributions

In this subsection, unless otherwise indicated, we will make no specific assumptions about individual demand other than those implied by quasi-concavity of individual utility functions. For the record, we first state Warr's [1983] well-known neutrality result for interior Cournot-Nash equilibria.

**Fact 5.1** *Let $(g_1^*, g_2^*)$ be an interior Cournot-Nash equilibrium (i.e. $g_1^*, g_2^* > 0$) corresponding to the income distribution $(m_1, m_2)$. Furthermore, let $\Delta m$ be such that $g_1^* - \Delta m, g_2^* + \Delta m \geq 0$. Then, $(g_1^* - \Delta m, g_2^* + \Delta m)$ is a Cournot-Nash equilibrium corresponding to the income distribution $(m_1 - \Delta m, m_2 + \Delta m)$.*

The corresponding neutrality result for interior Stackelberg equilibria is due to Varian [1994]. Note, however, that unlike Warr's result its applicability is restricted to transfers from the leader to the follower.

**Fact 5.2** *Let* $(g_1^S, g_2^S)$ *be an interior Stackelberg equilibrium (with individual* 1 *as leader) corresponding to the income distribution* $(m_1, m_2)$*. Let* $\Delta m > 0$ *be a positive transfer from the leader to the follower such that* $g_1^S - \Delta m \geq 0$*. Then* $(g_1^S - \Delta m, g_2^S + \Delta m)$ *is a Stackelberg equilibrium corresponding to the income distribution* $(m_1 - \Delta m, m_2 + \Delta m)$*.*

**Proof** In order to verify that $g_1^S - \Delta m$ is optimal for the leader given her income $m_1 - \Delta m$, we verify first that $g_2^S + \Delta m$ is the follower's best response to $g_1^S - \Delta m$, given the follower's income $m_2 + \Delta m$. By assumption $g_2^S > 0$, hence by (2.9),

$$g_2^S = \tilde{G}_2(m_2 + g_1^S) - g_1^S. \tag{5.1}$$

Furthermore, since the right hand side of (5.1) is positive and $\Delta m > 0$,

$$\tilde{G}_2(m_2 + \Delta m + (g_1^S - \Delta m)) - (g_1^S - \Delta m) > 0. \tag{5.2}$$

Therefore, by (5.1) and (5.2),[6]

$$\begin{aligned} g_2^R(m_2 + \Delta m, g_1^S - \Delta m) &= \tilde{G}_2(m_2 + \Delta m + (g_1^S - \Delta m)) - (g_1^S - \Delta m) \\ &= \tilde{G}_2(m_2 + g_1^S) - g_1^S + \Delta m \\ &= g_2^R(m_2, g_1^S) + \Delta m \\ &= g_2^S + \Delta m. \end{aligned}$$

Next, we verify that $g_1^S - \Delta m$ is indeed the utility maximzing contribution of the leader, given the follower's best response. Assume, by way of contradiction, that there would exist $g_1^0 \in [0, m_1 - \Delta m]$ such that

$$\begin{aligned} & u^1\left(m_1 - \Delta m - g_1^0, g_1^0 + g_2^R(m_2 + \Delta m, g_1^0)\right) \\ & \qquad > \; u^1\left(m_1 - \Delta m - (g_1^S - \Delta m), g_1^S + g_2^S\right). \end{aligned} \tag{5.3}$$

First, suppose that $g_2^R(m_2 + \Delta m, g_1^0) \leq \Delta m$. By non-negativity of $g_2^R(m_2, \cdot)$, monotonicity of $u^1$ and (5.3),

$$\begin{aligned} & u^1\left(m_1 - (g_1^0 + \Delta m), g_1^0 + \Delta m + g_2^R(m_2, g_1^0 + \Delta m)\right) \\ & \qquad \geq \; u^1\left(m_1 - (g_1^0 + \Delta m), g_1^0 + \Delta m\right) \\ & \qquad \geq \; u^1\left(m_1 - (g_1^0 + \Delta m), g_1^0 + g_2^R(m_2 + \Delta m, g_1^0)\right) \\ & \qquad > \; u^1\left(m_1 - g_1^S, g_1^S + g_2^S\right). \end{aligned}$$

[6] In order to keep track of the different income distributions, we include the follower's income as an argument in his reaction function in all what follows. Hence, $g_2^R(m_2, \cdot)$ denotes the follower's reaction function given his income $m_2$. Also, in order to economize notation, we omit the superscript "$^e$" in the denotation of current income throughout this section.

However, this shows that given the income distribution $(m_1, m_2)$, $g_1^0 + \Delta m$ would give the leader higher utility than $g_1^S$, and therefore contradicts the fact that $(g_1^S, g_2^S)$ is an initial equilibrium. Consequently, we can conclude that $g_2^R(m_2 + \Delta m, g_1^0) > \Delta m$. This implies

$$\begin{aligned} g_2^R(m_2 + \Delta m, g_1^0) - \Delta m &= \tilde{G}_2(m_2 + \Delta m + g_1^0) - g_1^0 - \Delta m \\ &= g_2^R(m_2, g_1^0 + \Delta m), \end{aligned}$$

and hence by (5.3),

$$u^1\left(m_1 - (g_1^0 + \Delta m), g_1^0 + \Delta m + g_2^R(m_2, g_1^0 + \Delta m)\right) > u^1(m_1 - g_1^S, g_1^S + g_2^S).$$

Again, this contradicts the fact that $(g_1^S, g_2^S)$ is an equilibrium corresponding to $(m_1, m_2)$. Hence, $g_1^S - \Delta m$ is optimal for the leader if the income distribution is $(m_1 - \Delta m, m_2 + \Delta m)$ and the proof of Fact 5.2 is complete.

Clearly, as a consequence of Facts 5.1 and 5.2 one has the following result.

**Corollary 5.1** *Let aggregate income be given by $M$. Suppose that for any given distribution of income there exists at most one interior Cournot-Nash equilibrium. Then, for any two distributions of income, consumption of private and public good are the same for both individuals in the corresponding interior Cournot-Nash equilibria. Similarly, if for any given distribution of income there exists at most one interior Stackelberg equilibrium, the same applies to interior Stackelberg equilibria.*

Note that the difference in the statements of Facts 5.1 and 5.2 plays no role for the statement in Corollary 5.1. Indeed, in order to verify Corollary 5.1 it suffices to consider transfers in one direction only, i.e. from individual 1 to individual 2.

The fact that the neutrality result for Stackelberg equilibria only applies to transfers from the leader to the follower may seem rather surprising. The reason is that an opposite transfer may destroy the interior equilibrium and induce an equilibrium in which the follower does not contribute. To illustrate this possibility, consider the following example.

**Example 4** Let $u^i(c_i, G) = c_i G$ for $i = 1, 2$, and assume that total income $M = m_1 + m_2$ equals 1. Computing the Stackelberg equilibria for any distribution of income, it can be verified that the equilibrium set looks as depicted in Figure 4: For $m_1 \leq 0.5$, the unique Stackelberg equilibrium is the boundary equilibrium $(0, \tilde{G}_2(m_2)) = (0, m_2/2)$. If $m_1$ increases beyond 0.5 we reach the region of interior equilibria. In any interior equilibrium, total provision of the public good is 0.25. Continuing with the redistribution from agent 2 to agent 1 we get to some critical value $\tilde{m}$ $(= \sqrt{0.5})$ such that for $m_1 = \tilde{m}$ there exist two equilibria, one interior ($g_1^S = \tilde{m} - 0.5$; $g_2^S = 0.25 - g_1^S$) and one boundary equilibrium in which the follower does not contribute ($g_1^S = \tilde{G}_1(\tilde{m})$; $g_2^S = 0$). If $m_1 > \tilde{m}$, only the boundary equilibrium survives. Hence, there is a "discontinuity" in the equilibrium manifold.

*Figure 4 here*

The "discontinuity" of the equilibrium correspondence exhibited by Example 4 is a general phenomenon, at least under concavity of the follower's Engel curve. Indeed, in that case the set of interior Stackelberg equilibria generated by redistributions is "bounded away" from the axis $g_2 = 0$. Specifically, one has the following result.

**Fact 5.3** *Let $M$ denote aggregate income, and suppose that the follower's Engel curve satisfies Assumptions I and II. Denote by $\mathcal{G}(M)$ the set of all interior equilibria generated by all redistributions of aggregate income $M$. If $\mathcal{G}(M)$ is non-empty, then there exists a constant $K > 0$ such that for any $(g_1^S, g_2^S) \in \mathcal{G}(M)$, $g_2^S \geq K$.*

**Proof** Let $(g_1^S, g_2^S)$ be an interior Stackelberg equilibrium corresponding to the income distribution $(m_1, m_2)$. For any $\delta > 0$, consider the redistribution $(m_1^\delta, m_2^\delta) = (m_1 + (g_2^S - \delta), m_2 - (g_2^S - \delta))$. By Fact 5.2, if there exists an interior equilibrium corresponding to $(m_1^\delta, m_2^\delta)$ it must be $(g_1^\delta, g_2^\delta) = (g_1^S + g_2^S - \delta, \delta)$. If $(g_1^\delta, g_2^\delta)$ is actually an equilibrium, the leader's first order condition gives[7]

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^\delta - g_1^\delta, g_1^\delta + g_2^R(m_2^\delta, g_1^\delta)\right) = \frac{1}{1 + \frac{\partial g_2^R}{\partial g_1}(m_2^\delta, g_1^\delta)}.$$

Hence, for $\delta \to 0$ one obtains,

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^0 - g_1^0, g_1^0\right) = \frac{1}{1 + \lim_{\delta\to 0} \frac{\partial g_2^R}{\partial g_1}(m_2^\delta, g_1^\delta)}, \tag{5.4}$$

where $m_1^0 = m_1 + g_2^S$ and $g_1^0 = g_1^S + g_2^S$. Now observe that by concavity of the follower's Engel curve, and hence by concavity of $g_2^R(m_2, \cdot)$ in the relevant range, the limit of the slope of the follower's reaction function in (5.4) must be strictly smaller than 0 (and, by Fact 3.2, greater than, or equal to, $-1$). Hence, in (5.4),

$$-1 \leq \lim_{\delta\to 0} \frac{\partial g_2^R}{\partial g_1}(m_2^\delta, g_1^\delta) < 0,$$

which implies

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^0 - g_1^0, g_1^0\right) > 1. \tag{5.5}$$

However, given that the follower contributes zero in (5.5), the leader's utility maximizing contribution $\tilde{G}_1(m_1^0)$ satisfies the following first order condition,

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) = 1. \tag{5.6}$$

[7] Assuming differentiability of $u^1(\cdot,\cdot)$ and $g_2^R(\cdot)$ in the relevant ranges, of course.

Hence, comparing (5.5) and (5.6), strict quasi-concavity of $u^1$ implies that $\tilde{G}_1(m_1^0) > g_1^0$ and

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > u^1(m_1^0 - g_1^0, g_1^0). \tag{5.7}$$

By continuity of $\tilde{G}_1(\cdot)$, and since by Corollary 5.1 utility is constant along $(g_1^\delta, g_2^\delta)$, (5.7) must also hold for sufficiently small $\delta > 0$. Consequently, there exists $K > 0$ such that for $\delta < K$, $(g_1^\delta, g_2^\delta)$ is not an equilibrium, and the proof is complete.

The driving force behind the discontinuity of the equilibrium correspondence revealed in Fact 5.3 is a discontinuity in the cost of providing an additional unit of the public good from the viewpoint of the leader: Up to the complete crowding out level $\bar{g}_1$ an additional dollar of contribution by the leader induces a change in total provision of the public good, $T(g_1) = g_1 + g_2^R(g_1)$, of less than one unit, since $T'(\cdot) = 1 + (g_2^R)'(\cdot)$. Beyond $\bar{g}_1$ each change in contribution induces an equal change in total provision. Concavity of the follower's Engel curve in $[0, \hat{m}_2]$ and inferiority beyond this critical income level together imply that $(g_2^R)'(\cdot)$ cannot approach 0 smoothly, so that the marginal change in $T(g_1)$ is strictly smaller than the change in $g_1$ in an interval $(\bar{g}_1 - \epsilon, \bar{g}_1)$, while it is equal to the change in $g_1$ in the interval $(\bar{g}_1, \infty)$. From the leader's point of view this is equivalent to a downward jump in the marginal cost of providing the public good at $\bar{g}_1$. Hence, if $\bar{g}_1$ maximizes the leader's utility over the interval $[0, \bar{g}_1]$, then there exists $g_1 > \bar{g}_1$ which provides strictly higher utility.

In concluding this subsection, we show how Facts 5.1 and 5.2 can be used in order to deduce an upper bound for total provision of the public good in equilibrium. Denote by $M$ total income, i.e. $M = m_1 + m_2$. Furthermore, for $i = 1, 2$, let $\tilde{G}_i^{max}$ denote the maximal standalone contribution of individual $i$ that can be obtained by redistributing wealth, i.e.

$$\tilde{G}_i^{max} := \max_{0 \le m_i \le M} \tilde{G}_i(m_i).$$

**Corollary 5.2** *In any interior Cournot-Nash equilibrium with aggregate income $M$, total supply of the public good is never larger than $\min\{\tilde{G}_1^{max}, \tilde{G}_2^{max}\}$. Moreover, total supply of the public good in an arbitrary Cournot-Nash equilibrium is never larger than $\max\{\tilde{G}_1^{max}, \tilde{G}_2^{max}\}$.*

**Proof** Let $(g_1^*, g_2^*)$ be an interior equilibrium corresponding to the income distribution $(m_1, m_2)$. By Fact 5.1, $(0, g_1^* + g_2^*)$ and $(g_1^* + g_2^*, 0)$ are Cournot-Nash equilibria corresponding to the income distributions $(m_1 - g_1^*, m_2 + g_1^*)$ and $(m_1 + g_2^*, m_2 - g_2^*)$, respectively. Hence,

$$g_1^* + g_2^* = \tilde{G}_2(m_2 + g_1^*) \le \tilde{G}_2^{max},$$

and

$$g_1^* + g_2^* = \tilde{G}_1(m_1 + g_2^*) \le \tilde{G}_1^{max}.$$

This proves the first assertion. From this, the second assertion follows at once.

The proof of the corresponding result for Stackelberg equilibrium is similar, noting that the argument is valid only for transfers from the leader to the follower.

**Corollary 5.3** *Let aggregate income be given by $M$, and let individual 1 be the leader in a Stackelberg game. In any interior Stackelberg equilibrium total supply of the public good is never larger than $\tilde{G}_2^{max}$. Moreover, total supply of the public good in an arbitrary Stackelberg equilibrium is never larger than* $\max\{\tilde{G}_1^{max}, \tilde{G}_2^{max}\}$.

## 5.2 Local Redistributions

This subsection investigates the consequences of local, i.e. sufficiently small, wealth redistributions, both in the simultaneous and in the sequential move game. We confine our attention to the most interesting cases, e.g. when a marginal redistribution takes place at incomes where the equilibrium correspondences are not l.h.c. Other cases can be analyzed along similar lines.

First, consider the case in which $(0, \tilde{G}_2(m_2))$ is a Cournot-Nash equilibrium of the simultaneous move game corresponding to the income distribution $(m_1, m_2)$. In this case, by Corollary 4.1, $(0, \tilde{G}_2(m_2))$ is also the unique Stackelberg equilibrium. If $\tilde{G}_2(m_2)$ is strictly greater than the total crowding out contribution of agent 2, the effect of a sufficiently small redistribution is straightforward: A small transfer $\Delta m$ given to the follower will result in the equilibrium $(0, \tilde{G}_2(m_2 + \Delta m))$. Hence, such a transfer can either reduce or increase total provision of public good, depending on the sign of the transfer and on whether the follower's Engel curve is increasing or decreasing at $m_2$. On the other hand, if $\tilde{G}_2(m_2) = \bar{g}_2(m_1)$, even a marginal transfer can qualitatively change the Cournot-Nash equilibrium set. For instance, suppose that $\tilde{G}_2(\cdot)$ is strictly decreasing at $m_2$, and consider a small positive transfer $\Delta m$ to the follower. By continuity, $(0, \tilde{G}_2(m_2 + \Delta m))$ must still be the unique Stackelberg equilibrium. However, if $\bar{g}_2(m_1)$ is non-decreasing at $m_1$ (which will happen whenever $\tilde{G}_1(\cdot)$ is non-increasing at $m_1 + \bar{g}_2(m_1)$), then there is no Cournot-Nash equilibrium in which the leader does not contribute.

Secondly, consider an income distribution $(m_1, m_2)$ at which the leader is indifferent between an interior Stackelberg equilibrium $(g_1^S, g_2^S)$ and the boundary equilibrium $(\tilde{G}_1(m_1), 0)$ (cf. Example 4 above). Also, assume that according to Assumption I, $\tilde{G}_2(\cdot)$ is strictly increasing for low income and decreasing for high income, and moreover, that it is concave in its increasing part. Then, by Fact 3.2, there is always at most one interior equilibrium on the "flat" part of the follower's reaction function. Since both allocations, $(g_1^S, g_2^S)$ and $(\tilde{G}_1(m_1), 0)$, are assumed to be equilibria one must have $\tilde{G}_1(m_1) > g_1^S$, and

$$u^1(m_1 - g_1^S, g_1^S + g_2^S) = u^1(m_1 - \tilde{G}_1(m_1), \tilde{G}_1(m_1)).$$

This implies that $\tilde{G}_1(m_1) > g_1^S + g_2^S$. Consequently, if one starts at an income distribution $(m_1 - \epsilon, m_2 + \epsilon)$ for some small $\epsilon > 0$, any transfer $\Delta m > \epsilon$ from the follower to the leader discontinuously increases total quantity of the public good supplied in Stackelberg equilibrium. Indeed, by Fact 5.2 and Corollary 5.1, total supply at the equilibrium corresponding to $(m_1 - \epsilon, m_2 + \epsilon)$ is $g_1^S + g_2^S$, whereas at the equilibrium corresponding to $(m_1 - \epsilon + \Delta m, m_2 + \epsilon - \Delta m)$ it is $\tilde{G}_1(m_1 - \epsilon + \Delta m)$

## 5.3 Redistributions with Identical Tastes

Consider now redistributions among individuals with identical tastes. Given Assumption I, the following result describes the *optimal* distribution of income, i.e. the distribution of income that maximizes the aggregate equilibrium amount of the public good. Since we assume identical tastes now, we may omit subscripts referring to individuals in all symbols related to individual Engel curves and reaction functions. For instance, in the following theorem $\hat{m}$ denotes (as before) the unique maximizer of the individuals' Engel curve $\tilde{G}(\cdot)$ (if a maximizer exists at all, i.e. if there is an "inferior" region; otherwise, of course, $\hat{m} = \infty$).

**Theorem 5.1** *Suppose that individual Engel curves satisfy Assumption I. Let $M$ denote aggregate income, and let $m^* := \min\{M, \hat{m}\}$. For any distribution of income, total supply of public good in equilibrium (Cournot-Nash or Stackelberg with individual 1 as leader) is never larger than $\tilde{G}(m^*)$. For the distribution $(M - m^*, m^*)$, the unique Stackelberg equilibrium is $(0, \tilde{G}(m^*))$ in which total supply attains its maximal value.*

**Proof** Consider the first assertion on the upper bound of total supply of public good. Clearly, $\tilde{G}(m^*)$ is the maximal standalone contribution of each individual. Hence, for interior equilibria the assertion follows at once from Corollaries 5.2 and 5.3. For boundary equilibria the assertion is trivial. Now assume that income is distributed according to $(M - m^*, m^*)$. If $M < \hat{m}$, clearly $(0, \tilde{G}(m^*))$ is the unique Stackelberg equilibrium. If, on the other hand $M \geq \hat{m}$, the same conclusion can be inferred from Fact 3.1 together with the observation that $\tilde{G}(\hat{m}) \geq \tilde{G}(M - \hat{m})$.

In the case of a strictly normal public good, the conclusion of Theorem 5.1 is well in line with Bergstrom, Blume and Varian's [1986] result that with identical tastes an equalizing transfer can never increase total supply of the public good in Cournot-Nash equilibrium. Indeed, in that case the most unequal distribution which gives everything to one person yields the highest possible amount of public good (both, in the unique Cournot-Nash and the unique Stackelberg equilibrium). On the other hand, if the public good becomes inferior at some income level, and if there is enough income to redistribute, then by Theorem 5.1 an increase of total provision of the public good in Stackelberg equilibrium may well require an equalizing transfer. For instance, if $M = 2\hat{m}$ the optimal distribution is the equal distribution. To illustrate the possible effects of redistribution consider again Examples 2 and 2.1. Obviously, the transition from the income distribution $(0.5, 1.4)$ (cf. Example 2.1) to the distribution $(0.9, 1)$ (cf. Example 2) involves an equalizing transfer. As a consequence of this transfer, total quantity of the public good supplied in the unique Stackelberg equilibrium rises from 0.6 to 1. Moreover, in this example total quantity of the public good supplied in *any* Cournot-Nash equilibrium after the transfer is strictly greater than total supply before the transfer.

## 6 Conclusion

In this paper, it has been shown that some of the standard results in the theory of private provision of public goods hinge on the questionable assumption of strict normality of the public good. In particular, it has been established that, once this assumption is relaxed, equalizing transfers between individuals may increase total supply of the public good, both in Stackelberg and in Cournot-Nash equilibrium. Furthermore, in contrast to the strictly normal case, total provision of the public good may be higher in Stackelberg equilibrium than in any Cournot-Nash equilibrium. Along our way, we also observed that Varian's neutrality result for interior Stackelberg equilibria only applies in one direction, namely to transfers from the leader to the follower.

Our results raise the question as to the welfare effects of income redistributions on the one hand, and of moving from a simultaneous to a sequential move game on the other. As to the welfare analysis of income redistributions, it has already been observed in Buchholz, Konrad and Lommerud [1996] that the "discontinuity" of the set of Stackelberg equilibria corresponding to different income distributions opens the possibility of strict Pareto improvements through income transfers. For instance, starting from an interior Stackelberg equilibrium a small transfer from the follower to the leader can make both individuals strictly better off, provided that the transfer induces the boundary equilibrium in which only the leader supplies the public good. Note that this is in sharp contrast to the case of Cournot-Nash equilibrium where such a possibility of Pareto improvement cannot exist by Warr's neutrality result.

Concerning the welfare comparison between Cournot-Nash and Stackelberg equilibria, it is easily verified that – unless the equilibria coincide – the leader's utility must be strictly higher in Stackelberg than in any Cournot-Nash equilibrium, while the opposite is true for the follower. Hence, no actual Pareto improvement can be obtained by moving from a simultaneous to a sequential move game, even if the Stackelberg equilibrium entails a greater amount of the public good. However, in the latter case there is room for *potential* Pareto improvements in the following sense: Given the Stackelberg quantity of the public good, there may exist a distribution of private consumption such that the resulting allocation Pareto-dominates the Cournot-Nash equilibrium. Note, however, that such a possibility of potential Pareto improvement may not *always* exist. For instance, it can be checked that the Cournot-Nash equilibrium in Example 2.1 is *efficient* although it involves a smaller amount of the public good than the corresponding Stackelberg equilibrium. The reason is that any Pareto improvement would necessarily entail a reduction of individual 1's private consumption which is not possible since 1's private consumption is already zero in equilibrium. Clearly, if the private good is *strictly* normal this can never happen, and any Cournot-Nash equilibrium necessarily entails underprovision of the public good. However, even in that case a Stackelberg equilibrium with a greater amount of public good may not be potentially Pareto superior to Cournot-Nash equilibrium. For instance, it may happen that one individual's private consumption in equilibrium is just too small to make the necessary adjustments.

## Appendix

**Proof of Theorem 3.1** Let $W \subseteq \mathbf{R}_+^2$ denote the set of all income pairs $(m_1, m_2)$ such that the corresponding sequential contribution game has more than one Stackelberg equilibrium. First, we show that $W$ is a closed set. Let $(m_1^0, m_2^0) \notin W$, i.e. suppose that there is a unique equilibrium corresponding to $(m_1^0, m_2^0)$. By our assumptions on the shape of $\tilde{G}_2(\cdot)$, we know from Facts 3.1 and 3.2 that there are only two candidates for equilibrium: $(g_1, g_2) = (\tilde{G}_1(m_1^0), 0)$, or $(g_1', g_2^R(m_2^0, g_1'))$ where $g_1'$ is the unique maximizer of the leader's utility along the positive part of the follower's reaction function. If $\tilde{G}_1(m_1^0) < \bar{g}_1(m_2^0)$, then clearly $(\tilde{G}_1(m_1^0), 0)$ cannot be an equilibrium. Since both $\tilde{G}_1(\cdot)$ and $\bar{g}_1(\cdot)$ vary continuously with income[8], it is clear that the same conclusion applies in an open neighbourhood of $(m_1^0, m_2^0)$. Hence, assume that $\tilde{G}_1(m_1^0) \geq \bar{g}_1(m_2^0)$. We distinguish two cases.

**Case 1** The unique equilibrium at $(m_1^0, m_2^0)$ is $(\tilde{G}_1(m_1^0), 0)$. In this case, one must have

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > u^1\left(m_1^0 - g_1, g_1 + g_2^R(m_2^0, g_1)\right) \tag{A.1}$$

for all $0 \leq g_1 < \bar{g}_1(m_2^0)$. By Fact 5.3, (A.1) implies that

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > u^1\left(m_1^0 - \bar{g}_1(m_2^0), \bar{g}_1(m_2^0) + g_2^R(m_2^0, \bar{g}_1(m_2^0))\right).$$

Indeed, note that by definition $g_2^R(m_2^0, \bar{g}_1(m_2^0)) = 0$. Hence, if

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) = u^1\left(m_1^0 - \bar{g}_1(m_2^0), \bar{g}_1(m_2^0) + g_2^R(m_2^0, \bar{g}_1(m_2^0))\right),$$

[8] Note that for $\bar{g}_1(\cdot)$ this is true due to $\bar{g}_1(m_2^0) > 0$ and the concavity assumption on $\tilde{G}_2(\cdot)$.

it is easily verified replicating the arguments of the proof of Fact 5.2, that any small positive transfer $\delta$ from the leader to the follower would result in an interior equilibrium $(\bar{g}_1(m_2^0) - \delta, \delta)$. However, this is not possible by Fact 5.3. Consequently,

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > \max_{0 \le g_1 \le \bar{g}_1(m_2^0)} u^1\left(m_1^0 - g_1, g_1 + g_2^R(m_2^0, g_1)\right). \tag{A.2}$$

By continuity, (A.2) must hold in an open neighbourhood of $(m_1^0, m_2^0)$.

**Case 2** The unique equilibrium at $(m_1^0, m_2^0)$ is $(g_1', g_2^R(m_2^0, g_1'))$ with $g_2^R(m_2^0, g_1')$ strictly positive. This implies

$$u^1\left(m_1^0 - g_1', g_1' + g_2^R(m_2^0, g_1')\right) > u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right).$$

By continuity this inequality is preserved under small perturbations of income. Hence, $(\tilde{G}_1(m_1), 0)$ cannot be an equilibrium in an open neighbourhood of $(m_1^0, m_2^0)$. Summarizing, this shows that the complement of $W$ is an open set, and hence $W$ is closed in $\mathbf{R}_+^2$.

Next, we show that $W$ has measure zero in $\mathbf{R}^2$. By Facts 3.1 and 3.2, $W = W_1 \cup W_2$ where $W_1$ is the set of all income pairs $(m_1, m_2)$ such that there are two boundary equilibria, $(0, \tilde{G}_2(m_2))$ and $(\tilde{G}_1(m_1), 0)$, and $W_2$ is the set of all income pairs such that, in addition to the equilibrium $(\tilde{G}_1(m_1), 0)$, there is an interior equilibrium. We will show that both sets, $W_1$ and $W_2$, have measure zero. First, consider $W_1$. The function $\tilde{G}_2(\cdot)$ is strictly increasing in $[0, \hat{m}_2)$, and strictly decreasing whenever positive in $(\hat{m}_2, \infty)$. Hence, for any fixed $m_1^0$ with $\tilde{G}_1(m_1^0) > 0$, there can exist at most two values of $m_2$ such that

$$u^1\left(m_1^0, \tilde{G}_2(m_2)\right) = u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right). \tag{A.3}$$

On the other hand, if $\tilde{G}_1(m_1^0) = 0$, (A.3) can be satisfied only if $\tilde{G}_2(m_2) = 0$ in which case the two boundary equilibria coincide. Hence, for any fixed $m_1^0$, there exist at most two values of $m_2$ such that $(m_1^0, m_2) \in W_1$, which immediately implies that $W_1$ has measure zero in $\mathbf{R}^2$.

Finally, consider the set $W_2$. For any $M \ge 0$, consider the set $I_M$ of income pairs $(m_1, m_2)$ such that $m_1 + m_2 = M$. Clearly, the leader's utility in the boundary equilibrium $(\tilde{G}_1(m_1), 0)$ is strictly increasing in $m_1$ along the line segment $I_M$. On the other hand, by Corollary 5.1, her utility is constant at any interior equilibrium in the set $I_M$. Hence, for any fixed $M$, the line segment $I_M$ can intersect $W_2$ at most once. This implies that $W_2$ must have measure zero in $\mathbf{R}^2$, and the proof of Theorem 3.1 is complete.

**Remark** Observe that the condition of strict monotonicity of $\tilde{G}_2(\cdot)$ in its non-increasing part is necessary in Theorem 3.1. For instance, it can be shown that for fixed $m_2^0$ with $\tilde{G}_2(m_2^0) > 0$,

$$u^1\left(m_1, \tilde{G}_2(m_2^0)\right) = u^1\left(m_1 - \tilde{G}_1(m_1), \tilde{G}_1(m_1)\right)$$

may hold for all $m_1$ in some open interval $(\bar{m}, \bar{\bar{m}})$. Consequently, if $\tilde{G}_2(\cdot)$ would be locally constant at $m_2^0$, generic uniqueness would no longer prevail. This also shows that one cannot expect generic uniqueness of Stackelberg equilibrium in the leader's income alone. In this sense, Theorem 3.1 seems to be the "optimal" uniqueness result one can hope for.

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