E C O N O M I C S   B U L L E T I N

Sequential contributions to public goods: on the structure of the equilibrium set

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Abstract

This note clarifies two issues in the context of Varian's (1994) model of sequential contributions to a public good. First, it is shown that private provision of a public good is not necessarily neutral with respect to income transfers when agents move sequentially rather than simultaneously. Secondly, we discuss uniqueness of equilibrium in the sequential set−up.

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1 Introduction

This note reconsiders Varian's (1994) private provision model in which two agents sequentially choose voluntary contributions to a public good. Our main finding is that there cannot exist subgame perfect equilibria in which the follower contributes an arbitrarily small positive amount. This result has several interesting consequences. First, it implies that in the sequential move game neutrality does not hold in full generality as in the simultaneous move game. Indeed, while income transfers from the leader to the follower that do not exceed the leader's original contribution are always neutral, opposite transfers can discontinuously increase total supply of the public good (even if the amount transferred falls short of the follower's original contribution). Secondly, the result can be used to derive a *generic* uniqueness property for equilibrium in the sequential move game. In particular, it is shown that under concavity of the follower's "Engel curve" for the public good the above result implies closedness of the set of income distributions for which equilibrium is not unique. Hence, in this case uniqueness of equilibrium is preserved under small perturbations of income.

2 The Model

In Varian's (1994) model there are two individuals, indexed by $i = 1, 2$, and two goods. Each individual's utility is given by a strictly monotonic and strictly quasi-concave utility function $u^i(c_i, G)$, where c_i denotes i's consumption of a private good and G the consumption of a purely public good.[1] We assume that utility functions are twice continuously differentiable. Each individual has an initial endowment of m_i units of the private good. For simplicity, let the price of the private good be equal to 1. Hence, one may think of m_i as consumer i's income. The public good is produced from the private good at a cost of one unit private good per unit of public good. Throughout, it is assumed that both goods are strictly normal at any level of wealth. The timing of decisions is such that individual 1 (the "leader") chooses first, anticipating individual 2's (the "follower's") best response. We denote by $\tilde{G}_i(m_i)$ consumer i's *standalone contribution*. i.e. the solution to

$$\begin{aligned}
&\max_{c_i, G} u^i(c_i, G) \\
&\text{s.t.} \\
&c_i + G = m_i \quad \text{and} \quad G \geq 0.
\end{aligned}$$

Since prices are held fixed in all what follows we may refer to the function $\tilde{G}_i(\cdot)$ as consumer i's *Engel curve* for the public good. It is easily verified that

[1] We note that Varian (1994) requires utility functions to be strictly concave rather than quasi-concave. However, for the results presented here that difference plays no role.

the follower's reaction function is given by (cf. Varian (1994, p.176))

$$g_2^R(g_1) = \max\{\tilde{G}_2(m_2 + g_1) - g_1, 0\}. \tag{1}$$

Hence, given that the leader has chosen to contribute the amount g_1 to the public good, the follower's optimal response is to contribute either zero or his standalone contribution at the fictitious income level $m_2 + g_1$, minus the leader's contribution. The leader's problem is given by

$$\begin{aligned} &\max_{c_1, g_1} u^1(c_1, g_1 + g_2^R(g_1)) \\ &\text{s.t.} \\ &c_1 + g_1 = m_1 \quad \text{and} \quad g_1 \geq 0. \end{aligned} \tag{2}$$

A pair (g_1^S, g_2^S) such that $g_2^S = g_2^R(g_1^S)$ and such that g_1^S solves (2) will be referred to as a "Stackelberg" equilibrium of the corresponding contribution game.

3 Non-neutrality of Income Transfers from the Follower to the Leader

Varian's (1994) analysis suggests that the well-known neutrality result obtained by Warr (1983) and Bergstrom, Blume and Varian (1986) for the simultaneous move game also applies to the present sequential move set-up. However, as will be shown presently, neutrality may fail if agents move sequentially. Although income transfers from the leader to the follower that do not exceed the leader's original contribution are neutral as stated in Varian (1994, Fact 4, p.178), a symmetric statement does *not* hold for transfers from the follower to the leader.[2] To illustrate this, consider the following example.

Example Let $u^i(c_i, G) = c_i G$ for $i = 1, 2$, and assume that total income $M = m_1 + m_2$ equals 1. Computing the Stackelberg equilibria for any distribution of income, it can be verified that the equilibrium set looks as depicted in Figure 1: For $m_1 \leq 0.5$, the unique Stackelberg equilibrium is the boundary equilibrium $(0, \tilde{G}_2(m_2)) = (0, m_2/2)$. If m_1 increases beyond 0.5 we reach the region of interior equilibria. In any interior equilibrium, total provision of the public good is 0.25. Continuing with the redistribution from agent 2 to agent 1 we get to some critical value $\tilde{m}$ $(= \sqrt{0.5})$ such that for $m_1 = \tilde{m}$ there exist two equilibria, one interior ($g_1^S = \tilde{m} - 0.5$; $g_2^S = 0.25 - g_1^S$) and one boundary equilibrium in which the follower does not contribute ($g_1^S = \tilde{G}_1(\tilde{m})$; $g_2^S = 0$). If $m_1 > \tilde{m}$, only the boundary equilibrium survives.

[2] The apparent conflict of this conclusion with Fact 4 in Varian (1994) is resolved by the observation that in employing first-order conditions, Varian's proof implicitly assumes that *both* agents contribute before *and* after a redistribution of income.

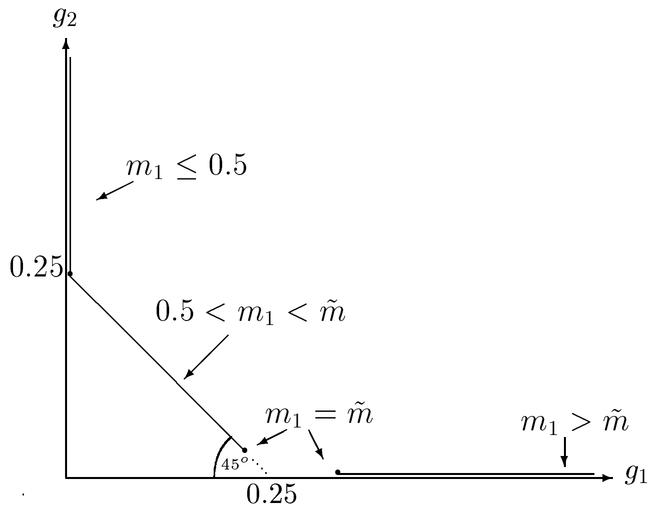

Figure 1: Set of Stackelberg equilibria generated by all income distributions with $m_1 + m_2 = 1$

Observe, that at $m_1 = \tilde{m}$ aggregate supply of the public good discontinuously jumps from 0.25 to $\tilde{G}_1(\tilde{m}) > 0.25$. The following result shows that the qualitative behaviour of the equilibrium correspondence exhibited in this example exemplifies a general phenomenon. Specifically, it demonstrates that there cannot exist interior Stackelberg equilibria in which the follower contributes an arbitrarily small amount.

Proposition 1 *Consider the set of all Stackelberg equilibria generated by redistributing a fixed aggregate income. There exists a constant $K > 0$ such that in any interior Stackelberg equilibrium (g_1^S, g_2^S) one has $g_2^S \geq K$.*

Proof Let (g_1^S, g_2^S) be an interior Stackelberg equilibrium corresponding to the income distribution (m_1, m_2). For any $\delta > 0$, consider the redistribution $(m_1^\delta, m_2^\delta) = (m_1 + (g_2^S - \delta), m_2 - (g_2^S - \delta))$. If there exists an interior equilibrium corresponding to (m_1^δ, m_2^δ), it can be verified that it must be $(g_1^\delta, g_2^\delta) = (g_1^S + g_2^S - \delta, \delta)$.[3] If (g_1^δ, g_2^δ) is actually an equilibrium, the leader's first order condition gives

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^\delta - g_1^\delta, g_1^\delta + g_2^R(m_2^\delta, g_1^\delta)\right) = \frac{1}{1 + \frac{\partial g_2^R}{\partial g_1}(m_2^\delta, g_1^\delta)}.$$

Hence, for $\delta \to 0$ one obtains,

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^0 - g_1^0, g_1^0\right) = \frac{1}{1 + \lim_{\delta \to 0} \frac{\partial g_2^R}{\partial g_1}(m_2^\delta, g_1^\delta)}, \tag{3}$$

[3]Indeed, this follows from the fact that transfers from the leader to the follower (that do not exceed the leader's original contribution) are neutral. This, in turn, can be proved along the lines of Fact 4 in Varian (1994).

where $m_1^0 = m_1 + g_2^S$ and $g_1^0 = g_1^S + g_2^S$. Now observe that by strict normality of both private and public consumption one has $0 < d\tilde{G}_2/dm_2 < 1$ everywhere. This gives

$$-1 < \lim_{\delta \to 0} \frac{\partial g_2^R}{\partial g_1}(m_2^\delta, g_1^\delta) < 0,$$

which in turn implies by (3),

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^0 - g_1^0, g_1^0\right) > 1. \tag{4}$$

However, given that the follower contributes zero (as he does when $\delta = 0$), the leader's utility maximizing contribution $\tilde{G}_1(m_1^0)$ satisfies the following first order condition,

$$\frac{\partial u^1/\partial G}{\partial u^1/\partial c}\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) = 1. \tag{5}$$

Hence, comparing (4) and (5), strict quasi-concavity of u^1 implies $\tilde{G}_1(m_1^0) > g_1^0$ and

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > u^1(m_1^0 - g_1^0, g_1^0). \tag{6}$$

By continuity of $\tilde{G}_1(\cdot)$, and since utility is constant along (g_1^δ, g_2^δ), (6) must also hold for sufficiently small $\delta > 0$. Consequently, there exists $K > 0$ such that for $\delta < K$, (g_1^δ, g_2^δ) is not an equilibrium.

It is clear that by Proposition 1 income transfers from the follower to the leader are not neutral even if the follower looses less income than his original contribution. Indeed, by Proposition 1 there exists an income distribution with an interior equilibrium (g_1^*, g_2^*) such that for all interior equilibria (g_1^S, g_2^S) corresponding to the different distributions of the same aggregate income one has $g_2^S \geq g_2^*$. Hence, starting from (g_1^*, g_2^*) *any* positive transfer from the follower to the leader induces the boundary equilibrium $(\tilde{G}_1(m_1), 0)$ with $\tilde{G}_1(m_1) > g_1^* + g_2^*$. This discontinuity also explains the possibility of *Pareto-improving* transfers in the sequential move set-up found by Buchholz, Konrad and Lommerud (1997). Starting at (g_1^*, g_2^*) a small transfer from the follower to the leader makes *both* individuals strictly better off. For the leader this is due to her increased income. But also the follower benefits from the transfer since total supply of the public good rises discontinuously while the follower's contribution jumps to zero. Thus, the follower is overcompensated for the loss of income.

The driving force behind the discontinuity of the equilibrium correspondence revealed by Proposition 1 is a discontinuity in the cost of providing an additional unit of the public good from the viewpoint of the leader: As in Varian (1994), let g_1^c denote the *complete crowding out* contribution, i.e. the smallest contribution of agent 1 such that agent 2 chooses to contribute zero. Up to the complete crowding out level g_1^c an additional dollar of contribution by the leader induces a change in total provision of the public good of less than one unit, since the follower will react in reducing his contribution. Beyond g_1^c each change in contribution induces an equal change in total provision. Strict normality everywhere implies that the follower's reaction curve cannot approach 0 smoothly. Hence, from the leader's point of view, there is a downward jump in the marginal cost of providing the public good at g_1^c. Therefore, if g_1^c maximizes the leader's utility over the interval $[0, g_1^c]$, there exists $g_1 > g_1^c$ which provides strictly higher utility.

4 Generic Uniqueness of Stackelberg Equilibrium

Proposition 1 can be used to derive that under concavity of the follower's Engel curve Stackelberg equilibrium is generically unique in the space of all income distributions.

There are two possible sources of non-uniqueness of Stackelberg equilibrium. The first is that there may be several maximizers of the leader's utility along the *positive* part of the follower's reaction curve. It is easily verified that this can, however, not happen if $\tilde{G}_2(\cdot)$ is a concave function, since then g_2^R is a concave function of g_1 in its positive part. The second source of possible non-uniqueness stems from the fact that a maximizer along the positive part of g_2^R may give the leader a utility just as high as the boundary solution $(\tilde{G}_1(m_1), 0)$ (in the above example, this happens at $m_1 = \tilde{m}$). The following result entails that the latter phenomenon is non-generic in the space of income distributions.

Proposition 2 *Suppose that the follower's Engel curve $\tilde{G}_2(\cdot)$ is a concave function. Then, the set of income pairs for which Stackelberg equilibrium is non-unique is a closed set. Furthermore, for any given aggregate income there is at most one distribution such that Stackelberg equilibrium is not unique.*[4] *For any such distribution there are exactly two equilibria.*

Proof Let $W \subseteq \mathbf{R}_+^2$ denote the set of all income pairs (m_1, m_2) such that the corresponding sequential contribution game has more than one Stackelberg equilibrium. First, we show that W is a closed set. Let $(m_1^0, m_2^0) \notin W$, i.e. suppose that there is a unique equilibrium corresponding to (m_1^0, m_2^0). By concavity of $\tilde{G}_2(\cdot)$, we know that there are only two candidates for equilibrium: $(g_1, g_2) = (\tilde{G}_1(m_1^0), 0)$, or $(g_1', g_2^R(m_2^0, g_1'))$ where g_1' is the unique maximizer of the leader's utility along the positive part of the follower's reaction function. Let $g_1^c(m_2^0)$ denote the leader's complete crowding out contribution when the

[4] In particular, non-uniqueness can only occur on a set of measure zero in the space of all income pairs.

follower's income is m_2^0. If $\tilde{G}_1(m_1^0) < g_1^c(m_2^0)$, then clearly $(\tilde{G}_1(m_1^0), 0)$ cannot be an equilibrium. Since both $\tilde{G}_1(\cdot)$ and $g_1^c(\cdot)$ vary continuously with income[5], it is clear that the same conclusion applies in an open neighbourhood of (m_1^0, m_2^0). Hence, assume that $\tilde{G}_1(m_1^0) \geq g_1^c(m_2^0)$. We distinguish two cases.

Case 1 The unique equilibrium at (m_1^0, m_2^0) is $(\tilde{G}_1(m_1^0), 0)$. Then,

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > u^1\left(m_1^0 - g_1, g_1 + g_2^R(m_2^0, g_1)\right) \tag{7}$$

for all $0 \leq g_1 < g_1^c(m_2^0)$. By Proposition 1 above, (7) implies that

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > u^1\left(m_1^0 - g_1^c(m_2^0), g_1^c(m_2^0) + g_2^R(m_2^0, g_1^c(m_2^0))\right).$$

Indeed, note that by definition of $g_1^c(m_2^0)$, $g_2^R(m_2^0, g_1^c(m_2^0)) = 0$. Hence, if

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) = u^1\left(m_1^0 - g_1^c(m_2^0), g_1^c(m_2^0) + g_2^R(m_2^0, g_1^c(m_2^0))\right),$$

any small positive transfer δ from the leader to the follower would result in an interior equilibrium $(g_1^c(m_2^0) - \delta, \delta)$. However, this is not possible by Proposition 1. Consequently,

$$u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right) > \max_{0 \leq g_1 \leq g_1^c(m_2^0)} u^1\left(m_1^0 - g_1, g_1 + g_2^R(m_2^0, g_1)\right). \tag{8}$$

By continuity, (8) must hold in an open neighbourhood of (m_1^0, m_2^0).

Case 2 The unique equilibrium at (m_1^0, m_2^0) is $(g_1', g_2^R(m_2^0, g_1'))$ with $g_2^R(m_2^0, g_1')$ strictly positive. This implies

$$u^1\left(m_1^0 - g_1', g_1' + g_2^R(m_2^0, g_1')\right) > u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right).$$

By continuity this inequality is preserved under small perturbations of income. Hence, $(\tilde{G}_1(m_1), 0)$ cannot be an equilibrium in an open neighbourhood of (m_1^0, m_2^0). Summarizing, this shows that the complement of W is an open set, and hence W is closed in $\mathbf{R}_+^2$.

In order to show the second part of the statement of Proposition 2, observe that $W = W_1 \cup W_2$ where W_1 is the set of all income pairs (m_1, m_2) such that there are two boundary equilibria, $(0, \tilde{G}_2(m_2))$ and $(\tilde{G}_1(m_1), 0)$, and W_2 is the set of all income pairs such that, in addition to the equilibrium $(\tilde{G}_1(m_1), 0)$, there is an interior equilibrium. First, consider W_1 and let $M \geq 0$ be any fixed

[5]Note that for $g_1^c(\cdot)$ this is true due to $g_1^c(m_2^0) > 0$ and the concavity assumption on $\tilde{G}_2(\cdot)$.

level of aggregate income. Suppose that $m_1^0 \leq M$ is such that $(m_1^0, M - m_1^0) \in W_1$, i.e.

$$u^1\left(m_1^0, \tilde{G}_2(M - m_1^0)\right) = u^1\left(m_1^0 - \tilde{G}_1(m_1^0), \tilde{G}_1(m_1^0)\right).$$

Now consider a positive transfer to the leader, so that her income becomes $m_1' > m_1^0$. Strict convexity of the leader's preferences together with strict normality of the public good implies that

$$u^1\left(m_1', \tilde{G}_2(M - m_1^0)\right) < u^1\left(m_1' - \tilde{G}_1(m_1'), \tilde{G}_1(m_1')\right),$$

hence by strict monotonicity of the leader's preferences and the fact that $\tilde{G}_2(\cdot)$ is increasing,

$$u^1\left(m_1', \tilde{G}_2(M - m_1')\right) < u^1\left(m_1' - \tilde{G}_1(m_1'), \tilde{G}_1(m_1')\right).$$

This shows that, for any fixed M, there exists at most one value of m_1 such that $(m_1, M - m_1) \in W_1$.

Finally, consider the set W_2 and let I_M be the set of income pairs (m_1, m_2) such that $m_1 + m_2 = M$. Clearly, the leader's utility in the boundary equilibrium $(\tilde{G}_1(m_1), 0)$ is strictly increasing in m_1 along the line segment I_M. On the other hand, her utility is constant at any interior equilibrium in the set I_M. Hence, for any fixed M, the line segment I_M can intersect W_2 at most once. This completes the proof of Proposition 2.

We conclude with the observation that, if the follower's Engel curve is not concave, there always exist preferences for the leader such that there are several equilibria in a whole range of income distributions.[6] In this sense, concavity of the follower's Engel curve is thus also a *necessary* condition for (generic) uniqueness of Stackelberg equilibrium.

5 References

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[6] The reason is simple. If the follower's reaction curve is not concave somewhere, one can find indifference curves for the leader such that there are several (even a continuum of) points of tangency with the "consumption possibility frontier."