Collusion in Beauty Contests

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May 2003

Abstract
During the recent sales of UMTS licenses in Europe some countries used auctions while others resorted to so-called Beauty Contests. There seems to be a wide consensus among economists that in these and other contexts like privatization an auction is the better selling mechanism. However, why exactly an auction should be preferred is unclear.

Here we present an argument why beauty contests or negotiations might be dominated by auctions, which is closely linked to the multi dimensionality on issues involved in this process. The important assumption we make is that bidding firms do not know the preferences of the government. As a consequence, during a beauty contest participants are uncertain about the final decision of the government. This uncertainty enables firms to collude.

JEL classification: C78; D44; D82; L51

Keywords: Auctions; Collusion; Beauty Contests;

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I would like to thank seminar participants at the universities of Bonn, Cologne, Magdeburg and Saarbruecken and at the CESifo Area Conference on Industrial Organization, Munich for helpful comments.
1 Introduction

During the recent sales of UMTS licenses in Europe some countries (among others Germany and Great Britain) used auctions while others (e.g. Italy and France) resorted to so-called Beauty Contests. In these contests firms make proposals for the implementation of the new mobile phone generation. On basis of these proposals the license owners are determined by the government.

There seems to be a wide consensus among economists that in these and other contexts like privatization an auction is the preferred selling mechanism. Maskin (1992) argues that both with respect to efficiency and revenue "auctions tend to fulfill these objectives better than do most common alternatives to auctions", among which are negotiations between the seller and individual buyers. McMillan (1995) claims that "of the alternative spectrum allocation methods ... auctioning works best." However, it is never clearly stated (or even shown rigorously in a model) why negotiations are undesired. McMillan makes the point that beauty contests usually lack transparency with the consequence that the winner is often the firm that hired the most effective lobbyists. Although this argument is intuitively appealing, it is not convincing. If regulators can undertake auctions (and thus make the allocation transparent), they should also be able to proceed with transparent and efficient negotiations. Or to put it differently: If lobbyists are successful to influence the beauty contest in their favor, why can’t they implement auction rules which favor them? Academics quite often resort to Bulow and Klemperer (1996) to argue that auctions dominate negotiations. Bulow and Klemperer show that an auction with one more bidder raises higher revenue than the optimal negotiation mechanism without this additional bidder. But, as the number of bidders is not the main issue here, this still implies that a negotiation can do better than an auction with the same number of contestants. In addition, there are many good arguments why beauty contests might fare better than auctions. One, as already pointed out by McMillan (1995) is the additional flexibility. While auctions finally come down to price competition,
negotiations allow to take many more aspects into consideration, like the degree of coverage, speed of introduction of the new generation of mobile phones, and so on. Thus it is not clear why auctions should be preferred.\footnote{Manelli and Vincent (1995) show in the context of procurement auctions that if quality is uncontractible, then simple take-it-or-leave-it offers might fare better than auctions, as auctions tend to favor the low cost and correspondingly low quality firms.}

Here we present an argument why beauty contests or negotiations might be dominated by auctions, which is closely linked to the multi dimensionality of issues involved in this process. A stylized description of beauty contests we employ here has the following form: Firms make offers which specify a multi dimensional vector of service levels, quality, speed of implementation and in some cases also price. These offers are discussed with the regulator compared to other firms, and modified by the participating firms.\footnote{McMillan (1995) writes that administrative hearings to allocate cellular telephone licenses sometimes lasted over a year.} On basis of the final offers the regulator decides on the winning firms. The important assumption we make is that even if firms know all the final offers, they do not know which firm is the winner, as they do not know the preferences of the regulator exactly. As usually these are not just economic but also political decisions, many factors influence the final decision which cannot be anticipated with certainty. In Canada, for example, subjective criteria like \textit{"the economic feasibility, the effects on the telecommunications industry and industry concentration"} are used (taken from McMillan, 1995). As we will show, this uncertainty about the decision of the regulator allows the participating firms to collude. The reasoning is as follows: If all firms offer low service levels they all have a chance to win with from their point of view very good conditions. If one firm were to improve its offer, it has to fear that others will follow suit. This might lead to a price and service war which results in much higher service levels and possibly higher prices, which makes it unattractive to improve the offer in the first place. Note that the stabilizing element in this collusion is that firms do not know how the regulator will finally decide. Thus with their initial offers all firms have a \textit{chance} of winning. Auctions differ to such a
mechanism in the following way: As auction rules clearly specify who wins a license, at any moment during the auction firms do not have any uncertainty whether they would receive a license or not if the auction were to stop at this point. Thus firms which at the present offers do not obtain a license have to bid more, which in turn puts pressure on other firms. Collusion is no longer possible to sustain.

The insight we obtain here can be applied to a variety of contexts of a similar form. In Section 4 of this paper we argue that blueprint contests are much more vulnerable against collusion than prototype contests. Furthermore we show that internet-market places might face more collusion than classical markets with a similar argument. In the context of procurement auctions, we show that the widespread use of e-biddings, where the buyer decides after the event took place who will get the order, is vulnerable to the same form of collusion as discussed above. Finally, it is argued that any form of bargaining with many parties potentially suffers from the same problem if bargaining is made openly.

Our work contributes to the literature of collusion in auctions (see e.g. Robinson, 1985, Graham and Marshall, 1987). Usually this literature assumes that before the auction takes place, a designated winner is selected.\(^3\) In addition, there must be some means to divide the gains of collusion between the participating bidders. This is different in the form of collusion we describe here. First, all participating firms have a chance of winning a license, thus there is (are) no predetermined winner(s). Second, during the beauty contest all firms have a positive expected profit, even if after the decision by the government only a few firms receive a license. This makes it unnecessary to divide the gains of collusion after the contest.

The paper is structured as follows: In the next section we present the model and solve the benchmark case of a first price sealed bid auction. In Section 3 we present our main argument for collusion. In Section 4 we apply the model to other economic circumstances. Finally, in Section 5 we conclude.

\(^3\)Graham and Marshall describe in this context a preauction knockput (PAKT) procedure which determines the final winner.
2 The Model

We simplify the structure of the problem by assuming that the government intends to sell only one license to one firm out of a group of two bidders. The argument could be generalized to \( k \) licenses with \( n > k \) bidders, but the formal derivation is much simpler for this special case. We will comment on this in the conclusions.

The two bidders compete for the license by offering a price \( p \) and a vector of service levels \( s_i \), \( i = 1, 2 \) to the government. \( s_i \) is a multi-dimensional vector which consists of e.g. speed of implementation, guarantees to provide telecommunication services to particular regions, and so on: \( s_i = (s^1_i, s^2_i, ...) \). If firm \( i \) with offer \((p, s_i)\) obtains a license, her profit is given by:

\[
v - p - c(s_i)
\]

\((1)\)

\( v \) is the expected gross profit from participating in the market, i.e. the discounted future cash flows. \( c(s_i) \) are the costs of delivering the services \( s_i \). Note that to concentrate the analysis on the possibilities to collude, we assume that the firms are equal with respect to their cost and profit structure. In section \( ? \) we provide an extension of the model where the valuation \( v \) differs among firms.

The government\(^5\) obtains utility from the services offered and the price paid. The important assumption we make is that if the two bidders make their final offers, then the ranking of these packages is private information to the government. This might be for two reasons: One could be that it is unclear to the bidders exactly by how much service differences are valued by the government. Is it better to have a fast implementation of a new standard, or is it better to have a wider coverage? The second reason is that firms differ in the view of the government. So even if both

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\(^4\)Sometimes the price is determined before the contest starts. This was the procedure which was used e.g. in France for the UMTS license allocation in 2001, where the government announced a price per license in advance. This case can be included in our analysis by fixing the price.

\(^5\)We speak in the following of the government, although in many cases a regulation authority is in charge of the process.
firms offer the same package the government would prefer one and it is unclear which one. Contacts between the government and some firms, pressure from local constituencies, etc. might make the government favor one firm over the other.\footnote{In this model we assume that the firms are ex-ante equal. In reality this is not the case. It is probably obvious to all participants that a government prefers a local firm to a foreign firm. However by how much is unclear. In Germany, for example a local consortium bid 1 billion marks less than a foreign consortium for houses owned by the government. In the end the local consortium obtained the offer (the case was later cancelled). The foreign consortium was probably aware that they had to bid more than their competitors, but not by how much. As we will discuss in Section 5, under these circumstances our model still applies.} Here we concentrate on the latter effect, mainly for simplicity. The same results will hold if it is unclear to all participants just by how much the government values different degrees of service levels.

To model this asymmetric information we employ a simplified version of Hotelling’s model of horizontal differentiation. Let firm 1 be positioned on position zero, and firm 2 at position one. Denote $\tau_1 = 0$ and $\tau_2 = 1$. Then the utility of the government from the contribution of firm $i$, if firm $i$ obtains a license, is defined as follows:

$$U = u(s_i) + p - \alpha|\theta - \tau_i|$$

$u(s_i)$ is the direct gain in utility through service levels and prices $s_i$. $\theta - \tau_i$ measures the distance of the government from the particular provider. This includes effects of sympathy, technical evaluation by the government of this firm, etc. The size of $\theta$ is private knowledge of the government. We assume that $\theta$ is taken from the uniform distribution $[0, 1]$. $\alpha$ measures the degree of heterogeneity between the firms. If $\alpha$ is small, the main things which count are the service levels and prices. If $\alpha$ becomes larger, then differences between firms from the perspective of the government matter more.
2.1 Benchmark case: "First price" sealed bid auction

Suppose firms can make only one bid \( b_i = (p_i, s_i) \) each simultaneously. The government then determines who will win the license. Call \( \text{prob}(b, b_{-i}) \) the winning probability of firm \( i \) if it offers price and service level \( b = (p, s) \) and the other firm bids \( b_{-i} = (p_{-i}, s_{-i}) \). Formally, this winning probability is given by

\[
\text{prob}(b, b_{-i}) = \text{prob}(u(s) + p - \alpha|\theta - \tau_i| \geq u(s_{-i}) + p_{-i} - \alpha|\theta - \tau_{-i}|) = \frac{u(s) - u(s_{-i}) + p - p_{-i} + \alpha}{2\alpha}
\]

The last equation only holds as long as the expressions lies between zero and one. Otherwise \( \text{prob}(b, b_{-i}) \in \{0, 1\} \). The best response function of firm \( i \) can now be calculated by

\[
\max_b \pi(b, b_{-i}) = (v - p - c(s))\text{prob}(b, b_{-i})
\]

We write the best response function as \( b^R(b_{-i}) \).

**Lemma** The best response function \( b^R(b_{-i}) \) exists and is unique in those cases where bidder \( i \) has a strictly positive chance of winning. The vector of services is independent of the bid of the other bidder and is given by \( s^R(b_{-i}) = s^* \), where \( s^* = (s_1^*, s_2^*, ...) \) satisfies \( c_i(s^*) = u_i(s^*) \).

That the optimal bid in services is uniquely defined, can be seen by noting that if e.g. \( u_i(s) > c_i(s) \) then an increase in the component \( s_i \) by \( \delta \) accompanied by a reduction in price by \( \delta u_i(s) \) keeps the winning probability the same and increases the profit in case of winning. For a given \( s^* \) the analysis boils down to choosing prices only, and the existence and uniqueness result is then analogue to the textbook case of Hotelling’s beach (see e.g. Tirole).

Using that in the Nash equilibrium the service levels and the price is the same for both firms we get

\[
p^N = v - c(s^*) - \alpha
\]

The expected rent of the firms is \( \frac{1}{2} \alpha \). Although only one firm obtains the license finally, both firms have a chance of receiving the license (50% in this case). So both
firms have a positive expected rent. If $\alpha$, the measure of differentiation between the firms, tends to zero, then the rent of the firms goes to zero as well. If both firms are identical to the government, then standard Bertrand-like competition leads to the full rent dissipation. This result resembles the well known result from the industrial organization literature. In the model of Hotelling’s beach firms are horizontally differentiated and compete in prices. Then the final price is equal to marginal costs plus a term which increases in the degree of differentiation.

3 Beauty Contest

The beauty contest can be described as follows. Firms make offers for this license, consisting of different degrees of service levels and prices. Then the government starts to negotiate. She plays back the offers of competing firms and tries to improve the conditions of the offering firm. At some stage this process stops and the government decides on the basis of the final offers who should obtain the license.

We model this process in the following way: The bidding contest consists of $t = 0, 1, 2, \ldots$ rounds. In each round $t$ each firm can make a new offer if it wishes. If firm $i$ makes a new bid $b_i = (p, s)$, then $b_i^t = b_i$ is the new running bid, otherwise $b_i^t = b_i^{t-1}$. The new offer can only be an improvement over the previous offer, i.e. we require that if $b_i^t \neq b_i^{t-1}$, then $u(s_i^t) + p_i^t \geq u(s_i^{t-1}) + p_i^{t-1}$. After each round, the offers by all firms are made public, and each firm can make a new offer. The beauty contest stops if in any single round no new offer is made. The government decides on the basis of the final offers who should obtain the license. Note that this process is very similar to an English auction. Firms bid against each other, and they can see the offers of the competitors. The main difference to an English auction is that if all final bids are made, the government decides who will win the license. Thus firms do not necessarily know whether they have won the license immediately after the bidding phase ends.\footnote{It is surely a strong assumption to allow firms to see the offers of their competitors after each round.}
Denote by $b_{\text{min}} = (p_{\text{min}}, s_{\text{min}})$ the minimum level of prices and services a firm can offer. These can be due to reservation prices or minimum requirements for the speed of implementation, etc. Then let $\pi^m$ be the expected profit each firm makes if both firms bid $b = b_{\text{min}}$. $m$ stands for the monopoly profit. With this definition we can state and prove our main proposition:

**Proposition** If $\alpha > 0$, then there exists an equilibrium in the beauty contest where both firms bid their minimum bid.

The proof is relegated to the Appendix. Here we provide the intuition for the result. In equilibrium both firms bid $b = b_{\text{min}}$ and stop bidding at this point. Thus each firm obtains the license with probability $\frac{1}{2}$ for very good conditions. If a firm observes that the competitor offers a different, better service level or larger price, then the firm responds by increasing its offer in the next round. In particular, a firm will set the service level and its price such that the chance of winning for the competitor will be $\frac{1}{2}$ again. In other words, a deviation from the equilibrium path by any firm triggers a price and service war. The behaviour constitutes a subgame perfect equilibrium by the following argument: In equilibrium each firm bids is minimum bid and obtains the monopoly profit. If it deviates, it will be penalized by its competitor and obtain less (possibly more service costs and a higher price with the same probability of winning). For the penalizing firm it is a best response to penalize the deviator because penalizing is own profit increasing.

The overall proof is slightly more elaborate as the penalizing strategy depends on whether prices and service levels are larger than in the Nash equilibrium or not.

The idea behind this collusive behavior is the following: Both firms make low offers. This gives them a chance to win a license for very good conditions. If any firm deviates by making a better offer to the government, it might in the end increase round. We will come back to this point in the next section, when we discuss applications of the model.
its chance of winning but at much worse conditions. Thus it does not pay out to
start the competition in the first place. It is the uncertainty about the decision
of the government which makes collusion feasible. Therefore \( \alpha > 0 \) is a necessary
requirement.\(^8\)

Note that this form of collusive behavior differs strongly from those usually
discussed in the literature. First, the participating firms do not need to determine
in advance who will be the winner in the contest. Both firms have a chance of
winning. Second, there is no need to divide the spoils of collusion after the contest.
As both firms have positive expected utility before the decision of the government is
taken, both firms are winners (in expectation). Third, I presume that this behavior
is perfectly legal, as firms just do not bid aggressively.

4 Applications

The idea behind this behavior can be applied to a variety of circumstances.

Beauty contests
All private or public tenders where prices and conditions are negotiated, offers by
competitors are displayed to the participating firms, the firms can react to these
offers, and most importantly, the final decision is not based on prices only, are
vulnerable to the same form of collusion as described above. A concrete example is
given next.

Prototype versus blueprint competition
In the US, defense contract awards can take two different forms (Alexander, 1997):
One is prototype competition where the participating firms create a prototype each.
Based on the quality of the prototype and the price demanded by the firms, the
Department of Defense decides which firm obtains the contract. Alexander (1997)
provides evidence that in many cases it is the prototype performance which is the

\(^8\)For \( \alpha = 0 \) collusion would not be possible if the government in case of indifference between
the offers always grants one predetermined firm the license.
determining factor in the final decision. As prototypes are hard to modify substantially, Alexander compares this situation to an all pay first-price-sealed-bid auction: Each firm can only make one prototype for which it incurs costs. The best prototype, i.e. the firm which invested the largest sum, wins the contest.

The alternative to prototype competition is blueprint competition. Here the competing firms make offers with a blueprint of the project. In general then a negotiation begins, blueprints are modified, and finally the decision is taken. Alexander describes that contracting officers tried to play bidders off against each other, which lead to the use of the acronyms ”BAFO” (best and final offers) and ”BARFO” (best and revised final offers”). As blueprints are more easily modified, and bidders can react to offers by their competitors, this procedure is quite similar to an English auction. But note however that in this case the assumptions of our model apply: Even if the final offers are made, the bidders probably still do not know who will win the contract, as the decision by the contracting officer might well take other issues like quality, etc. into consideration.

With the model given above, we might come to the conclusion that blueprint contests are much more vulnerable to collusion, even if no long term relationship exists between these firms. This is indeed what Alexander finds.9

**Dealer versus internet sales**

Klemperer (2000) compares the selling of cars by traders and via the internet. He

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9Alexander’s work (1997) is more in line with the classical literature on collusion in auction. She argues that competitors first agree who should be the winner of the contract, which might be a "favored player". Then she requires a mechanism how the winner of the contract divides the spoils with his competitors. This is done by the use of subcontracts, which are indeed much more often observed in blueprint than in prototype competitions. However, using subcontracts to divide the gains of collusion only works if the winner is indifferent between the other bidders as subcontractors and some other firms. Only in this case he can credibly argue that in case of defection he will not award the subcontract, while if the competitors abide, they will obtain the subcontract. As this subcontract has to be profitable for the competitors, this indifference assumption is hard to defend.
makes the argument that while traditional sellers compete like in a first-price-sealed-bid auction, internet trade should be considered similar to an English auction. Salesmen of cars usually make one price offer to the prospective buyer. This buyer has the chance to compare price offers between the sellers and then decide on this basis which car to buy. Internet traders on the other hand can observe the prices of their competitors and can react to them. Note that this scenario is similar to the one described in our model. Internet traders can collude on high prices, as they do not know which consumer will marginally favor which trader. The difference to our model is that the profit of collusion is not in expected terms, but deterministic. 

In effect, our result is related to the literature on strategic demand reduction (see e.g. Engelbrecht-Wiggans and Kahn, 1998; Ausubel and Cramton, 1998). Strategic demand reduction refers to the case where bidders reduce their demand during an auction to end the auction earlier. Thus bidders obtain fewer goods/licenses, etc. for very good conditions. One example is the Austrian auction of the UMTS licenses. Six bidders could bid on two or three frequency blocks each, out of a total of twelve blocks. This in principle four, five or six licenses were possible. As it turned out, all bidders decided very early in the auction that they would bid on two blocks only. Thus the auction stopped after one day, each bidder obtained a license with two frequency blocks for very low prices. In the example of car sales, the internet traders might accept fewer customers who however pay higher prices. In our model, the firms are content with a lower probability of winning, although for very good conditions.

E-procurement

In the context of e-procurement, i.e. procurement via the internet, one particular form of bidding contest seems to be employed quite often. Suppliers are asked to

\footnote{Klemperer also argues that internet sales are more open to collusion. He makes a comparison to the US and German auctions of radiospectrum, where bidders signalled who should win which objects. As the 'objects' in the case of internet traders are sales to the different customers, it is not clear whether such signal possibilities exist.}
participate in an English auction. They make their bids for a project, see the bids of the competitors, and can react during a prespecified time period. After the auction closes, the buyer decides which offer he accepts. For this decision he takes into account the final bids as well as other aspects like "total cost of ownership". As these other aspects are not exactly known to the bidders in advance, the situation is very similar to the one outlined above. Collusion among bidders is possible as all might have a chance to win the contest for very good conditions (for them).

Rent seeking contests The prime example of a contest where bidders are uncertain about their chances of winning are rent seeking contests like those introduced by ... While most models assume a static contest, the analysis in a dynamic framework would be similar to the model presented here. Firms make small efforts to obtain the rent. If they see that their competitors increase their efforts, they will follow suit and thus start a rent seeking war. In contrast to the present model, money invested into rent seeking is sunk at the point where the next round begins, while here bids are only valid (and have to be paid) in case of winning the contest. The argument in the rent seeking contest has been modelled by Leininger and

Bargaining
Recently, there has been an increased interest in the issue of multi-party bargaining (see e.g. Inderst and Wey, 2000). The model presented here shows that some of these results should be considered with caution. Consider the following two bargaining scenarios. A buyer would like to buy a good from one of two sellers. The bargaining procedure takes the following form: In each round, both sellers make an offer to the buyer. If in some round no new offer arises, the buyer decides on the basis of the final offers from whom he will buy the good. Now in scenario one, both sellers can see the offer of their competitor, in scenario two they cannot. Our analysis shows that scenario one is vulnerable to collusion, if the sellers do not know exactly how the buyer will finally decide. On the other hand scenario two will lead to the competitive outcome. Bargaining without information about the offer of the competitor is strategically equivalent to a first-price-sealed-bid auction. Our
discussion in subsection 2.1 shows that in this case collusion is not feasible.

5 Conclusion

We have argued that beauty contests are, compared to auctions, much more vulnerable to collusion. The basic argument is that firms might enter a beauty contest with a relatively bad offer. If all firms behave similarly, then each firm has a chance of winning a license. This uncertainty arises because firms do not know exactly what kind of considerations the government will take into account. It is unclear how strongly the government values quality, the fact that one firm is based in the home country, etc.

To collude on these low offers, firms should not have an incentive to improve their offer. This is given if other firms can observe during the beauty contest that one competitor offered much better conditions. In this case they can follow suit which in turn triggers a price or service war, which is detrimental for all participating firms, thus making deviation from the collusive behavior unattractive.

With auctions, collusion is no longer possible. If all firms make low offers, some firms know at this point that they will loose the contest if they do not react. So they have to improve their bid for sure. This leads to standard Bertrand competition which makes collusion infeasible.

The model we employ here is simplified in three respects. First, there are only two bidders. Second, firms are symmetric. Third, the government has private information on the ranking of the firms, not on the ranking of the service levels. All three assumptions were made for simplicity, none of them changes the conclusion of the paper. If there are more than two firms, and possibly also more than one license to be given away, the same reasoning still applies. Firms start the beauty contest with very low offers. Again, each firm has a chance of winning a license for very good conditions. This makes improving the offer and starting a price and service war unattractive. More interestingly is the second simplification, as it is probably
very realistic to assume that the government has desired providers and all bidders
know this. For example, in the UMTS beauty contest in France it was probably
clear to all participating firms that France Telecom had some advantage in receiving
the license. However, for our model to apply we only require that firms do not know
exactly the size of the advantage of the preferred firm. To make this point, assume
a beauty contest in prices only. Suppose the preferred firm makes an offer of 100,
and the other firms bid 110 for this license. If no one knows the exact advantage
of the preferred firm, this 10% difference in offers might lead to a stable outcome.
Still, all firms have a chance of winning the contest, while any deviation from this
price will again trigger a price war.\textsuperscript{11} The third simplification of our model also
gives rise to interesting extensions. If the government values different dimensions
of services differently, and these valuations are not precisely known to the firms,
then firms might differentiate on purpose to be able to support collusive behavior.
If each firm offers a different service package, then again all firms have a chance of
winning the license for potentially very good conditions. Thus service (or product)
differentiation might not arise because of technological differences, but because only
this differentiation enables firms to collude successfully.\textsuperscript{12}

6 Appendix

Proof of the Proposition

\textsuperscript{11}There is a piece of evidence which seems to confirm that firms have limited information to
how far a government favours local firms. In Sweden a beauty contest was used to sell the UMTS
licenses. The probably preferred winner, the national operator Telia, surprisingly lost the contest.
This came as a surprise also to Telia, who decided to start a law suit against this decision.
\textsuperscript{12}There is an extensive literature of whether product differentiation makes collusion easier to
sustain and whether the possibility to collude leads to more or less differentiation (Friedman
and Thisse, 1993; Jehiel, 1992; H"ackner, 1994; Ross, 1992; Rothschild, 1992; Werberfelt, 1989) .
However, in contrast to the present analysis, most of this work is done in the context of infinitely
repeated games.
The equilibrium strategies are such that on the equilibrium path, both bidders bid their minimum service level. If someone deviates this firm will be punished by the other firm. This punishment takes the form that a service level is offered which is such that the deviating firm does not obtain more than it would have received if it behaved according to its equilibrium strategy. The punishment is enforced by the threat that if someone does not punish this firm will obtain also less than it would have got by punishing correctly. We now proceed to prove this statement formally.

Let \( a^t = (s^t_1, s^t_2) \) be the action profile at stage \( t \), if the game has not stopped before. Define \( h^{k+1} = (a^1, a^2, ..., a^k) \) as the history at the end of stage \( k \). Denote by \( S_i = \{S^k_i\}_{k=0}^\infty \) the strategy of player \( i \), where each \( S^k_i \) maps all possible histories at the end of stage \( k-1 \) onto the set of all possible service levels, including no new bid which we denote as \( nb \). Note that large letter \( S^k_i \) denotes the strategy at stage \( k \), while small letter \( s^k_i \) is the actual service level offered at stage \( k \). Recall that the set of all possible service levels, i.e. bids, is the set of all \( s \) which are such that \( u(s) \geq u(s^{k-1}) \).

We discuss the equilibrium strategy for player \( i \). To do so, we distinguish between three phases:

(i) **Equilibrium phase**: Let \( S^0_i = s^0_i = s_{\text{min}} \). If \( a^0 = (s_{\text{min}}, s_{\text{min}}) \) make no new bid, i.e. \( S^1_i(s_{\text{min}}, s_{\text{min}}) = nb \). If both firms behave in this way, the game ends after two stages, and both firms obtain the expected profit \( \pi^m = 1/2(v - p - c(s_{\text{min}})) \). If at some stage \( k \) during the game both firms behaved according to their equilibrium strategy, which will be defined below, then let \( s^{k+1}_i = nb \). That is, if at one stage both firms did what they were supposed to do, then the auction will stop in the next stage.

(ii) **Punishment phase**: Suppose at some stage \( k \) the other firm \((-i)\) deviated from its equilibrium strategy by offering a service level \( s_{-i} \neq S^k_{-i}(h^k) \), while firm \( i \) has offered a service level in accordance with its equilibrium strategy. Let this be the \( q \)'th deviation from the equilibrium strategy, i.e. before this stage, \( q - 1 \) other deviations from the equilibrium strategies have occurred. Then the bid in stage \( k + 1 \)
will be $s_i^{k+1} = s^p(s_{-i}, h^{k+1})$ ($p$ stands for punishment) which is defined recursively as follows:

$$s^p(s_{-i}, h^{k+1}) = \arg\max_s \pi(s, s_{-i})$$

s.t. $\pi(s_{-i}, s) \leq \pi_{q-1}^{-1}(h^k)$

\[u(s) \geq u(s_i^k)\]  \hspace{1cm} (6)

where $\pi(s, s')$ is defined in equation (4). The profit level $\pi_{q-1}^{-1}(h^k)$ is determined recursively by defining $\pi_i^q = \pi(s^p(s_{-i}, h^{k+1}), s_{-i})$ and $\pi_{q-1}^{-1} = \pi(s_{-i}, s^p(s_{-i}, h^{k+1}))$. Therefore $\pi_{q-1}^{-1}(h^k)$ is the profit firm $-i$ would have received if it did not deviate.

In words, the punishment bid is the best possible offer which is such that the other firm who is to be punished only obtains at most the profit it would have obtained if it behaved according to its equilibrium strategy in the previous stage.

(iii) **Self-Punishment phase**: Suppose at some stage $k$ bidder $i$ has offered a service level $s_i \neq S_i^k(h^k)$ which is not in accord with its equilibrium strategy, while the other firm behaved as expected. Then the bid of firm $i$ in stage $k+1$ will be $s_i^{k+1} = nb$. In words, if one is about to be punished, then one accepts this punishment without bidding further.

Now the equilibrium strategy is recursively given by the combination of the three phases given above. If in any stage $k$, both firms behave according to their equilibrium strategy, then either the biddings ends (if both firms bid $nb$) or the strategy is defined such that both firms make no new bid in the next stage, in which case the bidding ends in the next stage. If the other firm offers a service level different than the one given by the equilibrium strategy, and oneself behaved according to the equilibrium strategy, then the new bid in round $k + 1$ is given by $s_i^p(s_{-i}, h^{k+1})$. If oneself in the last stage did not bid according to the equilibrium strategy, then one does not make a new bid in the next round.\(^{13}\)

\(^{13}\)To specify the equilibrium strategies completely, we need to define what the players will do if they both do not bid according to the equilibrium strategy in a given stage. As this is not important for the equilibrium discussion, we can specify this by saying that both will bid the higher of the two service levels after such a double-deviation.
Now by using the one-stage-deviation-principle it is easy to see that the strategies defined above indeed constitute a subgame perfect Nash equilibrium. Any deviation from the strategies triggers a response from the other bidder which gives no more profit than one would have obtained if one behaved according to the equilibrium strategies. Thus deviating is never profitable.
References


