Neoclassical Growth and the "Trivial" Steady State

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Abstract: If capital is an essential input, the neoclassical growth model has a steady state with zero capital. From this, one is inclined to conclude that an economy starting without capital can never grow. We challenge this view and claim that, if the production function satisfies the Inada conditions, a take-off is possible even though the initial capital stock is zero and capital is essential. Since the marginal product of capital is initially infinite, the "trivial" steady state becomes so unstable that the solution to the equation of motion involves the possibility of a take-off, even without capital. When it happens, the take-off is spontaneous; there is no causality.

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1 Introduction

Most specifications of the neoclassical growth model of Solow (1956) and Swan (1956) exhibit a trivial steady state, i. e. an unstable state with zero capital. Intuitively, it obtains in a closed economy void of capital if capital is essential to generate income. Without income, there are no savings and no investment. Thus, the evolution of capital is at a point of rest.

Based on this intuition, one is inclined to draw the conclusion that an economy in a state without capital remains there (see, e. g., Romer, 2001, p. 16). Our analysis challenges this view and claims that a neoclassical economy may take off even though the initial capital stock is zero and capital is essential. When this happens, the ignition of the process of capital accumulation is spontaneous, without a cause.

Our finding is based on a careful analysis of the instability associated with the trivial steady state. We show for a broad class of aggregate production functions (including the neoclassical production function) that the solution to the equation of motion for capital need not be unique when capital is zero. Assuming that capital is an essential input and that the Inada condition for capital is valid imply this finding. Together they impose opposing forces on the accumulation process when there is zero capital. On the one hand, since capital is essential, there is nothing to invest; on the other hand, due to the Inada condition, the contribution of a marginal increment in capital to the change in capital is infinite. The behavior of the trajectory for capital is then indeterminate. Depending on which of these two forces "gets the upper hand", the economy may either remain without capital, or take off.

Section 2 develops our main result for a (neo)classical economy that is equipped with a Cobb-Douglas production function. We link our finding to the lack of Lipschitz continuity in the equation of motion. This property allows for multiple solutions. In Section 3, we extend the setting to more general production functions and identify the tension between the essentiality of capital and the Inada condition as the driving force behind the spontaneous take-off.

Section 4 casts the notion of a spontaneous take-off in a dual economy, for example, in the spirit of Harris and Todaro (1970) or Hansen and Prescott (2002). Initially, there is no capital, and the rural sector employs the entire labor force. The evolution of capital may kick off spontaneously.[1] Following the take-off, the economy exhibits central features of an industrial revolution. Capital accumulates, and labor migrates from the rural sector to the industrial sector. The spontaneity of the take-off excludes a cause. It is therefore futile to ask why and when the take-off occurs. The focus is rather on necessary conditions that allow for spontaneity such as the possibility to adopt an appropriate industrial technology.

2 Neoclassical Growth under Cobb-Douglas

Consider a closed economy in continuous time, equipped with the aggregate production function

$$Y(t) = F(K(t), L(t)) = A\,K(t)^{\alpha}\,L(t)^{\beta}\,T^{1-\alpha-\beta}, \tag{1}$$

where $A > 0$ is total factor productivity, $K(t) \geq 0$ the capital stock at time t, $L(t) = e^{n\,t}$ the employed population at t (growing at rate $n \geq 0$), and T the available land.[2] Assume that $\alpha, \beta \in (0;\ 1)$. If, in addition, $0 < \alpha + \beta < 1$, then this production function (i) exhibits constant returns to scale, (ii) it has positive and diminishing returns, (iii) it satisfies the Inada conditions, and (iv) all of its inputs are essential. Swan (1956) calls this *a classical case* as opposed to *an unclassical case* for which he stipulates $\alpha + \beta = 1$. In the latter case, the amount of available land has no influence upon aggregate output, and the four properties hold with respect to capital and labor, i. e. the production function is neoclassical in the sense of Barro and Sala-í-Martin (2004, pp. 26–28).

The equation of motion for the capital stock is

$$\dot{K}(t) = s\,Y(t) - \delta\,K(t), \tag{2}$$

[1]The change in the qualitative behavior of the economy's dynamics bears some resemblance to the notion of a big push (see, e. g., Rosenstein-Rodan, 1943, and Murphy, Shleifer, and Vishny, 1989). The possibility of a big push obtains in settings with multiple equilibria. Therefore, it may or may not materialize. The solution to a complex coordination problem is the counterpart to the spontaneity of our framework.

[2]To allow for exogenous technological progress, one may simply replace n by $\tilde{n} = n + x$, where $x > 0$ is the growth rate of some factor multiplied with $L(t)$. All results of this paper extend to settings with exogenous labor-augmenting technical progress.

where $s \in (0;\ 1)$ is the savings rate and $\delta \geq 0$ the instantaneous depreciation rate. Without loss of generality we normalize and set $T = 1$. Then, the evolution of capital becomes

$$\dot{K}(t) = s\,A\,K(t)^{\alpha}\,e^{n\,\beta\,t} - \delta\,K(t). \tag{3}$$

Since our focus is on the trivial solution, we restrict attention to the initial value problem, with $K(t_c) = 0$ for some time t_c. This problem has two algebraic solutions,[3]

$$K_1(t) = \left(A\,\frac{s\,(1-\alpha)}{n\,\beta + \delta\,(1-\alpha)}\left(e^{n\,\beta\,t} - e^{n\,\beta\,t_c}\,e^{-(1-\alpha)\,\delta\,(t-t_c)}\right)\right)^{\frac{1}{1-\alpha}}, \quad \text{and}$$

$$K_2(t) = 0 \quad \text{for all } t.$$

We refer to the latter as the trivial solution. In addition, piece-wise combinations of K_1 and K_2 qualify as a solution as long as these are continuous and differentiable.

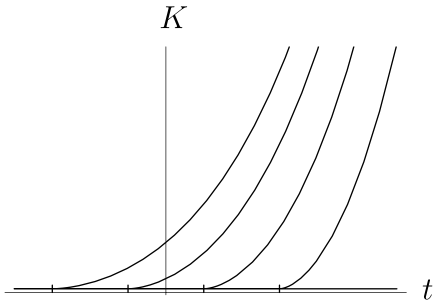
Observe that $K_1(t_c) = 0$. For $t < t_c$, $K_1(t)$ may either become positive or negative. In the former case, the implication is $\dot{K}_1(t) < 0$, which is contradictory to (3): when the capital stock is rather small, it cannot shrink since the additions to the capital stock exceed depreciation. In the latter case, $K(t) \geq 0$ is violated. Hence, $K_1(t)$ can only be part of a solution for $t \geq t_c$. Since $K_1(t_c) = K_2(t_c) = 0$ and $\dot{K}_1(t_c) = \dot{K}_2(t_c) = 0$, the path of capital is not unique at each critical date t_c: capital is zero before t_c and may either follow $K_1(t)$ or $K_2(t)$ after t_c. We may interpret t_c as the moment of a take-off and conclude that the solution to the differential equation (3) may take off at any time t_c, or never (see Figure 2).[4]

The behavior following the take-off is determined by the level of population at that time. If capital takes off late and $n > 0$, then the capital stock grows faster because its population is larger. For large t, $K_1(t)$ becomes approximately proportional to

[3] The solution $K_1(t)$ obtains because (3) is a Bernoulli equation that can be solved by appropriate substitution (see, e. g. Gandolfo, 1997, p. 436). For $\delta > 0$, there are additional algebraic solutions if we do not impose an initial value equal to zero. These solutions generate strictly positive levels of capital at all times. Generically, capital does not even get close to zero but converges to infinity for some $t < 0$.

[4] One may ask whether this outcome is an artefact of continuous-time modelling. The answer is that the two solutions, $K_1(t)$ and $K_2(t)$, obtain as the limits of the initial value problem with $K(t_c) > 0$ based on the discrete-time equation $K\,(t + \Delta t) - K(t) = (s\,A\,K(t)^{\alpha}\,e^{n\,\beta\,t} - \delta\,K(t))\,\Delta t$. Indeed, one finds that $\lim_{K(t_c)\to 0} \lim_{\Delta t \to 0} = K_1(t)$, but $\lim_{\Delta t \to 0} \lim_{K(t_c)\to 0} = K_2(t)$.

Figure 1: The Ambiguous Evolution of Capital.



All trajectories qualify as possible evolutions of capital. The later the take-off, the steeper is the trajectory for $t > t_c$.

$e^{n\,\beta\,t/(1-\alpha)}$. Hence, the asymptotic growth rate of capital is $n\,\beta/(1-\alpha)$. For the trivial solution $K_2(t)$, the growth rate is ill-defined.

The fact that the evolution of capital is not unique for $K = 0$ is linked to the missing Lipschitz continuity of the differential equation.[5] We know from Picard's Existence Theorem that a solution to a differential equation is unique if the equation is Lipschitz continuous. Here, the test for Lipschitz continuity fails,

$$\lim_{K(t)\to 0} \frac{\partial \dot{K}(t)}{\partial K(t)} = \lim_{K(t)\to 0} \frac{s\,A\,\alpha\,e^{n\,\beta\,t}}{K(t)^{1-\alpha}} - \delta = \infty. \tag{4}$$

Since $e^{n\,\beta\,t}$ is always positive, the fraction is unbounded for small $K(t)$.

The basic conclusion of this section is that the economy with zero capital at some time may either go on without accumulation forever or depart on a trajectory with positive growth of the capital stock, albeit with no cause. No first piece of capital is needed to trigger accumulation initially. The take-off happens spontaneously.

This possibility of a take-off seems not to be taken into account in the literature when commenting on the "trivial" steady state. For instance, Romer (2001, p. 16) writes, "If k [the capital intensity per unit of efficient labor] is initially zero, it

[5]A differential equation $\dot{K} = f(K,t)$ is said to satisfy the Lipschitz condition if $|f(K,t) - f(K',t)| < L\,|K - K'|$ within the definition interval for some finite constant L (see, for example, Aliprantis and Border, 1998). In particular, when $\partial f(K,t)/\partial K = \infty$ for some K and t, the differential equation cannot be Lipschitz continuous at this point since differentiability implies Lipschitz continuity.

remains there". This assessment may be based on Solow (1956, p. 70), who notes, "If $K = 0$, $r = 0$ [the capital intensity] and the system can't get started; with no capital there is no output and hence no accumulation. But this equilibrium is unstable: the slightest windfall capital accumulation will start the system off ...." In light of the preceding analysis, we conclude that under Cobb-Douglas, the instability of the trivial equilibrium is so pronounced that the system can get started, even without a slight windfall capital.

3 Essentiality and the Inada Conditions

We now turn to more general production functions. Our central finding is Proposition 1.

Proposition 1 *Consider the equation of motion (2) with $Y(t) = F(K(t), L(t))$ and $K(t_c) = 0$ at some time t_c. Then,*

1. *if $F(0, L) = 0$ and $\lim_{K\to 0} \partial F/\partial K = \infty$, capital may take off spontaneously or remain at zero,*
2. *if $F(0, L) = 0$ and $\lim_{K\to 0} \partial F/\partial K < \infty$, capital remains at zero,*
3. *if $F(0, L) > 0$, capital takes off immediately.*

Proof. In the main text below.

According to Case 1, a spontaneous take-off may occur for quite general production functions if capital is essential and the Inada condition (see Inada, 1963) is satisfied. Since the property of constant returns to scale in conjunction with the Inada condition implies essentiality (see, e. g. Barro and Sala-í-Martin (2004), p. 77), we conclude that Case 1 applies to all neoclassical production functions. The explanation of this result is as follows.

If $F(0, L) = 0$, then capital is essential and the trivial solution always satisfies the equation of motion: $K = 0$ for all t implies $\dot{K} = s\,F(K, L) - \delta\,K = s\,F(0, L) = 0$. The Inada condition for capital requires $\lim_{K\to 0} \partial F/\partial K = \infty$. It is usually imposed to exclude a stable trivial steady state. What matters here can be seen from the derivative of the equation of motion (2) with respect to K and its limit

$$\frac{\partial \dot{K}}{\partial K} = s\,\frac{\partial F}{\partial K} - \delta, \quad \text{and} \quad \lim_{K\to 0} \frac{\partial \dot{K}}{\partial K} = s\,\lim_{K\to 0} \frac{\partial F}{\partial K} - \delta.$$

Due to the Inada condition, $\partial \dot{K}/\partial K$ converges to infinity for small K. As a result, the differential equation is not Lipschitz continuous at $K = 0$, and its solution need not be unique. If capital is essential, then, besides the trivial solution, there may be solutions that spontaneously take off from zero.

Intuitively, this ambiguity arises from two opposing forces that affect the equation of motion at $K = 0$. On the one hand, no capital can be accumulated since capital is essential. On the other hand, the marginal product of capital is infinite. Roughly speaking, even a zero amount of capital can lead to positive output, and thereupon to accumulation. Which of these forces dominates at each date t_c is unpredictable. Either the essentiality of capital dominates and produces the trivial solution, i. e. capital remains zero, or the Inada condition gets the upper hand and triggers an instantaneous take-off.

In Case 2, F violates the Inada condition. Accordingly, the equation of motion is Lipschitz continuous; its solution is unique. Since essentiality implies $\dot{K} = 0$, a take-off is excluded. Case 3 states that a take-off must occur if capital is not essential. Here, however, the take-off is not spontaneous, but due to a strictly positive amount of investment.

The role of essentiality and the Inada condition can be illustrated for the CES production function $F(K, L) = \left[a\,(b\,K)^{\psi} + (1-a)((1-b)\,L)^{\psi}\right]^{1/\psi}$, where $\psi < 1$ determines the elasticity of substitution between capital and labor. Capital is essential for $\psi \leq 0$, i. e. for a sufficient degree of complementarity. Moreover, the Inada condition holds for $0 \leq \psi < 1$. Hence, Case 1 of the proposition only applies for $\psi = 0$; the production function is Cobb-Douglas. For $\psi < 0$, Case 2 applies, i. e. if capital is ever zero, it stays there. For $\psi > 0$, the production function satisfies the Inada condition, yet capital is not essential. According to Case 3, if capital is zero, it takes off instantaneously. Somewhat paradoxically, the analysis of the "trivial" steady state is most complex for the textbook example involving a Cobb-Douglas technology.

Observe that the intuition behind Proposition 1 can be used to allow for a spontaneous take-off from a state with strictly positive output. To see this, replace the equation of motion (2) by $\dot{K} = g(K) - \delta\, K$, where $g(K)$ relates aggregate output to gross investment. Denote $\bar{K}$ the initial amount of agricultural capital, and let $\bar{K}$ satisfy $g(\bar{K}) = \delta\, \bar{K}$. Then, the economy is initially in a stationary state with positive output, savings, and investment. If in addition $g'(\bar{K}) = \infty$, then the economy may either stay in the stationary state forever or take off.

4 A Spontaneous Take-off in a Dual Economy

This section probes the possibility of a spontaneous take-off in a two-sector economy. We demonstrate that such a take-off may occur and that it leads to a transition from a rural to an industrialized state. To make this point, we consider a competitive closed economy equipped with two technologies, one for agricultural production, the other for industrial manufacturing,

$$Y_A(t) = a\, L_A(t)^\gamma\, T_A^{1-\gamma}, \quad 0 < \gamma < 1,$$
$$Y_M(t) = m\, K(t)^\alpha\, L_M(t)^\beta\, T_M^{1-\alpha-\beta}, \quad 0 < \alpha, \beta < 1 \quad \text{and} \quad 0 < \alpha + \beta < 1.$$

Here, $Y_A(t)$ and $Y_M(t)$ are agricultural and industrial production at t, $L_A(t)$ and $L_M(t)$ are the labor forces employed in the respective sectors, and T_A, T_M are fixed amounts of land. To simplify, we assume that the surface T_A is only productive as arable land, whereas T_M can only be used for industrial purposes. Without loss of generality, we normalize $T_A = T_M = 1$. Note that in the manufacturing sector, capital is essential and satisfies the Inada condition. To keep things simple, we choose units such that the exchange rate between the agricultural and industrial good is one.

We abstract from population growth, hence $L_A(t) + L_M(t) = L$ for all t. Moreover, we assume that rural products must be consumed right away, i. e. they are not storable. Manufactured products can be either consumed or invested in future capital. Let $s \in (0;\ 1]$ denote the constant fraction of invested current industrial production.

With homogeneous labor searching for the highest wage and facing competitive firms, the equilibrium under full employment has

$$\frac{\partial Y_A}{\partial L_A} = a\,\gamma\,L_A^{\gamma-1} = a\,\gamma\,(L-L_M)^{\gamma-1} = \frac{\partial Y_M}{\partial L_M} = m\,\beta\,K^\alpha\,L_M^{\beta-1}. \tag{5}$$

To understand the structure of a take-off, we first consider the equilibrium with K either zero or small. Then (5) implies a low marginal product of the first marginal units of labor in manufacturing. Therefore, employment in manufacturing is negligible, and it is permissible to approximate $L_A \approx L$. Thus, near the take-off the marginal product of labor in agriculture is approximately $\partial Y_A/\partial L_A \approx a\,\gamma\,L^{\gamma-1}$, and from $m\,\beta\,K^\alpha\,L_M^{\beta-1} = a\,\gamma\,L^{\gamma-1}$ we obtain an approximate expression for L_M near the take-off,

$$L_M = \left(\frac{m\,\beta}{a\,\gamma}\,L^{1-\gamma}\,K^\alpha\right)^{\frac{1}{1-\beta}} := L_M(K). \tag{6}$$

Upon substituting the latter into the equation of motion, we successively find

$$\begin{aligned}\dot{K} &= s\,Y_M - \delta\,K = s\,m\,K^\alpha\,L_M^\beta - \delta\,K \qquad (7)\\ &= s\,m\,K^\alpha\left(\frac{m\,\beta}{a\,\gamma}\,L^{1-\gamma}\,K^\alpha\right)^{\frac{\beta}{1-\beta}} - \delta\,K\\ &= \Phi\,K^{\frac{\alpha}{1-\beta}} - \delta\,K, \qquad (8)\end{aligned}$$

where $\Phi > 0$ pools other parameters. Since $\alpha+\beta < 1$, we have $\alpha/(1-\beta) < 1$. Accordingly, (8) is structurally equivalent to (3) for $n = 0$, and we already know the properties of the solution.[6] There is the trivial solution with zero capital at all times, and there are solutions comprising a spontaneous take-off. When the take-off occurs, the (approximate) evolution of employment in manufacturing follows (6), i. e. L_M rises as K does.[7]

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[6]Positive population growth does not affect the evolution of capital near the take-off. It does, however, have an effect on the growth path in the long-run.

[7]As soon as a perceptible part of the labor force has left the agricultural sector, the marginal product of labor in this sector starts to grow. Then (6) overestimates L_M and, as a consequence, (8) overestimates the actual increments to the capital stock. However, near the take-off, our approximation exhibits the same quantiative and qualitative features as the actual evolution of the capital stock. To see this, let $L_\varepsilon := L - \varepsilon$ with ε positive and small. Then, for $0 \le L_M \le \varepsilon$, we have $L_\varepsilon \le L_A \le L$, and $\Phi_\varepsilon\,K^{\alpha/(1-\beta)} - \delta\,K \le \dot{K} \le \Phi\,K^{\alpha/(1-\beta)} - \delta\,K$, where Φ_ε is equal to Φ with L replaced by L_ε. Thus, we obtain a lower and an upper bound to the actual differential equation. Solving these differential equations gives a lower and an upper bound for the actual path of capital. Since these bounds coincide for $\varepsilon \to 0$, equation (8) and its solution are correct near a take-off.

The intuition for this result is closely linked to the key difference between the one-sector economy of Section 2 and the present two-sector economy. In the former, labor is inelastically supplied to the single sector. According to the Inada condition, the marginal product of the first marginal unit of capital is infinite and the Lipschitz continuity of the equation of motion is violated. In the latter, the first marginal unit of capital is only productive if the industrial sector is able to attract workers from the rural sector. As a result, the Inada condition is no longer sufficient for a spontaneous take-off. To see this, we write the production function of the industrial sector near the take-off using (6),

$$Y_M = F(K, L_M(K)). \tag{9}$$

To check the Lipschitz continuity of the equation of motion (7), we study

$$\lim_{K(t)\to 0} \frac{\partial \dot{K}(t)}{\partial K(t)} = \lim_{K(t)\to 0} s \left[\frac{\partial F}{\partial K} + \frac{\partial F}{\partial L_M}\frac{\partial L_M(K)}{\partial K}\right] - \delta. \tag{10}$$

There are two expressions whose limit behavior may cause a violation of Lipschitz continuity. First, consider the marginal product of capital,

$$\frac{\partial F}{\partial K} = m\,\alpha\,K^{\alpha-1}\,\left[L_M(K)\right]^{\beta}. \tag{11}$$

Using l'Hôpital's rule and (6), we obtain for the limit

$$\lim_{K(t)\to 0} \frac{\partial F}{\partial K} = \lim_{K(t)\to 0} \Phi' \,\frac{\partial L_M(K)}{\partial K}, \tag{12}$$

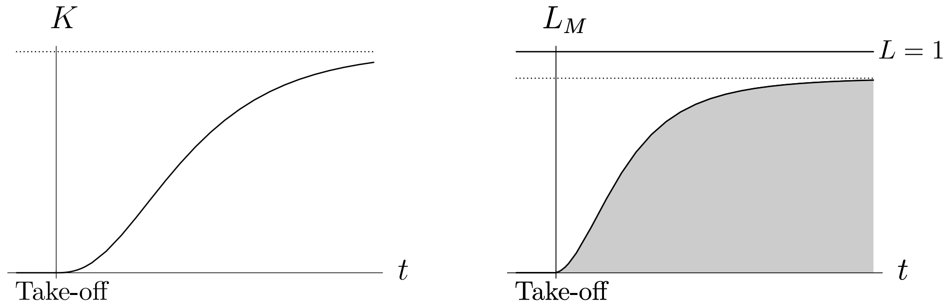
where $\Phi' > 0$ is a summary statistic. Hence, the ability of the industrial sector to attract workers from the rural sector determines the Lipschitz continuity, and therewith the possibility of a spontaneous take-off.

The second term in brackets is the indirect effect of capital on industrial output through the attraction of labor. Invoking the reasoning that led to (6), we have in equilibrium $\partial F/\partial L_M = a\,\gamma\,L^{\gamma-1}$. Using (12), we may therefore express (10) as

$$\lim_{K(t)\to 0} \frac{\partial \dot{K}(t)}{\partial K(t)} = \lim_{K(t)\to 0} s\left(\Phi' + a\,\gamma\,L^{\gamma-1}\right)\frac{\partial L_M(K)}{\partial K} - \delta = \lim_{K(t)\to 0} \Phi''\,\frac{\partial L_M(K)}{\partial K} - \delta, \tag{13}$$

where $\Phi'' > 0$. From (6), this limit is infinite if and only if $\alpha + \beta < 1$. This condition can be interpreted as a strengthened Inada condition. It guarantees an

Figure 2: The Dynamics of K and L_M.



Parameters are $\alpha = .4$, $\beta = .4$, $\gamma = .8$, $a = 10$, $m = 1$, $s = 1$, $\delta = .1$, $L = 1$, and $t \in [-10;\ 80]$. Assume that a take-off occurs. Without loss of generality we set $t_c = 0$. In the right figure, the white area is the fraction of agricultural workers, the shaded is the fraction of industrial workers.

infinite marginal product of capital even though the amount of employed labor in manufacturing vanishes as capital approaches zero.[8]

To see that our model has reasonable properties for the long-run, consider the stationary steady state. Intuitively, the evolution becomes stationary when labor stops switching sectors. At this point in time, capital growth must come to a halt since, otherwise, more capital would continue to raise the marginal product of labor and attract more workers from rural occupations. Hence, the steady state satisfies $\dot{K} = 0$ and (5). This gives an implicit equation for the limits of capital and labor, which for its part, leads to a unique steady-state solution with $K^* > 0$, $L_A^* > 0$, and $L_M^* > 0$ (see Figure 2).[9]

[8]To see that the Inada condition is indeed too weak to imply the possibility of a spontaneous take-off, consider the case $\alpha + \beta = 1$. Here, $L_M(K)$ is linear in K, and the limit of (13) is finite. Hence, the equation of motion (7) is Lipschitz continuous. The unique solution is exponential such that a take-off needs a slight windfall capital besides $\Phi - \delta > 0$.

[9]To confirm this reasoning, note that $\dot{K} = 0$ implies $s\, m\, K^\alpha\, L_M^\beta = \delta\, K$, hence $K^\alpha = (L_M^\beta\, s\, m/\delta)^{\alpha/(1-\alpha)}$. Substitution into (5) gives $a\,\gamma\,(L - L_M)^{\gamma-1} = m\,\beta\,(L_M^\beta\, s\, m/\delta)^{\alpha/(1-\alpha)}\, L_M^{\beta-1}$, or $L - L_M = c\, L_M^{(1-\alpha-\beta)/[(1-\alpha)\,(1-\gamma)]}$, where $c > 0$ is a constant. Since the exponent of L_M is positive, the solution for L_M^* is unique, and so is the solution for K^* and L_A^*.

5 Concluding Remarks

This paper contributes to the understanding of the dynamics of the seminal growth model of Solow (1956) and Swan (1956). We show that the evolution of an economy equipped with a neoclassical production function, yet void of capital, is not unique. The economy may either take off at any date, or remain without capital forever. Our findings suggests that this state is neither trivial nor steady.

In a dual economy, this feature may ignite a spontaneous regime shift that captures some stylized facts of an industrial revolution. All of a sudden, an entirely rural economy can start a manufacturing industry that gradually attracts labor. The process of capital accumulation begins, and over time, the agricultural sector loses significance. From the vantage point of our model, the vexing question about the timing of the Industrial Revolution has no answer. Yet, the analysis suggests preconditions for a take-off such as the access to an industrial technology that is sufficiently productive to attract labor from a rural occupation.

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