The Blessing of Wealth
The Curse of Poverty

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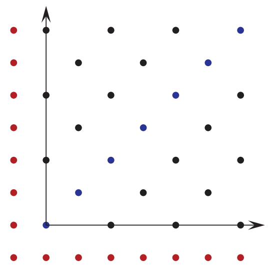
Introduction

- Why bats?
  - Gerald Wilkinson, “Reciprocal food sharing in the vampire bat.” *Nature* 308, 1984: 181–184.
  - ——, “Food sharing in vampire bats.” *Scientific American* 262, 1990: 76-82.
- Sustaining cooperation?
  - David Kreps, Paul Milgrom, John Roberts and Robert Wilson, “Rational cooperation in the finitely repeated prisoners’ dilemma.” *JET* 27, 1982.
  - Robert Axelrod, *The Evolution of Cooperation*, 1984.
  - Elinor Ostrom, *Governing the Commons: The Evolution of Institutions for Collective Action*, 1990.

Model



- Two bats each eat 1 per day
- Each bat hunts once a day.
- A hunt returns 2 with pr p, or 0.
- Bats maintain an inventory.
- A bat **may** share 1 with the other bat.
- A bat dies when she fails to eat.

The inventory space is $\mathcal{I}$, which describes each bat's current inventory.

$b_i = -1$ is death, an absorbing state.

Model

- Let Ω denote the sample space on which the processes are built: $\omega_t = (\omega_{1t}, \omega_{2t})$ where each ω_{it} describes the outcome of i's date-t hunt, success or failure.
- From each strategy profile (σ_1, σ_2) and the initial inventory $b = (b_{10}, b_{20})$ compute $\tau_b^i(\omega) = \inf\{t : b_{it} = -1\}$, the time at which i dies.
- Bat i's payoff function is

$$u_i(\sigma_1, \sigma_2) = \sum_{t=0}^{\tau_b^i(\omega)} \delta^t.$$

Dynamics

- An action for bat i is a choice to share S or withhold W 1 unit from bat j. $\mathcal{A} = \{S, W\}$.
- A state $q \in Q$ of the game is a quadruple $q = (b_1, b_2, \omega_1, \omega_2)$ where $(b_1, b_2) \in \mathcal{I}$ are the bat's inventory levels, and ω_i is describes the outcome of bat i's hunt. State q_t describes the date-t physical situation after hunting.

Taking $s = S = 1$ and $f = W = 0$, the dynamics are

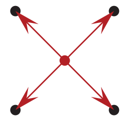
$$\longrightarrow \underbrace{\begin{matrix} b_{1t} \\ b_{2t} \end{matrix}}_{\mathbf{b}_t} \rightsquigarrow \underbrace{\begin{matrix} b_{1t}, \omega_{1t} \\ b_{2t}, \omega_{2t} \end{matrix}}_{q_t} \longrightarrow \underbrace{\begin{matrix} b_{1t} + \omega_{1t} - a_{1t} + a_{2t} - 1 \\ b_{2t} + \omega_{2t} - a_{2t} + a_{1t} - 1 \end{matrix}}_{\mathbf{b}_{t+1}} \rightsquigarrow$$

Strategies

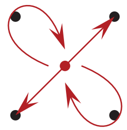
- A rule is a map $r : \mathcal{Q} \to \mathcal{A}$. $\mathcal{R}$ denotes the set of rules.
- A partial history is a structure $(q_0, a_{10}, a_{20}, \ldots, q_t, a_{1t}, a_{2t})$. Let $\mathcal{H}$ denote the set of partial histories. $h_t \in \mathcal{H}$ is the sequence of states and actions through date t.
- A strategy for bat i is a map $\sigma : \mathcal{H} \to \mathcal{R}$ where $\sigma(h_{t-1})$ is the rule employed at date t.

Some Rules

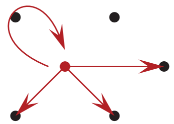
- Autarchy. Each bat contributes to and withdraws from her own inventory. No sharing takes place.



- Simple Sharing. A successful bat shares with an unsuccessful bat.



- Wealth-Based Sharing. A successful bat shares with the other bat if and only if she is wealthier than the other bat.



We only allow rules that share (or not) upon success.

Questions

The possibilities for cooperation:

- Is sharing over some or all of the state space optimal?

  Yes, on all of $\mathcal{I}$.

- For large δ, are there equilibria which support sharing on some or all of the state space?

  Depends on p, and only for some part of $\mathcal{I}$.

- For large δ, are there equilibria which achieve the welfare optima?

  No.

Folk Theorems

These questions are normally answered by folk theorems.

Requirements for the folk theorem:

- The set of feasible long-run average payoffs is state-independent.
- The long-run average min-max payoffs are state-independent.
- The dimension of the set of long-run average feasible payoffs is 2.

In our game.

- For $p < 1/2$ the only feasible long-run average payoff is 0.
- For $p > 1/2$, the maximal long-run average payoff is state-dependent.

Prajit Dutta, "A folk theorem for stochastic games" *JET* 66, 1995.

Value Functions

- Strategies determine a stochastic process on $\mathcal{I}$.
- τ^i_b is the first time the process hits -1.
- The value of being at $b \in \mathcal{I}$ is

$$V_i(b) = \mathsf{E}\left\{\sum_{t=0}^{\tau^i_b(\omega)} \delta^t\right\} = \frac{1 - \delta\,\mathsf{E}\left\{\delta^{\tau^i_b(\omega)}\right\}}{1-\delta}$$

$$\lim_{\delta\uparrow 1} V_i(b) = \begin{cases} 1 + \mathsf{E}\left\{\tau^i_b(\omega)\right\} & \text{if } \mathrm{pr}\left\{\tau^i_b(\omega) < \infty\right\} = 1, \\ \infty & \text{otherwise} \end{cases}$$

- The average discounted value at $b \in \mathcal{I}$ is

$$ADV_i(b) = 1 - \delta\,\mathsf{E}\left\{\delta^{\tau^i_b(\omega)}\right\},$$

$$\lim_{\delta\uparrow 1} ADV_i(b) = \mathrm{pr}\left\{\tau^i_b(\omega) = \infty\right\}.$$

The value to bat i of being in state (b_1, b_2), by recursion:

$$V_1^{aut}(b_1) = \begin{cases} 1 + \delta p V_1^{aut}(b_1 + 1) + \delta(1 - p) V_1^{aut}(b_1 - 1) & \text{for } b \geq 1, \\ p + \delta p V_1^{aut}(b_1 + 1) & \text{for } b = 0. \end{cases}$$

This is a linear second-order difference equation with two boundary conditions: $V_1^{aut}(-1) = 0$ and $\lim_{b_1 \to \infty} V_1^{aut}(b_1) = 1/(1 - \delta)$.

$$V_1^{aut}(b_1) = \frac{1}{1 - \delta}\left(1 - \frac{\mu^{b_1+1}}{\delta}\right)$$

Proofs

So how does one compute these stopping times?

For, at $b_1 > b_2 > 0$ substituting (19) into the defining equation yields by (10)

$$
\begin{aligned}
w(b_1, b_2) &= 1 - \delta + \delta p^2 \left[1 - \lambda_2^{b_2+1} + \lambda_2^{b_2+1} w(b_1 - b_2, 0)\right] \\
&\quad + 2\delta p (1-p) \left[1 - \lambda_2^{b_2} + \lambda_2^{b_2} w(b_1 - b_2, 0)\right] \\
&\quad + \delta (1-p)^2 \left[1 - \lambda_2^{b_2-1} + \lambda_2^{b_2-1} w(b_1 - b_2, 0)\right] \\
&= 1 - \left(\delta p^2 \lambda_2^2 + 2\delta p (1-p) \lambda_2 + \delta (1-p)^2\right) \lambda_2^{b_2-1} \left[1 - w(b_1 - b_2, 0)\right] \\
&= 1 - \lambda_2^{b_2} \left[1 - w(b_1 - b_2, 0)\right] = 1 - \lambda_2^{b_2} + \lambda_2^{b_2} w(b_1 - b_2, 0)
\end{aligned}
$$

and at $b_2 \geq b_1 > 0$ substituting (20) into the defining equation and using (10) yields

$$
\begin{aligned}
w(b_1, b_2) &= 1 - \delta + \delta p^2 \left[1 - \lambda_2^{b_1+1} + \lambda_2^{b_1+1} w(0, b_2 - b_1)\right] \\
&\quad + 2\delta p (1-p) \left[1 - \lambda_2^{b_1} + \lambda_2^{b_1} w(0, b_2 - b_1)\right] \\
&\quad + \delta (1-p)^2 \left[1 - \lambda_2^{b_1-1} + \lambda_2^{b_1-1} w(0, b_2 - b_1)\right] \\
&= 1 - \left(\delta p^2 \lambda_2^2 + 2\delta p (1-p) \lambda_2 + \delta (1-p)^2\right) \lambda_2^{b_1-1} \left[1 - w(0, b_2 - b_1)\right] \\
&= 1 - \lambda_2^{b_1} + \lambda_2^{b_1} w(0, b_2 - b_1),
\end{aligned}
$$

as required. Substituting (19) and (1) into the defining equation for $w(b_1, 0)$ and using (3), (16), and (18) yields for even $b_1 > 1$

$$
\begin{aligned}
w(b_1, 0) &= 1 - \delta + \delta p^2 \left[1 - \lambda_2 + \lambda_2 w(b_1, 0)\right] + \delta p (1-p) \left[1 - \lambda_2 + \lambda_2 w(b_1 - 2, 0)\right] \\
&\quad + \delta (1-p)^2 \left(1 - \frac{\mu_2^{b_1}}{\delta}\right) + \delta p (1-p) \left(1 - \frac{\mu_2^{b_1+2}}{\delta}\right) \\
&= 1 - \delta p \lambda_2 + \delta p^2 \lambda_2 w(b_1, 0) + \delta p (1-p) \lambda_2 w(b_1 - 2, 0) - (1-p) \frac{\mu_2^{b_1+1}}{\delta} \\
&= 1 - \delta p \lambda_2 - (1-p) \frac{\mu_2^{b_1+1}}{\delta} \\
&\quad + \delta p^2 \lambda_2 \left[1 - B\mu_2^{b_1+1} - (1 - B\mu_2 - w(0,0)) \gamma^{\frac{b_1}{2}}\right] \\
&\quad + \delta p (1-p) \lambda_2 \left[1 - B\mu_2^{b_1-1} - (1 - B\mu_2 - w(0,0)) \gamma^{(b_1-2)/2}\right] \\
&= 1 - \left[(1-p) \frac{\mu_2^2}{\delta} + \delta p^2 \lambda_2 B \mu_2^2 + \delta p (1-p) \lambda_2 B\right] \mu_2^{b_1-1} \\
&\quad - \delta p^2 \lambda_2 \left[1 - B\mu_2 - w(0,0)\right] \gamma^{b_1/2} \\
&\quad - \delta p (1-p) \lambda_2 \left[1 - B\mu_2 - w(0,0)\right] \gamma^{(b_1-2)/2} \\
&= 1 - \left[\frac{1-p}{\delta} + \frac{p(1-p)\lambda_2}{\delta(\mu_2 - p\lambda_2)}\right] \mu_2^{b_1+1} - \left[1 - B\mu_2 - w(0,0)\right] \gamma^{b_1/2} \\
&= 1 - \frac{(1-p)\mu_2}{\delta(\mu_2 - p\lambda_2)} \mu_2^{b_1+1} - \left[1 - B\mu_2 - w(0,0)\right] \gamma^{b_1/2} \\
&= 1 - B\mu_2^{b_1+1} - \left[1 - B\mu_2 - w(0,0)\right] \gamma^{b_1/2}
\end{aligned}
$$

Welfare

Take as a welfare function $W(b_1, b_2)$ the sum of the bats' expected lifetimes.

- It can be described as the fixed point of a Bellman operator.
- W is symmetric around the diagonal.
- W is strictly increasing.
- If $b_2 > b_2$, then $W(b_1, b_2) < W(b_1 + 1, b_2 - 1)$. Moving diagonally towards the main diagonal is welfare improving.

Results

- The welfare-optimal strategy is wealth sharing whenever both bats are alive.

We look for equilibria with a grim trigger. A successful bat may defect from sharing and revert to autarchy but with one more unit.

- Autarky everywhere is an equilibrium for any p.

- Autarchy is the only equilibrium for $p < 1/2$.

  Proof. $V_1^{aut}(b_1 + 1) + V_2^{aut}(b_2 + 1) > W(b_1, b_2)$.

  But if there is an equilibrium other than autarchy, then for all i,

$$V_i^{eq}(b_1, b_2) \geq V_i^{aut}(b_i + 1).$$

Results

- Bilateral sharing on all of $\mathcal{I}$ is not an equilibrium.

  Proof. Diagonals are invariant. The incentive constraint is violated at $\min\{b_1, b_2\} = 0$.

- If $p > 1/2$ and δ is sufficiently near 1, then bilateral sharing on the interior of $\mathcal{I}$ is an equilibrium.

- If $p > 1/2$ and δ sufficiently near 1, there is an equilibrium in which the wealth-sharing rule is used on the set $\{(b_1, b_2) : |b_1 - b_2| \leq 1,\ b_1, b_2 > 0\}$.

Summary

- Sharing is not possible in poor societies, $p < 1/2$.
- Sharing is possible in wealthy societies, $p > 1/2$, and moreso for wealthier societies.
- Nonetheless, even in wealthy societies the welfare optimum cannot be achieved.