Circulant Games

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Abstract

This paper presents a class of finite two-player normal-form $n \times n$ games we coin *circulant games.* In circulant games, each player's payoff matrix is a circulant matrix, i.e. each row vector is rotated by one element relative to the preceding row vector. We show that when the payoffs in the first row of each payoff matrix are strictly ordered, a single parameter fully determines the exact number and the structure of all Nash equilibria in these games. The parameter itself only depends on the position of the largest payoff in the first row of player 2's payoff matrix. The class of circulant games contains well-known games such as Matching Pennies, Rock-Paper-Scissors, as well as subclasses of coordination and common interest games.

1 Introduction

The multiplicity of Nash equilibrium outcomes for a given game has motivated many scholars to analyze the structure of Nash equilibria in general as well as for special subclasses of games. Consider for example the case of finite two-player normal-form $n \times n$ games. Provided that such a game is non-degenerate the number of Nash equilibria is finite and odd (see e.g. Shapley, 1974). Quint and Shubik (1997) have shown that for any odd integer number y between 1 and $2^n - 1$, there exist a game with exactly y Nash

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equilibria. Moreover, they conjectured that $2^n - 1$ is the tight upper bound on the cardinality of the set of Nash equilibria.

This conjecture can be considered as the starting point of the active search for the upper bound on the number of Nash equilibria (mixed or pure) in such games. Quint and Shubik (1997) showed that the conjecture holds for $n = 3$, Keiding (1997) and McLennan and Park (1999) proved it for $n = 4$. The conjecture was refuted by von Stengel (1997) who gave a counterexample for $n = 6$ with a total number of 75 Nash equilibria whereas $2^6 - 1 = 63$.[1] New upper bounds on the number of distinct Nash equilibria were established in Keiding (1998) and von Stengel (1999). However, special classes of games exist for which the conjecture is true as shown by Quint and Shubik (2002) for the class of coordination games.

In this paper, we investigate a class of finite two-player normal-form $n \times n$ games we coin *circulant games*. As any finite game can be fully represented by the associated players' payoff matrices, our concepts and ideas are expressed in terms of properties the payoff matrices have to fulfill. In circulant games, the players' payoff matrices are circulant, i.e. each row vector is rotated by one element relative to the preceding row vector. It is easy to show that all such games have a Nash equilibrium where players randomize between all pure strategies with equal probability (uniformly completely mixed Nash equilibrium). Our main theorems establish the exact number of (pure strategy) Nash equilibria when the first row of each payoff matrix is strictly ordered. We also provide necessary and sufficient conditions for the uniqueness of the uniformly completely mixed Nash equilibrium and for the existence of pure strategy Nash equilibria. As a consequence of our main results we obtain that the maximal number of Nash equilibria in these games is exactly $2^n - 1$. The number of pure strategy Nash equilibria is either 0, 1, 2, or n, and for a specific subclass a pure strategy Nash equilibrium always exists. Further, the best response correspondences induce an equivalence relation on each player's set of pure strategies. In any Nash equilibrium all strategies within one equivalence class are either played with strictly positive or with zero probability. Our proofs provide a recipe on how to derive the equivalence classes and allow us to characterize the support of all Nash equilibrium strategies.

[1] The game is constructed using pairs of dual cyclic polytopes with $2n$ suitably labeled facets in n-space. Such games were coined 'hard to solve'. The Lemke-Howson algorithm, the classical method for finding one Nash equilibrium of a two-player normal-form game, takes a number of steps exponential in the dimension n (see Savani and von Stengel, 2006).

The class of circulant games contains well-known games such as Matching Pennies, Rock-Paper-Scissors, and subclasses of common-interest and coordination games.[2] Recently, several other articles analyzed subclasses of games with a special focus on different notions of cyclicity. Duersch et al. (2012) consider symmetric two-player zero-sum normal-form games and define generalized rock-paper-scissors matrices ($gRPS$) in terms of best response cycles. In their setting, a game has a pure strategy Nash equilibrium if and only if it is not a $gRPS$. Bahel (2012) and Bahel and Haller (2013) examine zero-sum games that are based on cyclic preference relations on the set of actions and characterize the set of Nash equilibria. In the former paper, actions are distinguishable, i.e. one specific actions is the beginning of the cyclic relation, and there exists a unique Nash equilibrium. In the latter, the actions are anonymous, i.e. each action can be seen as the beginning of the cycle without affecting the relation, and depending on the number of actions the Nash equilibrium is unique or there exists an infinite number of Nash equilibria. To the best of our knowledge, games with circulant payoff matrices have not been studied so far.

The remainder of this paper is structured as follows. Section 2 introduces the class of (ordered) circulant games. In Section 3.2, we present the main theorems on the number and structure of Nash equilibria in ordered circulant games. Section 4 presents circulant games which are not ordered but exhibit similar properties as ordered circulant games. Section 5 concludes. All proofs are relegated to the appendix.

2 Circulant Games

Let $\Gamma = ((S_1, S_2), (\pi_1, \pi_2))$ be a finite two-player normal-form game where $S_i = \{0, 1, \ldots, n_i - 1\}$ denotes player i's set of pure strategies and $\pi_i : S_1 \times S_2 \to \mathbb{R}$ denotes player i's payoff function for $i = 1, 2$.[3] We will write player i's payoff function as an $n_1 \times n_2$ matrix. Player i's payoff matrix $A_i = (a^i_{kl})_{k \in S_1, l \in S_2}$ is then given by $a^i_{kl} = \pi_i(k, l)$. Thus in both matrices each row corresponds to a pure strategy of player 1 and each column to a pure

[2] As Quint and Shubik (2002) re-established the upper bound of $2^n - 1$ for the class of coordination games, we would like to point out that the class of circulant games is different from this class. The two classes do, however, have a non-empty intersection.

[3] We choose to label players' strategies from 0 to $n_i - 1$ as this will later simplify notation significantly.

strategy of player 2. Following the notation in e.g. Alós-Ferrer and Kuzmics (2013), we will also write $\pi_i(s|s')$ for player i's payoff if he chooses a strategy s and player $-i$ chooses strategy s'. The set of mixed strategies for player i is denoted by Σ_i. For $\sigma_i \in \Sigma_i$, $\sigma_i(s)$ denotes the probability that σ_i places on the pure strategy $s \in S_i$. The set of all pure strategies played with strictly positive probability is denoted by $\text{supp}(\sigma_i)$. Payoff functions are extended to the sets of mixed strategies by expected payoffs. Given a mixed strategy σ_{-i} of player $-i$, a *best response* for player i against σ_{-i} is a strategy σ_i such that $\pi_i(\sigma_i|\sigma_{-i}) \geq \pi_i(\sigma_i'|\sigma_{-i})$ for all $\sigma_i' \in \Sigma_i$. The set of best responses for player i against a strategy σ_{-i} of the other player is denoted by $BR_i(\sigma_{-i})$. A finite two-player normal-form game is *non-degenerate* if for any mixed strategy σ_i of player i with $\text{supp}(\sigma_i) = m$, player $-i$ has at most m best responses against σ_i. In what follows Γ_n denotes a finite two-player normal-form game in which $S_1 = S_2 = S^n = \{0, \ldots, n-1\}$.

The following two results are well-known and will be used throughout the paper.

Proposition 1 (Best Response Condition Nash, 1951)**.** *Let Γ be a finite two-player normal-form game. Let $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. Then σ_i is a best response to σ_{-i} if and only if for all $s_i \in S_i$*

$$\sigma_i(s_i) > 0 \Rightarrow \pi_i(s_i|\sigma_{-i}) = \max_{s \in S_i} \pi_i(s|\sigma_{-i}).$$

Proposition 2 (Shapley, 1974; Quint and Shubik, 1997)**.** *Let Γ be a finite non-degenerate two-player normal-form game with strategy set $S_1 = S_2 = S$. Then*

(i) Γ has a finite and odd number of Nash equilibria.

(ii) if $T_1, T_2 \subseteq S$ then Γ has at most one Nash equilibrium (σ_1, σ_2) such that $\text{supp}(\sigma_1) = T_1$ and $\text{supp}(\sigma_2) = T_2$.

Before we can introduce circulant games a couple of definitions are necessary.

Definition 1. A matrix $A \in \mathbb{R}^{n \times n}$ is *circulant* if it has the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ a_{n-2} & a_{n_1} & a_0 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$

and *anti-circulant* if

$$A = \begin{pmatrix} a_0 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_0 \\ a_2 & \cdots & a_{n-1} & a_0 & a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \end{pmatrix}$$

We are now ready to define a circulant game.

Definition 2. A two-player normal-form game Γ_n is a *circulant game* if each players' payoff matrix is either circulant or anti-circulant.

Note that if A_i is circulant then $a_{ij} = a_{j-i}$ and if A_i is anti-circulant then $a_{ij} = a_{i+j}$ where the indices are to be read modulo n, e.g. $-1 = n-1$, $n+1 = 1$, etc. In a circulant game, if player 1's payoff matrix is circulant then $\pi_1(s|s') = a^1_{s'-s}$ and if player 1's payoff matrix is anti-circulant then $\pi_1(s|s') = a^1_{s+s'}$. Similarly if player 2's payoff matrix is circulant then $\pi_2(s|s') = a^2_{s-s'}$ and if player 2's payoff matrix is anti-circulant then $\pi_2(s|s') = a^2_{s+s'}$. Throughout the paper the sum and difference of two strategies in a circulant game is to be read modulo n. Similarly, the multiplication of a strategy with an integer is to be read modulo n.

Since in a circulant game the sum of the payoffs in each row and each column is constant, if one player plays the completely uniformly mixed strategy, then all of the others player's pure strategies yield the same payoff. An immediate consequence of this is the following

Lemma 1. *Let Γ_n be a circulant game. Then $\sigma^* = (\sigma_1^*, \sigma_2^*)$ where $\sigma_i^*(s) = 1/n$ for all $s \in S^n$, $i = 1, 2$, is a Nash equilibrium of Γ_n.*

We can classify circulant games according to whether the players' payoff matrices "cycle" in the same or in opposite directions.

Definition 3. A circulant game is *iso-circulant* if the players' payoff matrices are either both circulant or both anti-circulant matrices. It is *counter-circulant* if one player's payoff matrix is circulant and the other player's payoff matrix is anti-circulant.

For $n = 2$ every iso-circulant game is also counter-circulant and vice versa, as any circulant 2×2 matrix is also anti-circulant. For $n \geq 3$, however, the class of iso-circulant games is disjoint from the class of counter-circulant games.

Example 1 (Matching Pennies).
The game given by

$$A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

is the well-know Matching Pennies game. As both players' payoff matrices are circulant (and anti-circulant), it is an iso-circulant (and also a counter-circulant) game and $[(1/2, 1/2), (1/2, 1/2)]$ is a Nash equilibrium of this game. As we will show later it is the unique one.

Example 2 (Rock-Paper-Scissors).
The game given by

$$A_1 = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

is Rock-Paper-Scissors. Strategies are ordered so that for both players strategy 0 corresponds to "Rock", strategy 1 corresponds to "Paper", and strategy 2 corresponds to "Scissors". This is an iso-circulant game and a Nash equilibrium of this game is $[(1/3, 1/3, 1/3), (1/3, 1/3, 1/3)]$. As we will see later it is the unique one.

Example 3 (4×4 Coordination Game).
The game given by

$$A_1 = \begin{pmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 5 \\ 3 & 2 & 5 & 4 \\ 2 & 5 & 4 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 5 \\ 3 & 2 & 5 & 4 \\ 2 & 5 & 4 & 3 \end{pmatrix}$$

is an iso-circulant game and the uniform probability distribution over all pure strategies, $[(1/4, 1/4, 1/4, 1/4), (1/4, 1/4, 1/4, 1/4)]$ constitutes a Nash equilibrium. It is not the only one. As we will see later this game has 15 Nash equilibria.

The following two games are examples of counter-circulant games. In both games player 1's payoff matrix is anti-circulant and player 2's payoff matrix is circulant.

Example 4.

$$A_1 = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 2 & 1 & 4 & 3 \\ 1 & 4 & 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

This is a counter-circulant game. The uniform probability distribution over all pure strategies $[(1/4, 1/4, 1/4, 1/4), (1/4, 1/4, 1/4, 1/4)]$ is a Nash equilibrium of this game. As we will see later this game has 3 Nash equilibria.

Example 5.

$$A_1 = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 5 \\ 3 & 2 & 1 & 5 & 4 \\ 2 & 1 & 5 & 4 & 3 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 2 & 1 & 5 & 4 \\ 4 & 3 & 2 & 1 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 5 & 4 & 3 & 2 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

This is a counter-circulant game. The uniform probability distribution over all pure strategies $[(1/5, 1/5, 1/5, 1/5, 1/5), (1/5, 1/5, 1/5, 1/5, 1/5)]$ is a Nash equilibrium of this game. As we will see later this game has 7 Nash equilibria.

Definition 4. A circulant game Γ_n is *ordered* with shift $1 \leq k \leq n$ if $a^1_0 > a^1_1 > \cdots > a^1_{n-1}$ and either $a^2_{n-k} > a^2_{n-k+1} > \cdots > a^2_{n-1} > a^2_0 > a^2_1 > \cdots > a^2_{n-k-1}$ or $a^2_{n-k} > a^2_{n-k-1} > \cdots > a^2_1 > a^2_0 > a^2_{n-1} > \cdots > a^2_{n-k+1}$.

In an ordered circulant game the entries in the first row of player 1's payoff matrix decrease when moving from left to right. The entries in the first row of player 2's payoff matrix decrease either when moving from the largest payoff to the right, or when moving from the largest payoff to the left. The shift k is determined by the position of the largest payoff in the first row of player 2's payoff matrix. A shift of $k = n$ corresponds to a_0^2 being player 2's largest payoff. A shift of $k = 0$ is of course possible but for notational convenience is formally represented by a shift of $k = n$. Ordered iso-circulant games with shift $k = n$ capture the class of ordered circulant coordination games. Matching Pennies (Example 1) is and example of an ordered iso-circulant (and counter-circulant) game with shift $k = 1$ as for player 2 $a_{n-1}^2 = a_1^2 = 1$ is the largest payoff. Relabeling the strategies in Example 2 such that for player 1, strategy 0 is 'Rock', strategy 1 is 'Scissors', and strategy 2 is 'Paper' and for player 2, strategy 0 is 'Scissors', strategy 1 is 'Rock', and strategy 2 is 'Paper' yields the following payoff matrices:

$$A_1 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

It is now easy to see that (this relabeled version of) Rock-Paper-Scissors is an ordered iso-circulant with shift $k = 1$, as for player 2 $a_{n-1}^2 = a_2^2 = 3$ is the largest payoff. The 4×4 coordination game (Example 3) is an example of an ordered iso-circulant game with shift $k = 4$ as for player 2 $a_{n-4}^2 = a_0^2 = 5$ is the largest payoff.

Example 4 is an ordered counter-circulant game with shift $k = 3$ as for player 2 $a_{n-3}^2 = a_1^2 = 4$ is the largest payoff. Example 5 is an ordered counter-circulant game with shift $k = 2$ as for player 2 $a_{n-2}^2 = a_3^2 = 5$ is the largest payoff.

3 Main Results

In this section we present the main results on the number and the structure of Nash equilibria in ordered circulant games. We start by presenting some preliminary lemmata that we require to state the main results. All proofs are relegated to the appendix.

3.1 Preliminaries

Lemma 2. *Let Γ_n be an ordered circulant game with shift k in which player 1's payoff matrix is anti-circulant and let $d = \gcd(k, n)$.*

(i) If Γ_n is iso-circulant, then in any Nash equilibrium (σ_1, σ_2), for all $s \in S^n$, $\sigma_i(s) = 0$ if and only if $\sigma_i(s+km) = 0$ for all $m = 0, \ldots, \frac{n}{d}-1$, $i = 1, 2$.

(ii) If Γ_n is counter-circulant, then in any Nash equilibrium (σ_1, σ_2), for all $s \in S^n$, $\sigma_1(s) = 0$ if and only if $\sigma_1(-s+k) = 0$ and $\sigma_2(s) = 0$ if and only if $\sigma_2(-s-k) = 0$

Given an ordered iso-circulant game Γ_n, we can define an equivalence relation $\sim$ on the set S^n by $s \sim s'$ if and only if $s = s' + mk$ for some $0 \le m \le \frac{n}{d} - 1$, where $d = \gcd(n, k)$. Denote the equivalence class of $s \in S^n$ by $I(s)$. Note that, $s' + m_1 k \ne s' + m_2 k$ for all $0 \le m_1 < m_2 \le \frac{n}{d} - 1$. Hence $I(s) = \{s + mk | 0 \le m \le \frac{n}{d} - 1\}$ contains n/d elements and there are d different equivalence classes. Let $I(S^n) = \{I(s) | s \in S^n\}$ be the set of equivalence classes. Suppose player 1's payoff matrix is anti-circulant. By Lemma 2(i) two strategies are equivalent if and only if in any Nash equilibrium either both are simultaneously played with positive probability or both are simultaneously played with zero probability.

For an ordered counter-circulant game let $C_1(s) = \{s, -s+k\}$ and $C_2(s) = \{s, -s-k\}$ for all $s \in S^n$. Note that any class $C_1(s)$ contains at least one and at most two elements. It contains one element if $-s+k \equiv s \mod n$ and two elements if $-s+k \not\equiv s \mod n$. The former occurs if and only if either $2s = k$ or $2s = n + k$. Thus there is a singleton class if and only if either $\frac{k}{2} \in S^n$ or $\frac{(n+k)}{2} \in S^n$, i.e. if either k or $(n + k)$ is an even number. In particular there can be at most two singleton classes. Similarly, any class $C_2(s)$ contains one element if $-s - k \equiv s \mod n$ and two elements if $-s - k \not\equiv s \mod n$. The former occurs if and only if either $2s = n - k$ or $2s = 2n - k$. Thus there is a singleton class if and only if either $n - k$ or $2n - k$ is an even number which holds if and only if either k or $(n + k)$ is an even number. We define $C_i(S^n) := \{C_i(s) | s \in S^n\}$, $i = 1, 2$. Suppose player 1's payoff matrix is anti-circulant. Then, by Lemma 2(ii), $s' \in C_i(s)$ if and only if in any Nash equilibrium either both are simultaneously played with positive probability or both are simultaneously played with zero probability.

For an ordered iso-circulant game, the set $I(S^n)$ is a partition of S^n by construction. It can be shown (Lemma B3 in the appendix) that this is also true in the case of ordered counter-circulant games and $C_1(S^n)$ and $C_2(S^n)$, respectively.

The following lemma covers the connection between the support of a strategy of player i and the best response of player $-i$ against that strategy.

Lemma 3. *Let Γ_n be an ordered circulant game in which player 1's payoff matrix is anti-circulant.*

(i) If Γ_n is iso-circulant then if $\sigma_i \in \Sigma_i$ and $I(s) \in I(S^n)$ are such that $\mathrm{supp}(\sigma_i) \cap I(s) = \emptyset$ then $BR_{-i}(\sigma_i) \cap I(-s) = \emptyset$.

(ii) If Γ_n is counter-circulant then if $\mathrm{supp}(\sigma_{-i}) \cap C_{-i}(s) = \emptyset$ for $C_{-i}(s) \in C_{-i}(S^n)$ then $BR_i(\sigma_{-i}) \cap C_i(-s) = \emptyset$.

The next lemma shows that only specific subsets of S^n can arise as the support of a Nash equilibrium strategy of player 1.

Lemma 4. *Let Γ_n be an ordered circulant game in which player 1's payoff matrix is anti-circulant.*

(i) If Γ_n is iso-circulant then for any union $U = \bigcup_{j=1}^m I(s^j)$ of elements of $I(S^n)$ there is a unique Nash equilibrium (σ_1, σ_2) such that $\mathrm{supp}(\sigma_1) = U$. Further, for any Nash Equilibrium (σ_1, σ_2) there is a union $U = \bigcup_{j=1}^m I(s^j)$ of elements of $I(S^n)$ such that $\mathrm{supp}(\sigma_1) = U$.

(ii) If Γ_n is counter-circulant then for any union $U = \bigcup_{j=1}^m C_1(s^j)$ of elements of $C_1(S^n)$ there is a unique Nash equilibrium (σ_1, σ_2) such that $\mathrm{supp}(\sigma_1) = U$. Further, for any Nash Equilibrium (σ_1, σ_2) there is a union $U = \bigcup_{j=1}^m C_1(s^j)$ of elements of $C_1(S^n)$ such that $\mathrm{supp}(\sigma_1) = U$.

3.2 The Number of Nash Equilibria

Theorem 1. *Let Γ_n be an ordered iso-circulant game with shift k and let $d = \gcd(k, n)$ denote the greatest common divisor of k and n. Then Γ_n has $2^d - 1$ Nash equilibria.*

Since by definition $k \leq n$, necessarily $\gcd(k, n) \leq n$. It follows that an ordered iso-circulant game can have at most $2^n - 1$ Nash equilibria. Further, an ordered iso-circulant game has a unique Nash equilibrium if and only if $\gcd(k, n) = 1$. Together with Lemma 1, this implies that if $\gcd(k, n) = 1$

then the unique Nash equilibrium is the one where both players place equal probability on each pure strategy. Some immediate consequences of these results are the following.

"Matching Pennies" (Example 1) is an ordered iso-circulant game with shift $k = 1$. Hence, $[(1/2, 1/2), (1/2, 1/2)]$ is the unique Nash equilibrium. "Rock-Paper-Scissors" (Example 2) is an ordered iso-circulant game with shift $k = 1$. Hence, the unique Nash equilibrium is $[(1/3, 1/3, 1/3), (1/3, 1/3, 1/3)]$.

Proposition 3. *Let Γ_n be an ordered iso-circulant game with shift k. Γ_n has n pure strategy Nash equilibria if and only if $k = n$. Further, Γ_n has no pure strategy Nash equilibrium if and only if $k \neq n$.*

By the last proposition an ordered iso-circulant game Γ_n has either 0 or n pure strategy Nash equilibria. The "4×4 coordination " (Example 3) is an ordered iso-circulant game with shift $k = 4$. As $\gcd(4, 4) = 4$, by Theorem 1, this game has $2^4 - 1 = 15$ Nash equilibria. By Proposition 3 four of these are in pure strategies.

Theorem 2. *Let Γ_n be an ordered counter-circulant game with shift k.*

(i) If n is odd, then Γ_n has exactly $2^{\frac{n+1}{2}} - 1$ Nash equilibria.

(ii) If both n and k are even, then Γ_n has exactly $2^{\frac{n}{2}+1} - 1$ Nash equilibria.

(iii) If n is even and k is odd, then Γ_n has exactly $2^{\frac{n}{2}} - 1$ Nash equilibria.

It follows that an ordered counter-circulant game can have at most $2^{\frac{n}{2}+1} - 1$ Nash equilibria. Further, an ordered counter-circulant game has a unique Nash equilibrium if and only if $n = 2$ and $k = 1$. Example 4 is an ordered counter-circulant game with shift $k = 3$. As n is even and k is odd, by Theorem 2(iii) the game has $2^2 - 1 = 3$ Nash equilibria. Example 5 is an ordered counter-circulant game with shift $k = 2$. As n is odd, by Theorem 2(i) the game has $2^3 - 1 = 7$ Nash equilibria.

Proposition 4. *Let Γ_n be an ordered counter-circulant game with shift k.*

(i) Γ_n has a exactly one pure strategy Nash equilibrium if and only if n is odd.

(ii) Γ_n has exactly two pure strategy Nash equilibria if and only if both n and k are even.

(iii) Γ_n *has no pure strategy Nash equilibrium if and only if* n *is even and* k *is odd.*

In Example 4 n is even and k is odd, and by Proposition 4(iii) none of its three Nash equilibria are in pure strategies. In Example 5 n is odd, and by Proposition 4(i) one of its seven Nash equilibria is in pure strategies.

It follows from (i) and (ii) in Proposition 4 that the class of ordered counter-circulant games with even shift is a class of games for which a pure strategy Nash equilibrium always exists.

3.3 The Structure of Nash Equilibria

By Lemma 4, there exists a straightforward way to characterize the support of all Nash equilibrium strategies for a given ordered circulant game. Moreover, once we know what to look for the weights of the strategies in the support can be easily derived.

Consider first the case of an ordered iso-circulant game with n and k, and let $d = \gcd(n, k)$. By Lemma A1(i) in the appendix we can transform the game so that player 1's payoff matrix is anti-circulant. The circulant structure of the payoff matrices allows us to define an equivalence relation on the set of pure strategies S for each player. For a pure strategy $s \in S^n$, the corresponding equivalence class $I(s) = \{s + mk | 0 \leq m \leq \frac{n}{d} - 1\}$ contains n/d elements and there are d different equivalence classes. In any Nash equilibrium all strategies within one equivalence class are either played with strictly positive or with zero probability. It follows from Lemma 4(i) that in any Nash equilibrium the support of either player's strategy is the union of classes in $I(S^n) = \{I(s) | s \in S^n\}$ and further that for any such union of classes in $I(S^n)$ there is a unique Nash equilibrium in which player 1's strategy has this union as its support. Further, if the mixed strategy profile (σ_1, σ_2) is a Nash equilibrium with $\text{supp}(\sigma_1) = \bigcup_{j=1}^{m} I(s^j)$ for some strategies $s^1, \ldots, s^m \in S^n$ then by Lemma 3(i) it follows that $\text{supp}(\sigma_2) = \bigcup_{j=1}^{m} I(-s^j)$. The actual probabilities put on each strategy of course depend on the actual payoffs, however, the structure of the supports is the same for all ordered iso-circulant games with the same shift and the same number of pure strategies.

Let us revisit the "4×4 coordination" game from Example 3. Here, $n = k = d = 4$ and hence there are four (singleton) classes: $I(0) = \{0\}$, $I(1) = \{1\}$, $I(2) = \{2\}$, and $I(3) = \{3\}$. Each class is part of a (pure strategy) Nash equilibrium in which $\text{supp}(\sigma_1) = I(s)$ and $\text{supp}(\sigma_2) = I(-s)$, and there are four such combinations. E.g., in one Nash equilibrium player 1 plays the strategy $s = 1$, i.e. plays the equivalence class $I(1)$ and player 2 plays $s = 3$, the equivalence class $I(-1) = I(3)$. Analogously, the three remaining pure strategy Nash equilibria are given by the profiles $(0, 0)$, $(2, 2)$, and $(3, 1)$. Further, each union of two classes is part of a (mixed strategy) Nash equilibrium in which $\text{supp}(\sigma_1) = I(s^1) \cup I(s^2)$ and $\text{supp}(\sigma_2) = I(-s^1) \cup I(-s^2)$. There are six such combinations. E.g., in one Nash equilibrium player 1 puts positive probability only on $I(0)$ and $I(1)$ and player 2 puts positive probability on $I(-0) = I(0)$ and $I(-1) = I(3)$. The probabilities are easily derived from the corresponding indifference conditions and the Nash equilibrium strategy profile is $[(1/4, 3/4, 0, 0), (3/4, 0, 0, 1/4)]$. The remaining five Nash equilibria in which the support of player 1's strategy is the union of two classes are given by the profiles $[(1/2, 0, 1/2, 0), (1/2, 0, 1/2, 0)]$, $[(3/4, 0, 0, 1/4), (1/4, 3/4, 0, 0)]$, $[(0, 1/4, 3/4, 0), (0, 0, 1/4, 3/4)]$, $[(0, 1/2, 0, 1/2), (0, 1/2, 0, 1/2)]$, $[(0, 0, 1/4, 3/4), (0, 1/4, 3/4,)]$. Analogously, there are four Nash equilibria in which the support of player 1's (and player 2's) strategy is the union of three classes: $[(1/4, 1/4, 1/2, 0), (1/2, 0, 1/4, 1/4)]$, $[(1/4, 1/2, 0, 1/4), (1/4, 1/2, 0, 1/4)]$, $[(1/2, 0, 1/4, 1/4), (1/4, 1/4, 1/2, 0)]$, $[(0, 1/4, 1/4, 1/2), (0, 1/4, 1/4, 1/2)]$. Finally, there is one Nash equilibrium where player 1's (and player 2's) strategy is completely mixed: $[(1/4, 1/4, 1/4, 1/4), (1/4, 1/4, 1/4, 1/4)]$.

Consider now the case of an ordered counter-circulant game with given n and k. We can apply Lemma A1(ii) in the appendix to transform this game so that player 1's payoff matrix is anti-circulant. As in the previous case, we can define an equivalence relation on set of pure strategies for each player. For all $s \in S$ let $C_1(s) = \{s, -s + k\}$ denote the corresponding equivalence class of player 1 and $C_2(s) = \{s, -s - k\}$ the one of player 2. Note that any class $C_1(s)$, $C_2(s)$ contains at least one and at most two elements. It follows from Lemma 4(ii) that in any Nash equilibrium the support of player 1's strategy is a union of classes $C_1(S^n)$ and that for any union of classes in $C_1(S^n) = \{C_1(s) | s \in S^n\}$ there is a Nash equilibrium in which the support of player 1's strategy has this union as its support. Further, if (σ_1, σ_2) is a Nash equilibrium with $\text{supp}(\sigma_1) = \bigcup_{j=1}^{m} C_1(s^j)$ for some strategies $s^1, \ldots, s^m \in S^n$ then by Lemma 3(ii) it follows that $\text{supp}(\sigma_2) = \bigcup_{j=1}^{m} C_2(-s^j)$.

Let us revisit the game in Example 4. Here, $n = 4$ and $k = 3$. There are two classes for player 1: $C_1(0) = C_1(3) = \{0, 3\}$ and $C_1(1) = C_1(2) = \{1, 2\}$. Correspondingly there are two classes for player 2: $C_2(0) = C_2(1) = \{0, 1\}$ and $C_2(2) = C_2(3) = \{2, 3\}$. There are two Nash equilibria in which the support of player 1's (and player 2's) strategy consists of a single class: $[(1/4,0,0,3/4),(1/4,3/4,0,0)]$,$[(0,3/4,1/4,0),(0,0,1/4,3/4)]$. Further there is one equilibrium in which player 1 (and player 2) plays a completely mixed strategy: $[(1/4,1/4,1/4,1/4),(1/4,1/4,1/4,1/4)]$.

4 Generalizations

By our definition there are games that are not ordered iso-circulant (counter-circulant) games, but that can be transformed into one by a simple relabeling of strategies. We chose to exclude those games from our definition for ease of exposition. However, the results presented above also apply for these games.

It is not necessary to insist on each row containing the same entries. All our proof go through if payoffs are transformed in a way that preserves the order of entries in each row and in each column of the payoff matrices.

Example 6. In the 3×3 with payoff matrices

$$A_1 = \begin{pmatrix} 3.1 & 1.9 & 0.8 \\ 1.5 & 0.9 & 3.4 \\ 0.5 & 3.2 & 2.1 \end{pmatrix}, A_2 = \begin{pmatrix} 0.7 & 2.2 & 3.5 \\ 1.8 & 2.6 & 0.1 \\ 3 & 0.5 & 2.8 \end{pmatrix}.$$

the order of payoffs in each row and in each column is the same as in Rock-Paper-Scissors (Example 2). The proof of Theorem 1 can easily be generalized to this case to show that this game has a unique Nash equilibrium. As the sum of payoffs in each row is not constant, however, the unique Nash equilibrium is not the strategy profile in which both players play the uniformly completely mixed strategies.

In this sense, our results on the number and the structure of Nash equilibria only depend on the order of payoffs in the rows and columns of the payoff matrices.

Our results further generalize to coordination games in which players obtain a strictly positive payoff if and only if they use the same strategy and a payoff of 0 otherwise i.e., so-called games of pure coordination. The resulting payoff matrices are of the form

$$A_1 = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \end{pmatrix} A_2 = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \end{pmatrix}$$

Proving that such games have $2^n - 1$ Nash equilibria works analogously to the proof of Theorem 1.

5 Conclusion

In this paper we introduce and investigate a class of two-player normal-form games we coin circulant games. Circulant games have a straightforward representation in form of circulant matrices. Each player's payoff matrix is fully characterized by a single row vector, which appears as the first row of the matrix. The remaining rows are obtained through cyclic permutations of the first line such that a row vector is rotated by one element relative to the preceding row vector. All circulant games have a Nash equilibrium where players randomize between all pure strategies with equal probability (uniformly completely mixed Nash equilibrium).

If the first row of each payoff matrix is strictly ordered (*ordered circulant games*), the circulant structure underlying the payoff matrices has some interesting implications. First, the best response correspondences induce a partition on each players' set of pure strategies into equivalence classes. In any Nash Equilibrium all strategies within one class are either played with strictly positive or with zero probability. Second, there exist a simple one-to-one correspondence between the players' equivalence classes. If some player puts zero probability on one class (i.e. plays all pure strategies within one class with zero probability), the other has one corresponding equivalence class he plays with zero probability. Finally, a single parameter k fully determines the strategy classes and the relation between the players' classes. The parameter itself only depends on the position of the largest payoff in the first row of player 2's payoff matrix. For a given ordered circulant game, knowing k and the number of pure strategies n suffices to calculate the number of Nash equilibria and to describe the support of all Nash equilibrium strategies. As an immediate consequence of our main results we reestablish $2^n - 1$ as the tight upper bound on the number of Nash equilibria in these games.

The class of ordered circulant games contains well-known games such as Matching Pennies and Rock-Paper-Scissors. Our approach shifts the focus of these two games away from their zero-sum property towards the circulant structure of the corresponding payoff matrices. Hereby, we shed new light on their connection. Matching Pennies is simply the two-strategy variant of Rock-Paper-Scissors. Within our framework the two games belong to the same sub-class of ordered circulant games. Both games are characterized by $k = 1$ and the only Nash equilibrium is the uniformly completely mixed one. The common denominator that connects these games is the balanced payoff structure induced by the circulant matrices with a shift of $k = 1$. Moreover, this reinterpretation is robust in the sense that only relative payoffs matter. We can write down Rock-Paper-Scissors in many variants, including asymmetric evaluations of wins or losses, variants that cannot be transformed into zero-sum games. Yet, the balanced structure is preserved and the best players can do is to randomize between all pure strategies with equal probability.

A different way to interpret our results is from the perspective of Social Choice Theory. As for example in Gibbard (1974), we can decompose social states into different components representing private spheres of the individuals. If we require a collective choice rule to respect liberalism with respect to private spheres, then individuals should be decisive over their own private spheres (i.e. choose their strategies) and should not intervene with private spheres of the other individuals (i.e. given the other players' strategies). In this context, social states correspond to strategy profiles, and private spheres to players' strategies. If the there is no pure strategy equilibrium in the corresponding game, the collective choice set is empty (see e.g. Gaertner, 2006, Chapter 4.4). Hence we can interpret the subclass of ordered circulant games with pure strategy Nash equilibria as a domain restriction on the collective choice rule such that the choice set is not empty.

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A Transformation of Games

Lemma A1. *(i) Let Γ_n be an iso-circulant game in which both players' payoff matrices are circulant. There is a permutation of row vectors that fixes the first row in both matrices and transforms both player's payoff matrices into anti-circulant matrices.*

(ii) Let Γ_n be a counter-circulant game in which player 1's payoff matrix is circulant. There is a permutation of row vectors that fixes that first row in both matrices and transforms player 1's payoff matrix into an anti-circulant matrix and player 2's matrix into a circulant matrix.

Proof. (i) A matrix A is anti-circulant if and only if $A = PC$, where C is a circulant matrix and

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

(Davis, 1979, p. 162, Corollary). The matrix P switches rows i and $n+1-i$ and fixes the first row. Using this result, we obtain that PA_1 and PA_2 are anti-circulant matrices since both A_1 and A_2 are circulant matrices.

(ii) Using the matrix P defined as in (i), we obtain that PA_1 is anti-circulant (Davis, 1979, p. 162, Corollary). As A_2 is anti-circulant, $A_2 = PC$ for some circulant matrix C (Davis, 1979, p. 162, Corollary). Hence $PA_2 = P(PC)$ and since $P = P^{-1}$ (Davis, 1979, p.28, equ. (2.4.22)), we obtain that PA_2 is a circulant matrix. □

B Proofs

Lemma B2. *Let Γ_n be an ordered circulant game with shift k in which player 1's payoff matrix is anti-circulant.*

(i) For all $\sigma_2 \in \Sigma_2$ and all $s \in S^n$ if $\sigma_2(s) = 0$ then $-s \notin BR_1(\sigma_2)$.

(ii) If Γ_n is iso-circulant, then for all $\sigma_1 \in \Sigma_1$ and all $s \in S^n$ if $\sigma_1(s) = 0$ then $(-s-k) \notin BR_2(\sigma_1)$.

(iii) If Γ_n is counter-circulant, then for all $\sigma_1 \in \Sigma_1$ and all $s \in S^n$ if $\sigma_1(s) = 0$ then $(s-k) \notin BR_2(\sigma_1)$.

Proof. (i) Let $\sigma_2 \in \Sigma_2$ be such that $\sigma_2(s) = 0$ for some $s \in S^n$. Since player 1's payoff matrix is anti-circulant $\pi_i(s|s') = a^1_{s+s'}$. By definition, for player 1 $a^1_{s'-s} < a^1_{s'-s-1}$ for all $s' \neq s$. Hence, as $\sigma_2(s) = 0$,

$$\begin{aligned}\pi_1(-s|\sigma_2) &= \sum_{s' \neq s} \sigma_2(s') a^1_{s'-s} \\ &< \sum_{s' \neq s} \sigma_2(s') a^1_{s'-s-1} = \pi_1(-s-1|\sigma_2),\end{aligned}$$

which implies that $-s \notin BR_1(\sigma_2)$.

(ii) Let $\sigma_1 \in \Sigma_1$ and $s \in S^n$ be such that $\sigma_1(s) = 0$. Since player 2's payoff matrix is anti-circulant, $\pi_2(s|s') = a^2_{s'+s}$ for $s, s' \in S$. Since Γ_n is an ordered circulant game, either $a^2_{n-k} > a^2_{n-k+1} > \cdots > a^2_{n-1} > a^2_0 > a_1 > \cdots > a^2_{n-k-1}$ or $a^2_{n-k} > a^2_{n-k-1} > \cdots > a^2_1 > a^2_0 > a_{n-1} > \cdots > a^2_{n-k+1}$ and hence either $a^2_{s'} < a^2_{s'-1}$ or $a^2_{s'} < a^2_{s'+1}$ for all $s' \neq n-k$. We will only prove the result for the former case as the proof for the latter works analogously (by using the inequality $a^2_{s'} < a^2_{s'+1}$ for all $s' \neq n-k$ instead of $a^2_{s'} < a^2_{s'-1}$). Since $a^2_{s'} < a^2_{s'-1}$ we obtain $a^2_{s'-s-k} < a^2_{s'-s-k-1}$ for all $s' \neq s$ and hence

$$\begin{aligned}\pi_2(-s-k|\sigma_1) &= \sum_{s' \neq s} \sigma_1(s') a^2_{s'-s-k} \\ &< \sum_{s' \neq s} \sigma_1(s') a^2_{s'-s-k-1} = \pi_2(-s-k-1|\sigma_1),\end{aligned}$$

so $(-s-k) \notin BR_2(\sigma_1)$.

(iii) Let $\sigma_1 \in \Sigma_1$ and $s \in S^n$ be such that $\sigma_1(s) = 0$. Since player 2's payoff matrix is circulant, $\pi_2(s|s') = a^2_{s'-s}$ for $s, s' \in S$. Since Γ_n is an ordered circulant game, by definition either $a^2_{n-k} > \cdots > a^2_{n-1} > a^2_0 > \cdots > a^2_{n-k-1}$ or $a^2_{n-k} > a^2_{n-k-1} > \cdots > a^2_0 > a^2_{n-1} > \cdots > a^2_{n-k+1}$ and hence $a^2_{s'} < a^2_{s'-1}$ or $a^2_{s'} < a^2_{s'+1}$ for all $s' \neq n-k$. We will only prove the result for the former case as the proof for the latter works analogously. Since $a^2_{s'} < a^2_{s'-1}$ we obtain $a^2_{s-k-s'} < a^2_{s-k-1-s'}$ for all $s' \neq s$ and hence

$$\begin{aligned}\pi_2(s-k|\sigma_1) &= \sum_{s'\neq s} \sigma_1(s') a^2_{s-k-s'} \\ &< \sum_{s'\neq s} \sigma_1(s') a^2_{s-k-1-s'} = \pi_2(s-k-1|\sigma_1),\end{aligned}$$

so $(s-k) \notin BR_2(\sigma_1)$. □

The lemma makes a statement about best responses of player i given that player $-i$ plays some strategy with zero probability provided that player 1's payoff matrix is anti-circulant. By (i) if player 2 plays a strategy s with probability 0 then for player 1 strategy $-s$ cannot be a best response. Similarly, (ii) and (iii) state that if in an iso-circulant (counter-circulant) game player 1 places probability 0 on strategy s then $-s-k$ $(s-k)$ cannot be a best response for player 2. The idea behind the proof is very simple. For player 1 strategy $-s-1$ yields a strictly higher payoff than $-s$ against any pure strategy of player 2 except s. Hence, if player 2 places probability 0 on s then $-s$ cannot be a best response. Analogously, for player 2 strategy $-s-k-1$ $(s-k-1)$ yields a strictly higher payoff than $-s-k$ $(s-k)$ against any pure strategy of player 1 except s.

Proof of Lemma 2. (i) The "if" part is trivial. To see the "only if" part let (σ_1, σ_2) be a Nash equilibrium of Γ_n and let $s \in S^n$ be such that $\sigma_1(s) = 0$. By Lemma B2(ii), $\sigma_2(-s-k) = 0$ and consequently by Lemma B2(i) $\sigma_1(s+k) = 0$. Iterating this argument yields $\sigma_1(s+mk) = 0$ for all $m = 0, \ldots, \frac{n}{d} - 1$. If $\sigma_2(s) = 0$ the argument works analogously.

(ii) By Lemma B2(i) and (iii) for any Nash equilibrium (σ_1, σ_2) and any $s \in S^n$ we obtain

$$\sigma_1(s) = 0 \Rightarrow \sigma_2(s-k) = 0 \Rightarrow \sigma_1(-s+k) = 0$$

and

$$\sigma_1(-s+k) = 0 \Rightarrow \sigma_2(-s) = 0 \Rightarrow \sigma_1(s) = 0.$$

Analogously, for player 2, we obtain

$$\sigma_2(s) = 0 \Rightarrow \sigma_1(-s) = 0 \Rightarrow \sigma_2(-s-k) = 0$$

and

$$\sigma_2(-s-k) = 0 \Rightarrow \sigma_1(s+k) = 0 \Rightarrow \sigma_2(s) = 0.$$

□

Lemma B3. *Let Γ_n be an ordered counter-circulant game. For $i = 1, 2$ the set $C_i(S^n)$ is a partition of S^n.*

Proof. We will prove the result for $i = 1$ as the proof for $i = 2$ works analogously. Since $s \in C_1(s)$ for all $s \in S^n$, it follows that $\bigcup_{s \in S^n} C_1(s) = S^n$. If there is $\overline{s} \in C_1(s) \cap C_1(s')$ for some $s, s' \in S^n$, then then since $\overline{s} \in C_1(s)$ either $\overline{s} = s$ or $\overline{s} = -s + k$. If $\overline{s} = s$ then $C_i(s) = C_i(\overline{s})$. If $\overline{s} = -s + k$ then $-\overline{s} + k = s - k + k = s$. In any case it follows that $C_1(\overline{s}) = C_1(s)$. Using the same argument one obtains $C_1(\overline{s}) = C_1(s')$ and hence that $C_1(s) = C_1(s')$. □

Proof of Lemma 3. (i) First, let $s \in S^n$ be such that $\mathrm{supp}(\sigma_1) \cap I(s) = \emptyset$. By Lemma B2(ii), $-s - (m+1)k \notin BR_2(\sigma_1)$ for all $0 \le m \le n/d - 1$. As $\{-s - (m+1)k | 0 \le m \le n/d - 1\} = I(-s)$ we obtain $BR_2(\sigma_1) \cap I(-s) = \emptyset$.

Next, let $s \in S^n$ be such that $\mathrm{supp}(\sigma_2) \cap I(s) = \emptyset$. By Lemma B2(i), $-s - mk \notin BR_1(\sigma_2)$ for all $0 \le m \le n/d - 1$. As $\{-s - mk | 0 \le m \le n/d - 1\} = I(-s)$ we obtain $BR_1(\sigma_2) \cap I(-s) = \emptyset$.

(ii) If $\mathrm{supp}(\sigma_{-i}) \cap C_{-i}(s) = \emptyset$ for $C_{-i}(s) \in C_{-i}(S^n)$, then, since $C_{-i}(s) = \{s, -s + (-1)^{i-1}k\}$, by Lemma B2(i) and (iii), $-s, s + (-1)^{i-1}k \notin BR_i(\sigma_{-i})$. Hence $BR_i(\sigma_{-i}) \cap C_i(-s) = \emptyset$. □

Lemma B4. *Let Γ_n be an ordered iso-circulant game in which both players' payoff matrices are anti-circulant. For every $s \in S^n$, there is a Nash equilibrium (σ_1, σ_2) such that $\mathrm{supp}(\sigma_1) = I(s)$ and $\mathrm{supp}(\sigma_2) = I(-s)$.*

Proof. Given $\overline{s} \in S^n$, define $\sigma_1(s) = d/n$ for all $s \in I(\overline{s})$ and $\sigma_2(s) = d/n$ for all $s \in I(-\overline{s})$. By construction $\mathrm{supp}(\sigma_1) = I(\overline{s})$ and $\mathrm{supp}(\sigma_2) = I(-\overline{s})$. By Lemma 3(i), any strategy outside $I(\overline{s})$ cannot be a best response for player 1 against σ_2 and any strategy outside $I(-\overline{s})$ cannot be a best response for player 2 against σ_1. Further, $\pi_1(s|\sigma_2) = \sum_{m=0}^{n/d-1} \frac{d}{n} a_{s+\overline{s}+mk} = \pi_1(s'|\sigma_2)$ for all $s, s' \in I(\overline{s})$ and analogously $\pi_2(s|\sigma_1) = \pi_2(s'|\sigma_1)$ for all $s, s' \in I(-\overline{s})$. Proposition 1 yields that (σ_1, σ_2) is a Nash equilibrium of Γ_n. □

The next lemma establishes a result for ordered counter-circulant games which is analogous to Lemma B4 for ordered iso-circulant games.

Lemma B5. *Let Γ_n be an ordered counter-circulant game in which player 1's payoff matrix is anti-circulant and let $\sigma = (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$.*

(i) $C_i(s)$ is a singleton if and only if $C_{-i}(-s)$ is a singleton.

(ii) For every $s \in S^n$, there is a Nash equilibrium (σ_1, σ_2) such that $\text{supp}(\sigma_1) = C_1(s)$ and $\text{supp}(\sigma_2) = C_2(-s)$.

Proof. (i) Suppose that $C_i(s)$ is a singleton. By construction, $s \equiv -s + (-1)^{i-1}k \mod n$ which is equivalent to $-s \equiv s + (-1)^i k \mod n$. This holds if and only if $C_{-i}(-s)$ is a singleton.

(ii) Note that this follows from (i) and Lemma 3(ii) if $C_1(s)$ is a singleton set. Hence, suppose that $C_1(s) = \{s, -s+k\}$ contains two elements. Then, by (i), $C_2(-s) = \{-s, s-k\}$ contains two elements and neither $2s = k$ nor $2s = n+k$. Choose $\sigma_1(s)$ as the solution to $xa^2_{-2s} + (1-x)a^2_{-k} = xa^2_{-k} + (1-x)a^2_{2s-2k}$, i.e.

$$\sigma_1^s(s) = \frac{a^2_{2s-2k} - a^2_{n-k}}{a^2_{2s-2k} - a^2_{n-k} + a^2_{n-2s} - a^2_{n-k}}.$$

By definition a^2_{n-k} is player 2's largest payoff implying that $a^2_{2s-2k} - a^2_{n-k} < 0$ since $2s \neq n+k$ and that $a^2_{n-2s} - a^2_{n-k} < 0$ since $2s \neq k$. Hence $\sigma_1(s) \in]0,1[$.

Choose $\sigma_2^s(-s)$ as the solution to $xa^1_0 + (1-x)a^1_{2s-k} = xa^1_{-2s+k} + (1-x)a^1_0$, i.e.

$$\sigma_2^s(-s) = \frac{a^1_0 - a^1_{2s-k}}{a^1_0 - a^1_{2s-k} + a^1_0 - a^1_{-2s+k}}.$$

By definition a^1_0 is player 1's largest payoff. Hence as $2s \neq k$ $a^1_0 - a^1_{2s-k} > 0$ and $a^1_0 - a^1_{-2s+k} > 0$ implying that $\sigma_2(-s) \in]0,1[$. By Lemma 3(ii) and Proposition 1, (σ_1, σ_2) is a Nash equilibrium. □

The set $C_1(S)$ can be viewed as a partition of the strategy set for player 1 while $C_2(S)$ is a partition of the strategy set for player 2. By (i) a class $C_1(s)$ of player 1 "corresponds" to a class $C_2(-s)$ of player 2 in the sense that if player 1 puts probability 0 on all strategies in $C_1(s)$ then none of the strategies in $C_2(-s)$ are a best response for player 2 and vice versa. Part (ii) states that two "corresponding" classes contain the same number of elements.

By (iii) for every class $C_1(s)$ there is always a Nash Equilibrium such that player 1's strategy has this class as its support while player 2's strategy has support $C_2(-s)$. The equilibrium constructed to prove (ii) is such that player 1 chooses his strategy (with support $C_1(s)$) such that player 2 is indifferent between all strategies in $C_2(-s)$ (and vice versa). As Γ_n is a non-degenerate game, by Proposition 2(ii) this is the unique equilibrium (σ_1, σ_2) such that $\text{supp}(\sigma_1) = C_1(s)$ and $\text{supp}(\sigma_2) = C_2(-s)$.

In Example 4 there are 2 classes $C_1(s)$ (and by Lemma B4(i) also $C_2(s)$) in S^n. The classes are $C_1(0) = \{0, 3\}$, $C_1(1) = \{1, 2\}$, and $C_2(0) = \{0, 1\}$, $C_2(2) = \{2, 3\}$. As there are no singleton classes, there are no pure strategy equilibria.

In Example 5 there are 3 classes $C_1(s)$ (and by Lemma B4(i) also $C_2(s)$) in S^n. These are $C_1(0) = \{0, 2\}$, $C_1(1) = \{1\}$, $C_1(3) = \{3, 4\}$, and $C_2(0) = \{0, 3\}$, $C_2(2) = \{1, 2\}$, $C_2(4) = \{4\}$. The game has one pure-strategy Nash equilibrium: $(1, 4)$.

Having determined the support of an equilibrium strategy for player 1 Lemmata B4 and B5 can be used to determine the support of player 2's equilibrium strategy.

Proposition B1. *For the two-player normal-form game Γ_n let $\overline{S}_1 = \{[s]_1 | s \in S^n\}$ and $\overline{S}_2 = \{[s]_2 | s \in S^n\}$ be partitions of S^n such that $|\overline{S}_1| = |\overline{S}_2|$. If Γ_n, $\overline{S}_1$, and $\overline{S}_2$ satisfy*

(a) for all Nash equilibria (σ_1, σ_2), and all $s, s' \in S$, if $s' \in [s]_i$ then $\sigma_i(s) = 0$ if and only if $\sigma_i(s') = 0$,

(b) for all $\sigma_i \in \Sigma_i$, $i = 1, 2$, $\text{supp}(\sigma_i) \cap [s]_i = \emptyset$ for $[s]_i \in \overline{S}_i$ implies $BR_{-i}(\sigma_i) \cap [-s]_{-i} = \emptyset$,

(c) for all $s \in S^n$, Γ_n has a Nash equilibrium (σ_1, σ_2) with $\text{supp}(\sigma_1) = [s]_1$ and $\text{supp}(\sigma_2) = [-s]_2$,

then

(i) for any $M \subseteq \overline{S}_1$ Γ_n has a unique Nash equilibrium (σ_1, σ_2) with $\text{supp}(\sigma_1) = \bigcup_{[s]_1 \in M} [s]_1$;

(ii) Γ_n has exactly $2^{|\overline{S}_1|} - 1$ Nash equilibria.

This proposition is the central part of the proof of Theorems 1 and 2. Using the results in the previous sections it follows immediately that its hypotheses (in particular properties (a)-(c)) are satisfied by ordered iso-circulant and ordered counter-circulant games.

Proof. (i) Given $\emptyset \neq M \subseteq \overline{S}_1$ let $-M := \{[-s]_2 | [s]_1 \in M\} \subseteq$ and let Γ_n^M be the reduced game where player 1's set of strategies is $\bigcup_{[s]_1 \in M} [s]_1$ and player 2's set of strategies is $\bigcup_{[s]_1 \in M} [-s]_2$ (and the payoff functions are restricted accordingly).

Claim A: Let $M' \subseteq M \subseteq \overline{S}_1$ be a nonempty subset of $\overline{S}_1$ and let $(\sigma_1^{M'}, \sigma_2^{M'})$ be a completely mixed Nash equilibrium of $\Gamma_n^{M'}$. Then (σ_1^M, σ_2^M) defined by $\sigma_1^M(s) = \sigma_1^{M'}(s)$ if $[s]_1 \in M'$ and $\sigma_1^M(s) = 0$ otherwise, and $\sigma_2^M(s) = \sigma_2^{M'}(s)$ if $[s]_2 \in -M'$ and $\sigma_2^M(s) = 0$ otherwise is a Nash equilibrium in Γ_n^M.

Since $(\sigma_1^{M'}, \sigma_2^{M'})$ is a completely mixed Nash equilibrium of $\Gamma_n^{M'}$, all strategies in $\bigcup_{[s]_1 \in M'} [-s]_2$ yield the same payoff for player 2 against σ_1^M. By hypothesis (b), since $\text{supp}(\sigma_1^M) = \bigcup_{[s] \in M'} [s]$, all strategies outside $\bigcup_{[s]_1 \in M'} [-s]_2$ cannot be a best response for player 2 against σ_1^M. Analogously all strategies in $\bigcup_{[s]_1 \in M'} [s]_1$ yield the same payoff for player 1 against σ_2^M, and since $\text{supp}(\sigma_2^M) = -\bigcup_{[s]_1 \in M'} [-s]_2$, strategies outside $\bigcup_{[s]_1 \in M'} [s]_1$ cannot be a best response for player 1 against σ_2^M. Hence, by Proposition 1, (σ_1^M, σ_2^M) is a Nash equilibrium in Γ_n^M. This proves the claim.

Claim B: For any $\emptyset \neq M \subseteq \overline{S}_1$, the reduced game Γ_n^M has exactly one completely mixed Nash equilibrium.

Let $\emptyset \neq M \subseteq \overline{S}_1$ be such that $|M| = m$. We will prove the claim by induction over m. Note first, that by hypothesis (b), in any Nash equilibrium (σ_1, σ_2) of Γ_n^M, $\text{supp}(\sigma_1)$ is a union of elements of M.

For $m = 1$, this follows by hypothesis (c). For $m > 1$, by induction hypothesis we obtain that for all $\emptyset \neq M' \subsetneq M$ the reduced game $\Gamma_n^{M'}$ has a unique completely mixed Nash equilibrium. By Claim A, for every $\emptyset \neq M' \subsetneq M$ there is a Nash equilibrium (σ_1^M, σ_2^M) in Γ_n^M with $\text{supp}(\sigma_1^M) = \bigcup_{[s] \in M'} [s]$. As by Proposition 2(ii) for any $\emptyset \neq M' \subsetneq M$ there can be at most one Nash equilibrium (σ_1, σ_2) in Γ_n^M with $\text{supp}(\sigma_1) = M'$ we obtain that there is exactly one such Nash equilibrium. This implies that Γ_n^M has at least $2^m - 2$ Nash equilibria.

Suppose there is no completely mixed Nash equilibrium in Γ_n^M. Then Γ_n^M has exactly $2^m - 2$ Nash equilibria. From hypotheses (a) and (b) it follows that Γ_n is non-degenerate and hence that Γ_n^M is non-degenerate. By Proposition 2(i) Γ_n^M must have an odd number of Nash equilibria, which contradicts the fact that $2^m - 2$ is even. Hence there is at least one completely mixed Nash equilibrium and again because Γ_n^M is non-degenerate by Proposition 2(ii) there is exactly one. This proves the claim.

By Claim B, for $\emptyset \neq M \subseteq \overline{S}_1$, Γ_n^M has exactly one completely mixed Nash equilibrium (σ_1^M, σ_2^M). By Claim A, this induces a Nash equilibrium (σ_1, σ_2) in Γ_n with $\text{supp}(\sigma_1) = \bigcup_{[s]_1 \in M} [s]_1$. Any Nash equilibrium $(\sigma_1', \sigma_2') \neq (\sigma_1, \sigma_2)$ with $\text{supp}(\sigma_1') = \bigcup_{[s]_1 \in M} [s]_1$ would induce a completely mixed Nash equilibrium in Γ_n^M different from (σ_1^M, σ_2^M), a contradiction. Hence Γ_n has exactly one Nash equilibrium (σ_1, σ_2) with $\text{supp}(\sigma_1) = \bigcup_{[s]_1 \in M} [s]_1$.

(ii) From (i) it follows that for any $\emptyset \neq M \subseteq \overline{S}_1$ there is a unique Nash equilibrium (σ_1, σ_2) in Γ_n such that $\text{supp}(\sigma_1) = \bigcup_{[s]_1 \in M} [s]_1$. Further, by hypothesis (a), for any Nash equilibrium (σ_1, σ_2) of Γ_n there is $\emptyset \neq M \subseteq \overline{S}_1$ such that $\text{supp}(\sigma_1) = \bigcup_{[s]_1 \in M} [s]_1$. As $\overline{S}_1$ has $2^{|\overline{S}_1|} - 1$ nonempty subsets, Γ_n has exactly $2^{|\overline{S}_1|} - 1$ Nash equilibria.

□

Proof of Lemma 4. (i) To see the first part, let $M = \bigcup_{j=1}^m I(s^j)$ be a union of elements of $I(S^n)$. By Lemma 2(i) and Lemma B4, Γ_n and $\overline{S}_1 = \overline{S}_2 = I(S^n)$ as defined in section 3.1 then satisfy the hypotheses of Proposition B1. Hence, there is a unique Nash equilibrium (σ_1, σ_2) with $\text{supp}(\sigma_1) = M$.

To prove the second part, let (σ_1, σ_2) be a Nash equilibrium. By Lemma 2(i), $\text{supp}(\sigma_1)$ is a union of elements in $I(S^n)$.

(ii) Too see the first part, let $M = \bigcup_{j=1}^m C_1(s^j)$ be a union of elements of $C_1(S^n)$. By Lemma B3, $C_1(S^n)$ and $C_2(S^n)$ as defined in section 3.1 are partitions of S^n. Further, by Lemma B5(i), $|C_1(S^n)| = |C_2(S^n)|$ and by Lemma 2(ii), Lemma 3(ii), and B5(ii), Γ_n, $\overline{S}_1 = C_1(S^n)$, and $\overline{S}_2 = C_2(S^n)$ satisfy properties (a)-(c) in Proposition B1. It follows that there is a unique Nash equilibrium (σ_1, σ_2) with $\text{supp}(\sigma_1) = M$.

To prove the second part, let (σ_1, σ_2) be a Nash equilibrium. By Lemma 2(ii), $\text{supp}(\sigma_1)$ is a union of elements in $C_1(S^n)$. □

Proof of Theorem 1. If Γ_n is an ordered iso-circulant game in which both players' payoff matrices are anti-circulant then by Lemma 2(i) and Lemma

B4, Γ_n and $\overline{S}_1 = \overline{S}_2 = I(S^n)$ as defined in section 3.1 then satisfy the hypotheses of Proposition B1. As $|I(S^n)| = d$, it follows that Γ_n has $2^d - 1$ Nash equilibria. If Γ_n is an iso-circulant game in which both players' payoff matrices are circulant, there is a permutation of row vectors that transforms both players' payoff matrices into anti-circulant matrices while fixing the first row in both matrices (Lemma A1(i)). This permutation, which is essentially a relabeling of the players' strategies, does not affect the number of equilibria. Hence, the proof of Theorem 1 is complete. □

Proof of Theorem 2. If Γ_n is an ordered counter-circulant game in which player 1's payoff matrix is anti-circulant and player 2's payoff matrix is circulant then by Lemma B3, $C_1(S^n)$ and $C_2(S^n)$ as defined in section 3.1 are partitions of S^n. Further, by Lemma B5(i), $|C_1(S^n)| = |C_2(S^n)|$ and by Lemma 2(ii), Lemma 3(ii), and B5(ii), Γ_n, $\overline{S}_1 = C_1(S^n)$, and $\overline{S}_2 = C_2(S^n)$ satisfy properties (a)-(c) in Proposition B1 and hence Γ_n has $2^{|C_1(S^n)|} - 1$ Nash equilibria.

To prove (i)-(iii) it hence suffices to determine $|C_1(S^n)|$. Note that any class $C_1(s)$ contains either one or two elements. It contains one element if and only if $-s+k \equiv s$ which occurs if and only if either $2s = k$ or $2s = n+k$. Further, there are at most two singleton classes.

(i) If n is odd, then either $n - k$ is odd (if k is even) or $2n - k$ is odd (if k is odd). Hence there is one singleton class in $C_1(S^n)$ and since all other elements of $C_1(S^n)$ contain two elements, $|C_1(S^n)| = (n-1)/2+1 = (n+1)/2$.

(ii) If both n and k are even, then both k and $n+k$ are even and $k/2, (n+k)/2 \in S^n$. Hence there are two singleton classes in $C_1(S^n)$ and since all other elements of $C_1(S^n)$ contain two elements, $|C_1(S^n)| = (n-2)/2+2 = (n+2)/2$.

(iii) If n is even and k is odd, then $n+k$ is odd and hence neither $k/2 \in S^n$ nor $(n+k)/2 \in S^n$. Hence there is no singleton class and hence all elements of $C_1(S^n)$ contain 2 elements, implying that $|C_1(S^n)| = n/2 = n/2$.

If Γ_n is a count-circulant game in which player 1's payoff matrix is circulant and player 2's payoff matrix is anti-circulant, there is a permutation of row vectors that transforms player 1's payoff matrix into an anti-circulant matrix. Applying the same permutation of row vectors to player 2's payoff matrix yields a different version of the same game in which strategies are differently labeled and player 1's payoff matrix is anti-circulant and player 2's payoff matrix is circulant (Lemma A1(ii)). This permutation does not affect the number of Nash equilibria and hence the proof of Theorem 2 is complete. □

Proof of Proposition 3. Note first that if both players' payoff matrices are circulant then by Lemma A1(i) the game can be transformed into a different version of the same game in which both players' payoff matrices are anti-circulant by a permutation of row vectors. Since such a permutation does not affect the number of pure strategy Nash equilibria, we assume wlog that both players' payoff matrices are anti-circulant.

To see the "if" part suppose $k = n$. Then by construction, each class $I(s)$ is a singleton set and there are n disjoint classes. Hence by Lemma B4, Γ_n has at least n pure strategy Nash equilibria. By Lemma 2(i), in any pure strategy Nash equilibrium (σ_1, σ_2), $\text{supp}(\sigma_1) = I(s)$ for some $s \in S$ and hence Γ_n has exactly n pure strategy Nash equilibria.

To prove the "only if" part let Γ_n have n pure strategy Nash equilibria and let (s_1, s_2) be one of them. By Lemma 2(i), $I(s_1)$ must be a singleton set. By construction, $I(s_1)$ is a singleton set if and only if $k = n$.

This proves the first part of the theorem.

To see the second part, note that by construction of the classes $I(s)$ is a singleton set if and only if $k = n$ for any $s \in S$. Further by Lemma 2(i) and Lemma B4, Γ_n has a pure strategy Nash equilibrium if and only if there is a singleton equivalence class $I(s)$. Hence, Γ_n has no pure strategy Nash equilibrium if and only if $k \neq n$. □

Proof of Proposition 4. Note first that if player 1's payoff matrix is circulant then by Lemma A1(i) the game can be transformed into a different version of the same game in which player 1's payoff matrix is anti-circulant by a permutation of row vectors. Since such a permutation does not affect the number of pure strategy Nash equilibria, we assume wlog that player 1's payoff matrix is anti-circulant.

(i) By Lemma 2(ii) and Lemma B5(ii), Γ_n has one pure strategy Nash equilibrium if and only if one of the classes $C_1(s)$ is a singleton set, which by construction happens if and only if n is odd.

(ii) By Lemma 2(ii) and Lemma B5(ii), Γ_n has two pure strategy Nash equilibria if and only if two of the classes $C_1(s)$ is a singleton set, which by construction happens if and only if both n and k are even.

(iii) By Lemma 2(ii) and Lemma B5(ii), Γ_n has no pure strategy Nash equilibrium if and only if none of the classes $C_1(s)$ is a singleton set, which by construction happens if and only n is even and k is odd.

□