

Paths to Stability in the Assignment Problem*

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Abstract

Many markets and social processes involve bilateral relationships where each agent of one side of the market can be matched to any agent of the other side of the market but cannot be matched to any agent from the same side. Examples for such two-sided matching markets include marriage markets (women and men), college admissions markets (colleges and students), auction markets (buyers and sellers), and labor markets (workers and firms). Next, two-sided matching markets can be partitioned in two main categories: markets without side payments (e.g., marriage and college admissions markets) and markets with side payments (e.g., auction and labor markets). Side payments/prices are a natural feature of many economic situations. Here, we study a simple two-sided one-to-one matching market with side payments: a labor market with finitely many heterogeneous workers and firms. To keep the model simple we impose a unit-demand condition such that each workers accepts at most one job and each firms hires at most one worker.

Two-sided matching markets with side payments –*assignment problems*– have first been analyzed by Shapley and Shubik (1971). In an assignment problem, indivisible objects (e.g., auctioned items or jobs) are exchanged with monetary transfers (e.g., prices or salaries) between two finite sets of agents (e.g., buyers/sellers or workers/firms). Agents are heterogeneous in the sense that each object may have different values to different agents. Each agent either demands or supplies exactly one unit. Thus, agents can form pairs (and singletons) to exchange the corresponding objects (execute an outside option) and at the same time make monetary transfers that are related to the values created by pairs (singletons).

An outcome for an assignment problem specifies a matching between the two sides of the market and, for each agent, a payoff. An outcome is stable if it is individually rational and there is no blocking pair, that is, there are no two agents that are not matched with each other, but in fact would prefer to be. For instance, in a labor market a worker and a firm form a blocking pair if both could get higher payoffs than the payoffs they obtain by being matched to their current partners. Moreover, an outcome is in the *core* if no coalition of agents can improve their payoffs by rematching among themselves. Shapley and Shubik (1971) showed that (a) the core of an assignment problem and the set of stable outcomes are the same, (b) there always exists a stable outcome for any assignment problem, (c) the set of stable outcomes is a complete lattice with two extreme points, each of them corresponding to an outcome where all the agents of the same side of the market (e.g., the workers) receive their maximal stable payoffs while the agents of the other side (e.g., the firms) receive their minimal stable payoffs, and (d) at any stable outcome the matching between the workers

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and the firms is optimal (i.e., the sum of all values induced by the corresponding matching is maximal). Sotomayor (2003) and Wako (2006) proved that if there is only one optimal matching, then the core contains infinitely many stable outcomes. Conversely, the core is a singleton only when multiple optimal matchings exist.

Dynamic changes in real world (two-sided) matching markets are frequently observed. This indicates that outcomes often are not stable. For instance, in a labor market, a worker might switch to a new job if that increases his salary. On the other side of the market, a firm might hire a different workers for a position if the new employee has a higher qualification/productivity compared to his predecessor. A blocking path for an assignment problem is a finite sequence of outcomes where each outcome is obtained from the previous one by satisfying a blocking pair taking into account that agents behave myopically, i.e., agents do not forecast how their decision to block an outcome might influence the future evolution of the market. We are interested in the question if starting from any (unstable) outcome, there always exists a blocking path that will lead to stability in finitely many steps.

The literature on stability in two-sided matching markets was initiated by Gale and Shapley (1962) who proposed a centralized procedure, the famous deferred acceptance algorithm, to find a stable outcome for any marriage or college admissions problem (with responsive preferences). The deferred acceptance algorithm turned out to be the key element for many centralized market clearing houses, e.g., for the National Resident Matching Program (Roth, 1984), for school choice programs (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005), and for auctions (Demange et al., 1986; Gul and Stacchetti, 2000; Milgrom, 2000; Ausubel, 2006; Sun and Yang, 2009).¹

However, many markets are decentralized (e.g., marriage markets and labor markets). In a decentralized process agents meet randomly and decide myopically whether to form pairs (or not). Knuth (1976) showed that for marriage markets such a process may cycle, i.e., a decentralized process may not converge to a stable outcome. Roth and Vande Vate (1990) show that for marriage markets there always exists a blocking path starting from any unstable outcome that leads to a stable outcome in finitely many steps. Assuming that each blocking pair is selected with strict positive probability, this result implies that a decentralized blocking process converges to stability with probability one.² Chen et al. (2011) analyze a similar decentralized blocking process for labor markets with discrete side payments. As in Roth and Vande Vate (1990), Chen et al. (2011) construct a finite blocking path to stability and show that a decentralized blocking process converges to stability with probability one.

We consider assignment problems with continuous side payments and ask whether, starting from an arbitrary (unstable) outcome, there always exists a blocking path that leads to stability in finitely many steps. In contrast to the discretized version of Chen et al. (2011), the existence of blocking paths to stability cannot always be guaranteed. We identify a necessary and sufficient condition for an assignment problem to guarantee the existence of blocking paths to stability (and show how to construct such a path whenever this is possible). More precisely, we prove the existence of a finite blocking path to stability for all assignment problems for which there exists an interior stable outcome (a stable outcome such that all agents that are matched to agents on the other side of the market receive a payoff larger their worst stable payoff and smaller than their best stable payoff).

¹See also Crawford and Knoer (1981), Kelso and Crawford (1982), and Crawford (2008) for centralized processes in labor markets and Roth and Sotomayor (1990) for an excellent survey on two-sided matching theory until 1990.

²Diamantoudi et al. (2004) establish a similar result for (one-to-one matching) roommate problems and Klaus and Klijn (2007) for matching markets with couples.

We obtain the following results in our paper: We first introduce the classical assignment model with continuous side payments (Shapley and Shubik, 1971). Then, we define a generic blocking path and we show with a few examples that a finite blocking path to stability might not exist for all assignment problems. Our main result is that, for all assignment problems that satisfy our necessary condition, a stable outcome can always be obtained through a finite sequence of outcomes, each outcome being obtained from the previous one by satisfying a blocking pair. We conclude by discussing some relevant points. First, we consider a specific blocking path where each time a blocking pair is satisfied the blocking agents equally split the surplus they create. We ask whether such a blocking path leads to stability in finitely many steps. Second, we discuss the probabilistic interpretation of the blocking path result we have obtained. More precisely, we investigate whether a decentralized process that selects randomly each possible blocking pair converges to stability. Third, we discuss in more details the paper by Chen et al. (2011) and show how their results and our results are related.

JEL classification: C71, C78, D63.

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