QUANTUM SPEEDUP IN TESTING CAUSAL HYPOTHESES

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Quantum Machine Learning Plus, Innsbruck, 17-21 2018

CIFAR CANADIAN INSTITUTE FOR ADVANCED RESEARCH
JTF
RGC
Croucher Foundation 裘槎基金會
National Natural Science Foundation of China NSFC
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CAUSAL INFERENCE (CLASSICAL)

The problem: discovering causal relations among a set of variables (cf. Pearl, Spirtes-Glymour-Scheines)

Basic idea: *A* is *a cause* for *B* iff
intervening on A has an effect on the statistics of B

Caveat: "correlation does not imply causation":

no way to infer a causal relation
from a *single* probability distribution *p(a,b)*.
It is necessary to probe different settings for *a*

CAUSAL INFERENCE (GENERAL)

Recently, various extensions of the notions of
"causal relation" and "causal network"
to quantum theory and beyond.

Basic idea (modulo variations across frameworks):
Variables: physical systems.
Causal relations: variable *A* is *a cause* for variable *B* iff
changing the state of A induces a change of the state of B

Leifer (2006), GC-D'Ariano-Perinotti (2008),
Coecke-Spekkens (2012), Leifer-Spekkens (2013),
Henson-Lal-Pusey (2014), Pienaar-Brukner (2015), Costa-Shrapnel (2016),
Portmann-Matt-Maurer-Renner-Tackmann (2017),
Allen-Barrett-Horsman-Lee-Spekkens (2017), MacLean-Ried-Spekkens-Resch (2017).

MOTIVATIONS FOR QUANTUM EXTENSION

- **Foundational:**
  -understanding interplay between causality and quantum probability
  -find new principles for quantum theory

- **Practical:**
  -identifying new quantum advantages
  -identifying working principles for new quantum devices, develop a "technology" of quantum causality.

PLAN OF THIS TALK

Formulate and analyze the quantum version of the task of **testing causal hypotheses.**

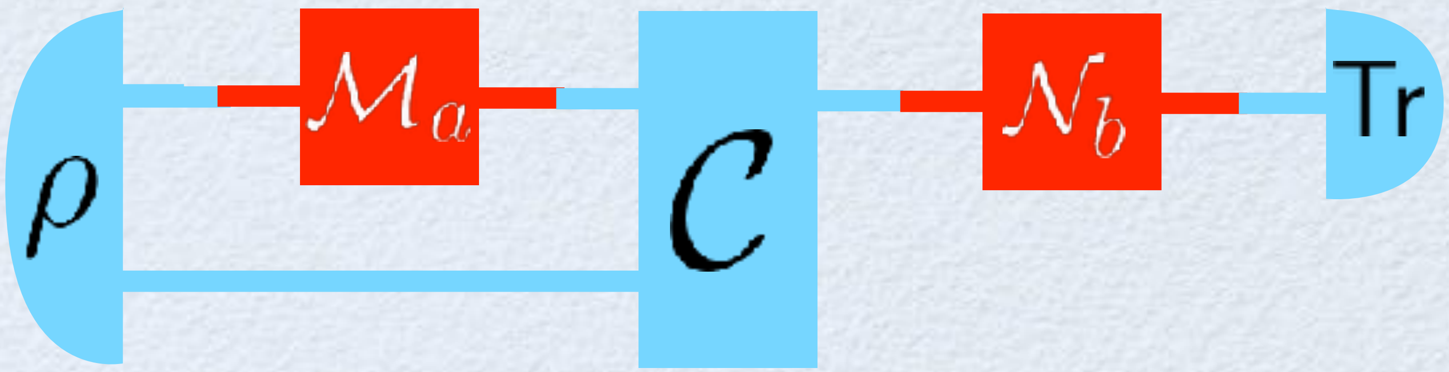
In this task, one has a **set of candidate hypotheses** on the causal relations occurring in a process and the goal is to **identify the correct hypothesis.**

PROLOGUE

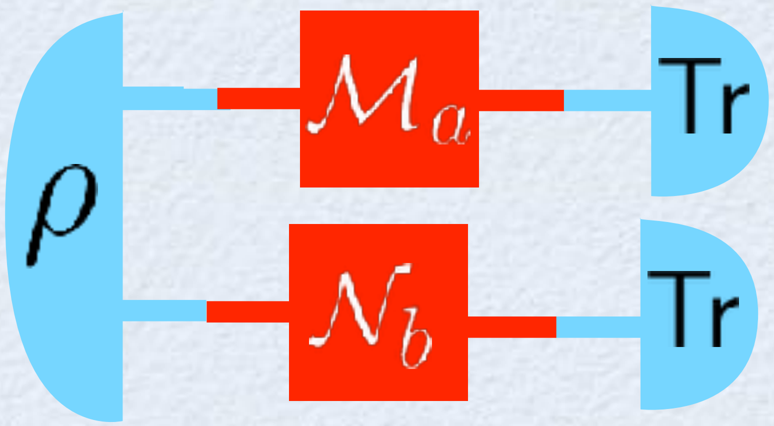
AN INTRIGUING EXAMPLE

Task: distinguish between

- Situation (1): A causes B



- Situation (2): A and B have a common cause



Fact: *for some specific* ρ and $\mathcal{C}$ it is possible to distinguish between (1) and (2) using *only projective measurements.*

Fitzsimons, Jones, and Vedral, Scientific Reports 5, 18281 (2015).
Ried, Agnew, Vermeyden, Janzing, Spekkens, and Resch, Nature Physics 11, 414 (2015).

QUESTION

In the classical world, projective measurements correspond to ***passive observational strategies***, where ***no intervention*** is allowed.

Question:
Can we find advantages in the situation where ***arbitrary interventions*** are allowed?

TESTING
CAUSAL HYPOTHESES:

A THEORY-INDEPENDENT FRAMEWORK

CAUSAL DISCOVERY VS CAUSAL HYPOTHESIS TESTING

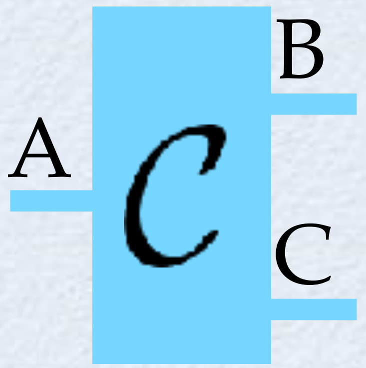
Causal discovery. *Input:* variables *A*, *B*, *C*, …
Output: the causal relations among them.

Causal hypothesis testing: *Input:* variables *A*, *B*, *C*, …
and a set of hypotheses on the causal relations among them.
Output: the correct hypothesis

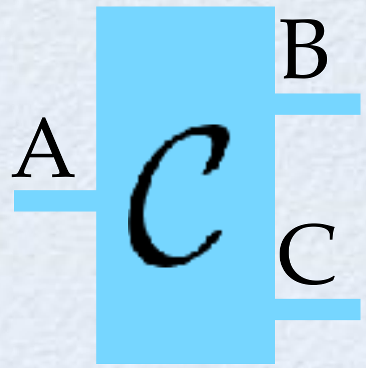
CAUSAL HYPOTHESES

Causal Hypothesis: an hypothesis **on the causal structure** of the process connecting the variables.

e.g.



(H1) A causes B but not C



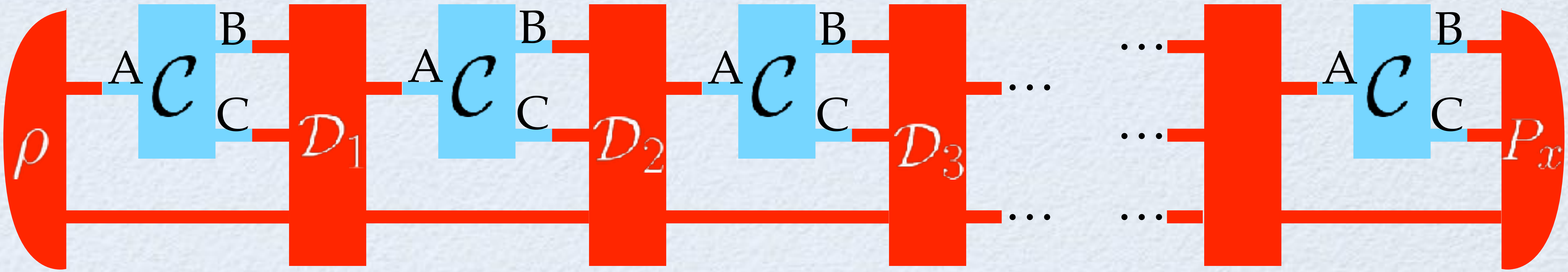
(H2) A causes C but not B

NB: causal hypotheses can be formulated **independently of the underlying theory.**

TESTING CAUSAL HYPOTHESES

The experimenter can probe the **same process**
for a **finite number of times**, performing **arbitrary interventions.**

Most general intervention:



x= guess for the correct hypothesis

Special cases: process tomography, parallel queries, etc…

DISCRIMINATION RATE

Goal of causal hypothesis testing:
minimize the probability of choosing the wrong hypothesis.

Worst-case approach: since the process C is **unknown**
(a part from the fact that it is compatible with one and only one of the given hypotheses)
we will consider the
worst-case error probability $p_{\rm err}(N)$

Discrimination rate: $$R = \lim_{N\to\infty} \frac{-\log p_{\rm err}(N)}{N}$$

quantifies the distinguishability of the hypotheses

EXAMPLE:

IDENTIFYING
THE CAUSAL INTERMEDIARY

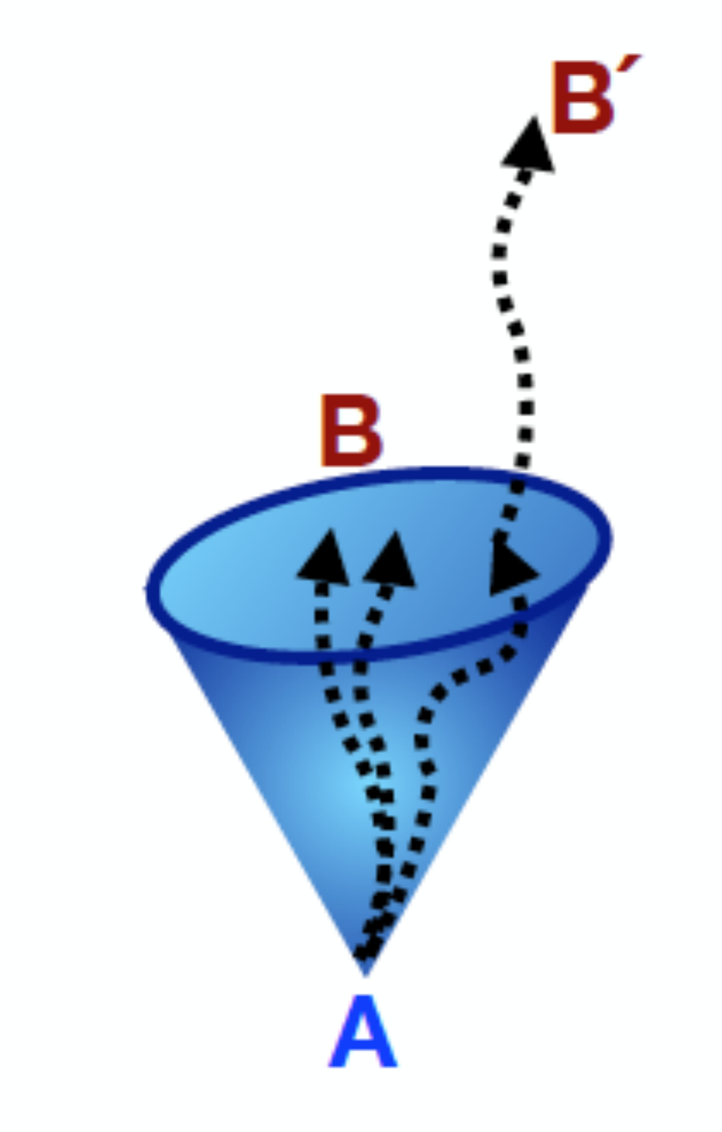
CAUSAL INTERMEDIARIES

Variable B is a **(complete) causal intermediary** for variable A, if "all the causal influences of A" propagate through B.

More formally:
Variable B is a *causal intermediary* for variable A if

- B is an effect of A
- every other effect of A, say B', is an effect of B (assuming that B' takes place after B)

Example:
variable A localized at a spacetime point and variable B localized in a section of the forward light cone based at A.



IDENTIFYING THE CAUSAL INTERMEDIARY

Variables: A, B, and C

Hypothesis (1): B is a causal intermediary of A,
while C fluctuates uniformly at random.

Hypothesis (2): C is a causal intermediary of A,
while B fluctuates uniformly at random.

Problem: decide which hypothesis is correct.

CLASSICAL SOLUTION

SETTINGS

Assume that the random variables A, B, and C have all the **same dimension *d*.**

With this assumption, Hypotheses (1) and (2) become:

Hypothesis (1): b is a permutation of a,
and c is uniformly random

Hypothesis (2): c is a permutation of a,
and b is uniformly random

NAIVE CLASSICAL STRATEGY

Initialize the input variable A to a certain value,
and observe the values taken by the output variables B and C.
Repeat for N times, possibly trying different values of A.

Example for *N*=8, *d*=2

	1	2	3	4	5	6	7	8
A	0	0	1	1	0	0	0	1
B	1	1	0	0	1	1	1	0
C	0	0	1	1	1	0	0	1

PROBABILITY OF ERROR (NAIVE STRATEGY)

Error occurs when both variables B and C
take values that are compatible with permutations.

In that unlucky case, the probability of error is 1/2.

If we try v different values for A,
the probability to be unlucky is

$$p_{\text{unlucky}} = \frac{\left|\{\text{injective functions from } v \text{ element set to } d \text{ element set}\}\right|}{d^N}$$

$$= \frac{d(d-1)(d-2)\cdots(d-v+1)}{d^N}$$

DISCRIMINATION RATE (NAIVE STRATEGY)

Choosing v=1, the error probability of the naive strategy is minimal:

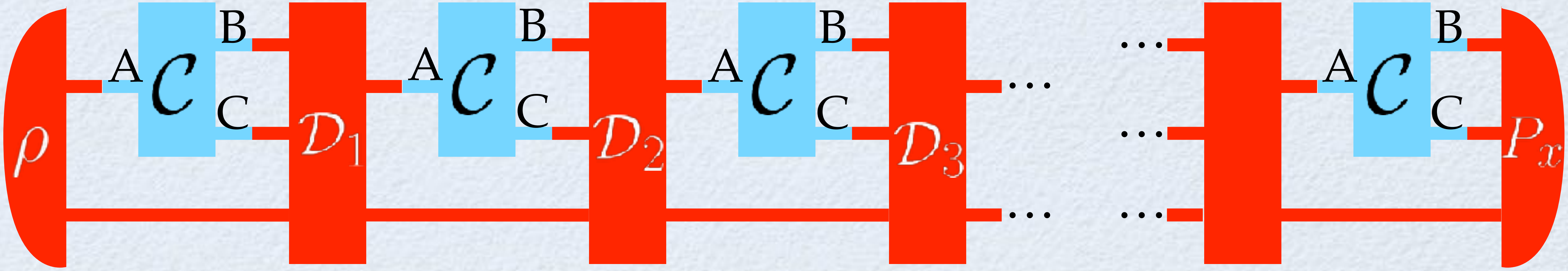
$$p_{\mathrm{err}}(N) = \frac{p_{\mathrm{unlucky}}}{2} = \frac{1}{2d^{N-1}}$$

Discrimination rate: $$R = \lim_{N\to\infty} \frac{-\log p_{\mathrm{err}}(N)}{N}$$

$$= \log d$$

GENERAL CLASSICAL STRATEGIES

We have found the rate of the naive classical strategy.
What about general strategies?



Theorem [Hayashi, IEEE TIT, 55, 3807 (2009)]:
The optimal *asymptotic rate* in distinguishing *two classical channels* can be attained by a parallel strategy.

Applying this theorem to a fixed pair of channels,
we obtain that $\log d$ is an **upper bound to the rate.**

IN SUMMARY

For classical variables of dimension *d*,
the optimal rate in identifying a complete causal intermediary is

$$R_{\mathrm{C}} = \log d$$

Attained by the naive strategy
"initialize variable *A* for N times to the same value"

QUANTUM
SOLUTION

SETTINGS

Assume that the quantum systems A, B, and C have all the **same dimension *d***.

With this assumption, Hypotheses (1) and (2) become:

Hypothesis (1):

$$\mathcal{C}_{A\to BC}(\rho_A) = (U\rho U^\dagger)_B \otimes \left(\frac{I}{d}\right)_C$$

for some unknown unitary U

Hypothesis (2):

$$\mathcal{C}_{A\to BC}(\rho_A) = \left(\frac{I}{d}\right)_B \otimes (V\rho V^\dagger)_C$$

for some unknown unitary V

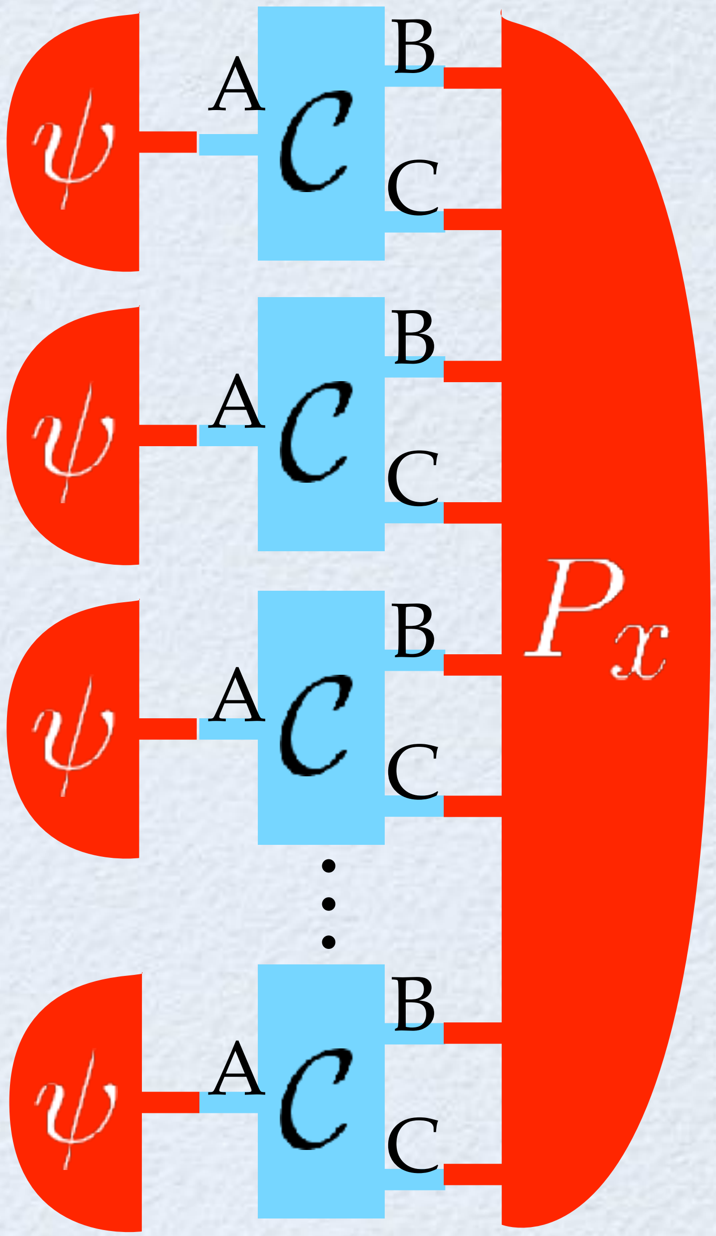
NAIVE QUANTUM STRATEGY

Initialize the input system A
in a fixed state,
repeat for N times,
measure the output state.

Error probability:

$$p_{\rm err}(N) = \frac{\binom{N+d-1}{d-1}}{2d^N}$$

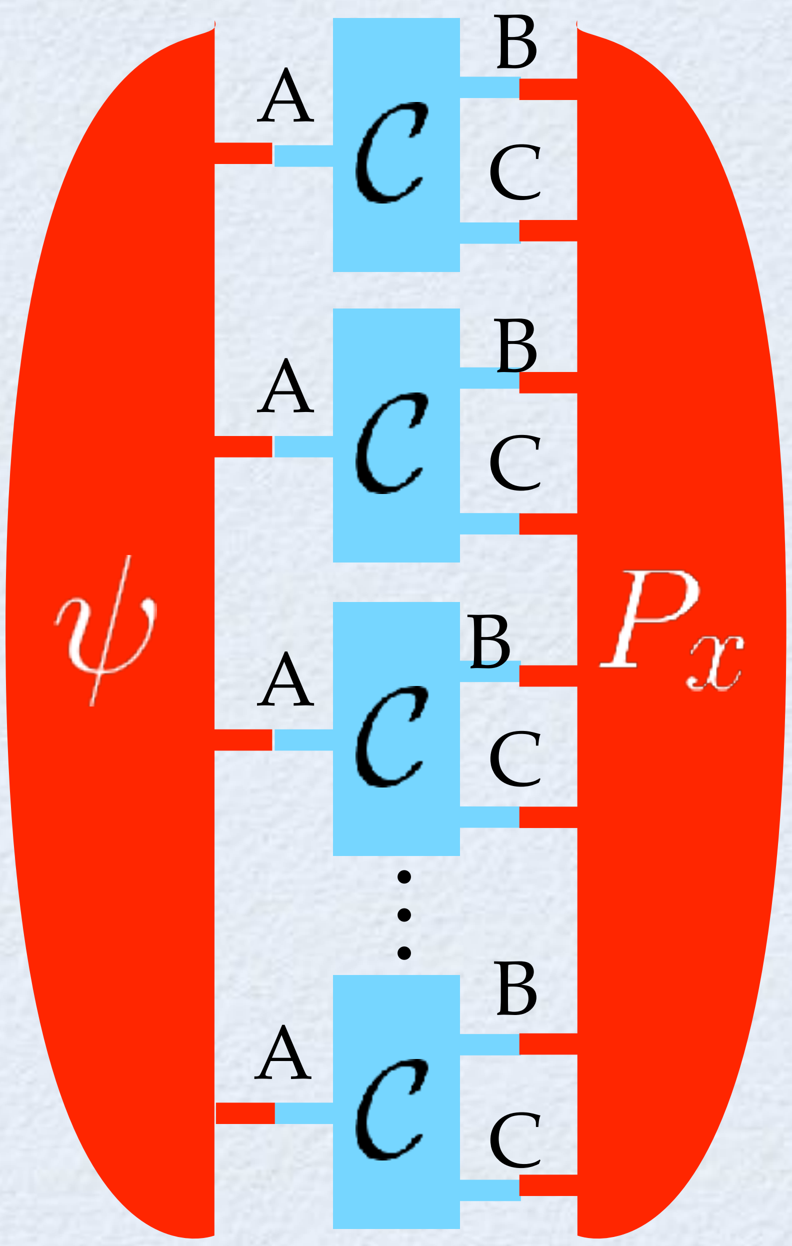
Worse than the classical error probability.
But at least, same rate: log *d*



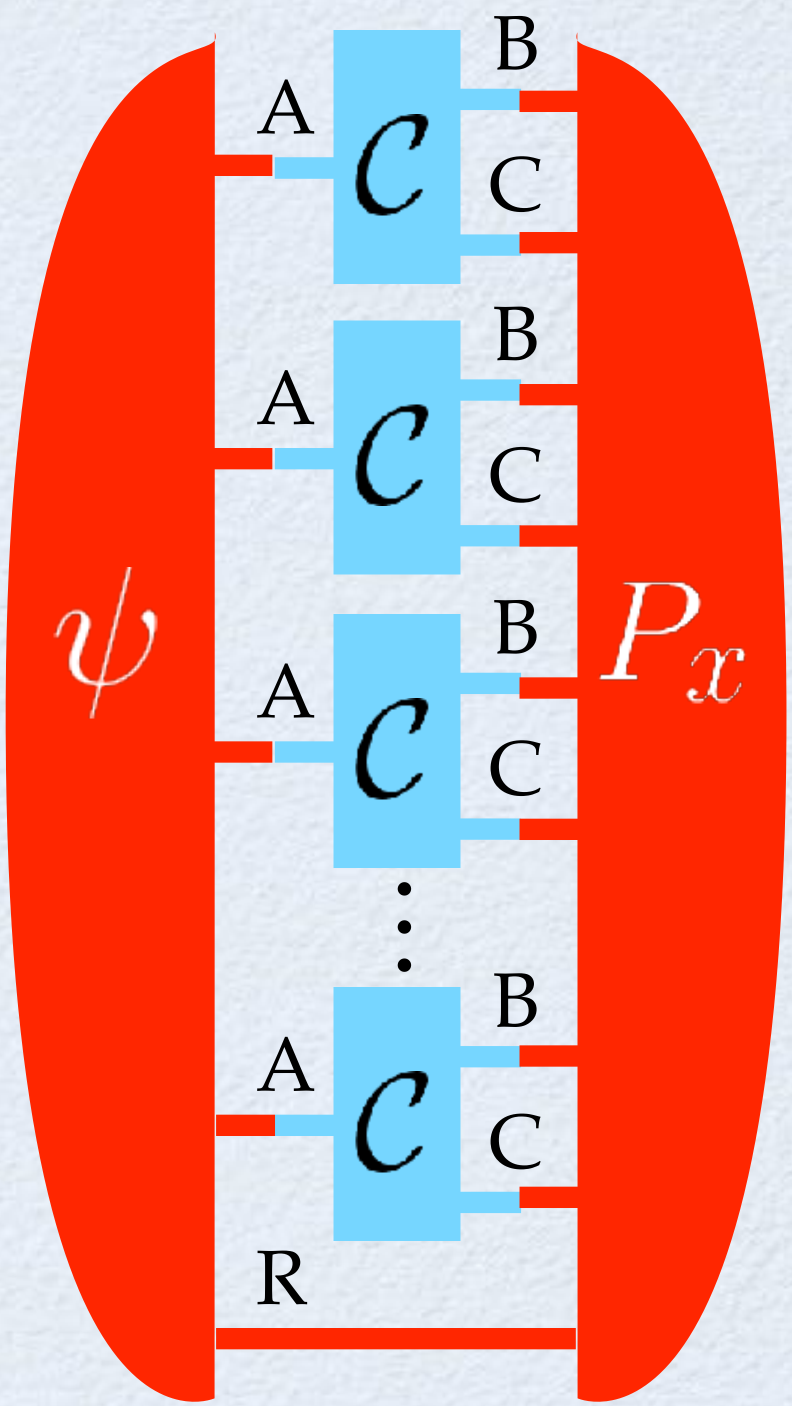
OPTIMAL
PARALLEL STRATEGIES

PARALLEL STRATEGIES

Without reference:



With reference:



OPTIMAL PARALLEL STRATEGIES WITHOUT REFERENCE

OPTIMAL STRATEGY WITHOUT REFERENCE

For simplicity, assume $d = 2$ and N even, say N=2p.

Divide the N input variables in p pairs.
Prepare each group in the singlet state $|\Psi^-\rangle = \frac{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle}{\sqrt{2}}$

Key intuition: invariance of the singlet

$$(U \otimes U)|\Psi^-\rangle = |\Psi^-\rangle \qquad \forall U$$

we can test the causal structure without extracting any information about the functional dependence between cause and effect.

ERROR PROBABILITY

For general dimension *d*,
divide the *N* input variables in groups of *d*
and prepare each group in the SU(*d*) singlet

$$|S_d\rangle = \frac{1}{\sqrt{d!}} \sum_{k_1,k_2,\cdots,k_d} \epsilon_{k_1 k_2 \dots k_d} |k_1\rangle|k_2\rangle \cdots |k_d\rangle$$

Perform the Helstrom measurement on the output.

Error probability: $p_{\text{err}}(N) = \frac{1}{2d^N}$

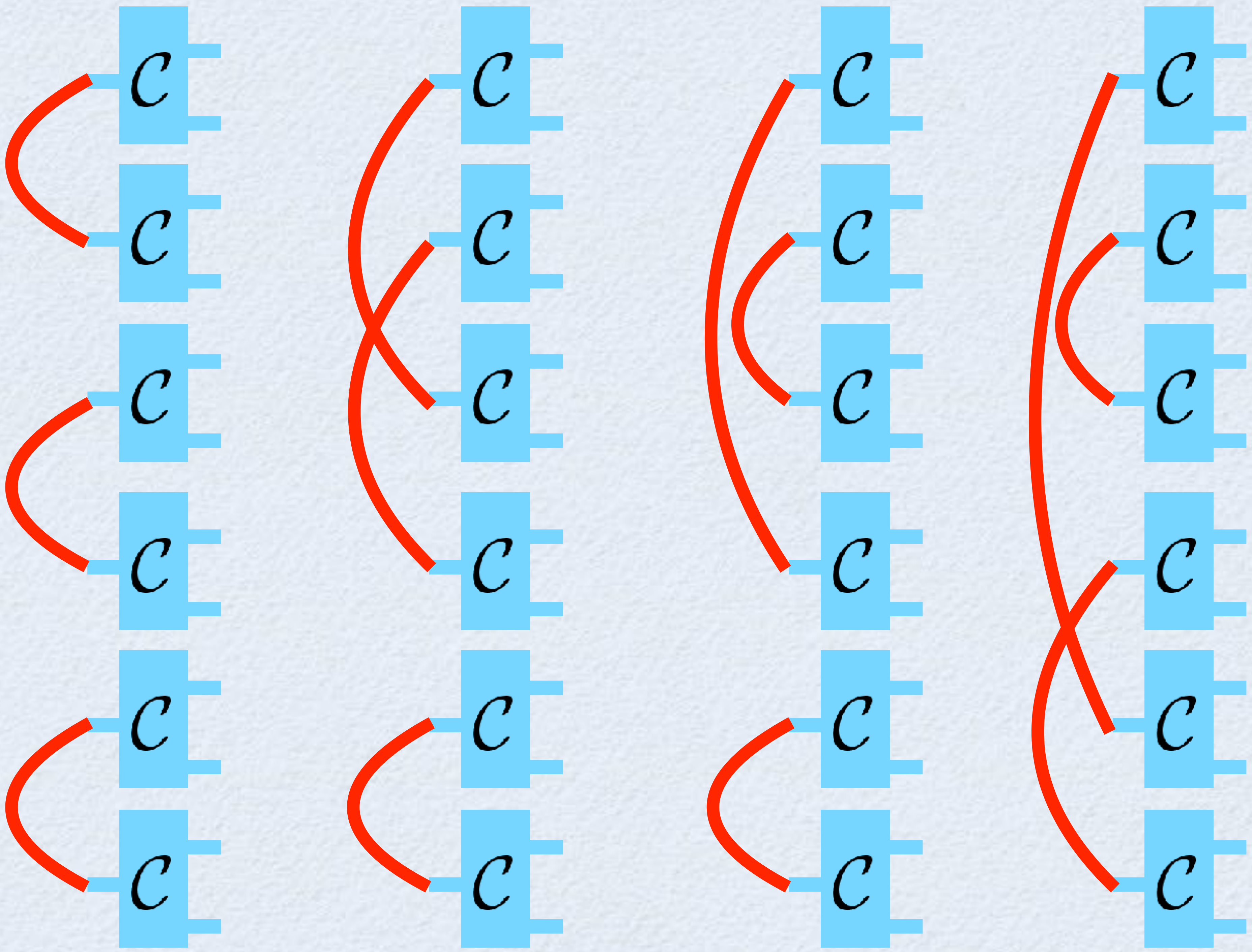
Better than classical value $p_{\text{err}}(N) = \frac{1}{2d^{N-1}}$

but rate is still log *d*

OPTIMAL PARALLEL STRATEGIES WITH REFERENCE

EQUIVALENT STRATEGIES

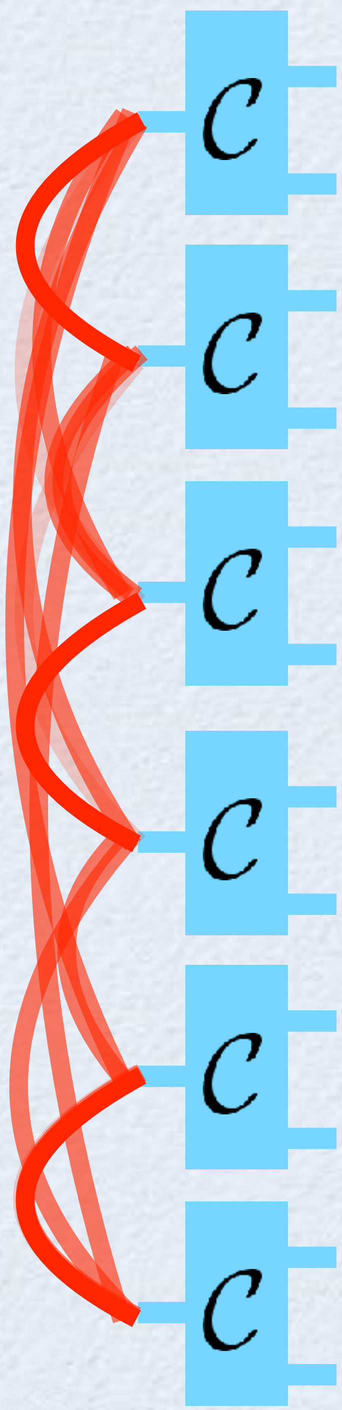
d=2, N even. Many ways to partition the inputs into pairs:



etc,

etc…

IDEA: EQUIVALENT STRATEGIES IN SUPERPOSITION



In dimension *d*:

$$|\Psi\rangle = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \left(|S_d\rangle^{\otimes N/d} \right)_i \otimes |i\rangle_R$$

where *i* labels the way to group the systems and *L* is the number of groupings in the superposition

ERROR PROBABILITY

When there are *r* linearly independent groupings, the error probability is

$$p_{\mathrm{err}}^{\mathrm{Q}}(r) = \frac{r}{2d^N}\left(1 - \sqrt{1 - r^{-2}}\right) \xrightarrow{r \gg 1} \frac{1}{4rd^N}$$

Picking the maximum *r*, we obtain the rate

$$R_{\mathrm{Q}} = -\lim_{N \to \infty} \frac{\log p_{\mathrm{err}}^{\mathrm{Q}}}{N} = 2\log d$$

Twice the classical rate!

IN SUMMARY

For quantum variables of dimension *d*,
the optimal parallel strategy identifies
the complete causal intermediary with rate

$$R_{\mathrm{Q}} = 2 \log d$$

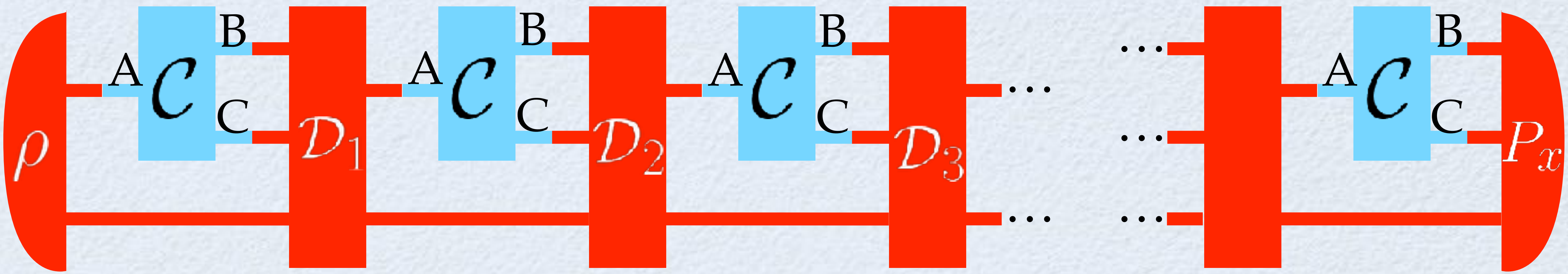
twice the classical rate.

The quantum rate is attained by preparing
singlets in a superposition of different groupings.

GENERAL
QUANTUM STRATEGIES

GENERAL CLASSICAL STRATEGIES?

We have found the rate of the best parallel strategies.
What about general strategies?

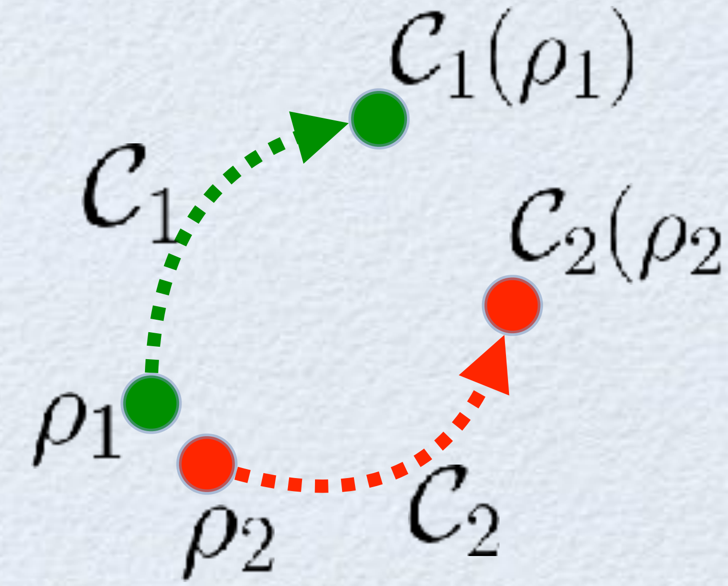


In principle, we should optimize over all **quantum testers**

GC-D'Ariano-Perinotti, PRL 101, 180501 (2008)
Gutoski-Watrous, Proc. STOC, p. 565-574 (2007).

However, the optimization is hard.

A TRICK



Define the **fidelity divergence** of two channels

$$\partial F(\mathcal{C}_1, \mathcal{C}_2) = \inf_{R} \inf_{\rho_1, \rho_2} \frac{F\left[(\mathcal{C}_1 \otimes \mathcal{I}_R)(\rho_1), (\mathcal{C}_2 \otimes \mathcal{I}_R)(\rho_2)\right]}{F(\rho_1, \rho_2)}$$

Fact: if we try to distinguish between two channels with N queries, the error probability satisfies

$$p_{\rm err}^{\rm seq}(\mathcal{C}_1, \mathcal{C}_2; N) \geq \frac{\partial F(\mathcal{C}_1, \mathcal{C}_2)^N}{4}$$

Upper bound on the rate: $R_{\rm Q}^{\rm seq}(\mathcal{C}_1, \mathcal{C}_2) \leq -\log \partial F(\mathcal{C}_1, \mathcal{C}_2)$

OPTIMAL RATE

The **fidelity divergence** between the channels

$$\mathcal{C}_{A\to BC}(\rho_A) = (U\rho U^\dagger)_B \otimes \left(\frac{I}{d}\right)_C$$

and

$$\mathcal{C}_{A\to BC}(\rho_A) = \left(\frac{I}{d}\right)_B \otimes (V\rho V^\dagger)_C$$

is $\partial F = \dfrac{1}{d^2}$

Hence, we have the bound $R_{\mathrm{Q}} \leq 2\log d$

IN SUMMARY

For quantum variables of dimension *d,*
the rate

$$R_{\mathrm{Q}} = 2 \log d$$

is optimal, and it is attained by preparing
singlets in a superposition of different groupings.

EXTENSION
TO
K CAUSAL HYPOTHESES

CAUSAL INTERMEDIARY: *K* CANDIDATES

Variables: A, B_1, B_2 … B_k

Hypothesis (i): i=1, …, *k* B_i is a causal intermediary of A, and all the other variables fluctuate uniformly at random.

Problem: decide which hypothesis is correct.

OPTIMAL RATES

Classical: **log** d

Quantum without reference: **log** d
(attained with singlets)

Quantum with reference: **2 log** d
(attained with superposition of singlets, optimal among all quantum strategies)

Note: rates are independent of the number of hypotheses k

SUMMARY AND OUTLOOK

CONCLUSIONS

- Theory-independent framework for **testing causal hypotheses**
- Instance of the problem: **identifying the causal intermediary.**
- Classical solution: rate log d
- Quantum solution: rate 2 log d, achieved by **superposition of singlet states in equivalent configurations**

OUTLOOK

- Is it always true that quantum theory does better (or at least, not worse) than classical theory in the task of causal hypothesis testing?
- Is quantum theory optimal for causal hypothesis testing?
- If not, which physical principles determine the power in identifying causal hypotheses?
- What about indefinite causal order? How well can we test non-standard hypotheses on the causal structure?

Reference for this work: https://arxiv.org/abs/1806.06459

FACULTY OPENING AT HKU CS

A tenure track faculty position in Quantum Information Theory is now open at the Computer Science Department of The University of Hong Kong.

Target areas include (but are not limited to)
quantum complexity theory
quantum simulations
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Deadline for applications: 31 October 2018

More information @
https://www.cs.hku.hk/people/vacancies.jsp