Convergence analysis of Strang splitting for Vlasov-type equations

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Motivation

Equations

1. Vlasov equation

\[ \partial_t f(t, x, v) + v \cdot \nabla f(t, x, v) + F \cdot \nabla_v f(t, x, v) = 0, \]

with force term \( F \) and particle-density \( f \).

2. Coupling to the EM field (Maxwell or Poisson equations)

3. Approximations (e.g. gyrokinetic equations, linear Vlasov equation)

Applications

1. Plasma simulations
   (e.g. Tokamaks or plasma-laser interactions)

2. Especially if fluid models are not sufficient
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Vlasov-type equations

Definition (Vlasov-type equation)

\[
\begin{align*}
\partial_t f(t) &= (A + B)f(t) \\
f(0) &= f_0,
\end{align*}
\]

where $A$ is a linear operator and $Bf = B(f)f$ with $B(f)$ linear.

Scope

1. Abstract evolution equation
2. Includes Vlasov–Poisson, Vlasov–Maxwell, and gyrokinetic equations as a special case
3. No discretization in space
Introduction

Convergence VP Numerical experiments

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**Strang splitting**

**Definition (Strang splitting)**

\[ S = e^{\frac{h}{2}A}e^{hB_{h/2}}e^{\frac{h}{2}A}, \]

where \( B_{h/2} \) is a linear approximation of order 1 to \( Bf \).

**Strang splitting for grid-based Vlasov solvers**

2. Vlasov–Maxwell equations (Mangeney et al. 2002)
3. Computationally interesting since for

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solution can be represented as a translation.
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Convergence

Convergence follows from consistency and stability

Stability

Stability follows from probability conservation

Consistency

1. Expansion of the exact solution (Gröbner–Alekseev formula)
2. Expansion of the splitting operator
3. Estimation of the (possibly) unbounded remainder terms
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Expansion of the exact solution

\[
f(h) = E_B(h, f_0) + \int_0^h \partial_2 E_B(h - \tau, f(\tau)) AE_B(\tau, f_0) \, d\tau
\]
\[
+ \int_0^h \int_0^\tau \partial_2 E_B(h - \tau, f(\tau)) A \partial_2 E_B(\tau - \sigma, f(\sigma)) AE_B(\sigma, f_0) \, d\sigma \, d\tau
\]
\[
+ \int_0^h \int_0^\tau_1 \int_0^\tau_2 \left( \prod_{k=0}^{2} \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1})) A \right) f(\tau_3) \, d\tau_1 \, d\tau_2 \, d\tau_3,
\]

where \( \tau_0 := h \).

Expansion of the splitting operator

\[
Sf_0 = e^{hBh/2} f_0 + \frac{h}{2} \left\{ A, e^{hBh/2} \right\} f_0 + \frac{h^2}{8} \left\{ A, \left\{ A, e^{hBh/2} \right\} \right\} f_0 + R_3 f_0.
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**Consistency**

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+ \int_0^h \int_0^{\tau_1} \int_0^{\tau_2} \left( \prod_{k=0}^{2} \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1})) A \right) f(\tau_3) \, d\tau_1 \, d\tau_2 \, d\tau_3,
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Bounds

1. **Compare the terms by employing a quadrature rules**

2. We have to estimate e.g.

   \[
   \left[ e^{hB_{h/2}} - \partial_2 E_B(h, f_0) \right] A f_0 = \partial_2 \left[ e^{hB_{h/2}} - E_B(h, f_0) \right] A f_0
   \]

3. Use Gröbner–Alekseev formula to get

   \[
   E_B(h, f_0) - e^{hB_{h/2}} f_0 = \int_0^h e^{(h-\tau)B_{h/2}} (B - B_{h/2}) E_B(\tau, f_0) \, d\tau
   \]

4. Gives a condition on \( B - B_{h/2} \).

5. Note that \( f_0 \) is an arbitrary initial value for the Strang splitting operator.
Introduction  Convergence  VP  Numerical experiments

Consistency

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Consistency

**Theorem (Consistency)**

Suppose that $B_{h/2}$ is an approximation of order 1 to $B$ and the estimates

\[
\| A^i e^{(h-s)B_{h/2}} (R_2 - i(B) - R_2 - i(B_{h/2})) E_B(s, f_0) \|_X \leq C, \quad i \in \{0, 1, 2\} \tag{1}
\]
\[
\| (R_1(B_{h/2}) - R_1(\partial_2 B f_0)) A f_0 \|_X \leq C, \tag{2}
\]
\[
\| A^{\delta i_2} B_{h/2}^{1+\delta i_0} \varphi_{1+\delta i_0}(h B_{h/2}) A^{1+\delta i_1} f_0 \|_X \leq C, \quad i \in \{0, 1, 2\} \tag{3}
\]
\[
\| A^{\delta i_2} R_{1+\delta i_0} (\partial_2 E_B(\cdot, f_0)) A^{1+\delta i_1} f_0 \|_X \leq C, \quad i \in \{0, 1, 2\} \tag{4}
\]

hold uniformly in $t$ and in $s \in [0, h]$. In addition, suppose that for all $k_j \in \mathbb{N}$, with $\sum_{j=1}^{i+1} k_j = 3 - i$, the estimates

\[
\left\| \left( \prod_{j=1}^{i} D_j^{k_j} \partial_2 E_B(s_j, f(\sigma_j)) A \right) \partial_{s_{i+1}}^{k_{i+1}} E_B(s_{i+1}, f_0) \right\|_X \leq C, \quad i \in \{1, 2\} \tag{5}
\]
\[
\left\| \left( \prod_{k=1}^{3} \partial_2 E_B(s_k - \sigma_k, f(\sigma_k)) A \right) f(s) \right\|_X \leq C, \tag{6}
\]
\[
\left\{ A, \left\{ A, \left\{ A, e^{\frac{s}{2}} A e^{h B_{h/2}} e^{\frac{s}{2}} A \right\} \right\} \right\} f_0 \right\|_X \leq C, \tag{7}
\]

hold uniformly in $t$ as well as in $s \in [0, h]$, $s_j \in [0, h]$, and $\sigma_j \in [0, h]$, where $D_j^{k_j}$ is a differential operator of order $k_j$ in the variables $s_j$ and $\sigma_j$. Then Strang splitting is consistent of order 2.
Vlasov–Poisson equations

Definition (Vlasov–Poisson equations in 1+1 dimensions)

\[
\begin{align*}
\partial_t f(t, x, v) &= -v \partial_x f(t, x, v) - E(f(t, \cdot, \cdot), x) \partial_v f(t, x, v) \\
\partial_x E(f(t, \cdot, \cdot), x) &= \int_{\mathbb{R}} f(t, x, v) \, dv - 1 \\
f(0, x, v) &= f_0(x, v),
\end{align*}
\]

Theorem (Uniqueness, existence, and regularity)

Assume that \( f_0 \in C_{\text{per}, c}^m \) is non-negative, then \( f \in C^m(0, T; C_{\text{per}, c}^m) \) and \( E(f(t, \cdot, \cdot), x) \) as a function of \((t, x)\) lies in \( C^m(0, T; C_{\text{per}}^m) \).

In addition, we can find a \( Q(T) \) such that for all \( t \in [0, T] \) and \( x \in \mathbb{R} \) it holds that \( \text{supp} f(t, x, \cdot) \subset [-Q(T), Q(T)] \).
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Derivative with respect to the initial value

Problem

1. To bound

\[ \int_0^h \int_0^{\tau_1} \int_0^{\tau_2} \left( \prod_{k=0}^2 \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1}))A \right) f(\tau_3) \, d\tau_1 \, d\tau_2 \, d\tau_3, \]

investigate \( \partial_2 E_B(t, u_0)g \) as a function of \( u_0 \) and \( g \)

2. Methods of characteristics expresses solution in the form

\[ V'_{u_0}(t) = E(u(t, \cdot, \cdot), x) \]
\[ u(t, x, v) = u_0(x, V_{u_0}(t)(x, v)). \]

Theorem

The following function is well defined

\[ C^m_{\text{per}, c} \times C^n_{\text{per}, c} \rightarrow C^\text{min}(m-1,n)_{\text{per}, c} \]
\[ (u_0, g) \mapsto \partial_2 E_B(t, u_0)g. \]
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\[ C_{\text{per},c}^m \times C_{\text{per},c}^n \rightarrow C_{\text{per},c}^{\min(m-1,n)} \]
\( (u_0, g) \mapsto \partial_2 E_B(t, u_0) g. \)
**Convergence**

**Consistency**
1. Application of $A$, $B$, $B_{h/2}$ are maps from $C^m_{\text{per,c}}$ to $C^{m-1}_{\text{per,c}}$.
2. Control derivatives of $Bf(t)$ with respect to time.
3. Control the derivative $\partial_2 E_B(t, u_0)g$.

**Stability**
Rewrite splitting step, for example, as

$$e^{-hE(f_{h/2}, x)} \partial_v f_0(x, v) = f_0(x, v - E(f_{h/2}, x)h).$$

**Theorem (Convergence)**
Suppose $f_0 \in C^3_{\text{per,c}}$, $f_0$ is non-negative and $f_{h/2}$ is an approximation to $f(h/2)$ of order 1. Then Strang splitting for the Vlasov–Poisson equations is convergent of order 2 (with respect to the $L^1$ norm).
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**Definition (Landau damping in 1+1 dimensions)**

The Vlasov–Poisson equations together with the initial value

\[ f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} (1 + \alpha \cos(0.5x)) \]

on the domain \([0, 4\pi] \times \mathbb{R}\).
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on the domain \([0, 4\pi] \times \mathbb{R}\).

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### Landau damping

1. **Popular test problem**
2. **Linear Landau damping** (\(\alpha = 0.01\))
3. **Non-linear Landau damping** (\(\alpha = 0.5\))
Definition (Landau damping in 1+1 dimensions)

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Figure: Approximation compared to a reference solution at $t = 1$. Discontinuous Galerkin approximation in space (order 2, $N_x, N_v = 80$).
Linear Landau damping

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Convergence analysis of Strang splitting

Thank you for your attention