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WEAKLY-BOUND STATES OF THREE RESONANTLY-INTERACTING PARTICLES

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It is shown that if the pair forces of three identical particles are sufficiently resonant, a family of bound states of low energy is produced. The quantum numbers of all the states are the same: for spinless bosons 0^+ and for nucleons $\frac{1}{2}^+$, $T = \frac{1}{2}$. The dimension of the states is larger than the radius of the pair forces. The most favorable conditions for the appearance of a family of levels occur for three spinless neutral bosons: the conditions are less favorable for charged particles and particles with spin and isospin. The possibility of existence of such levels in a system of three particles (in the C^{12} nucleus) and of three nucleons (H^3) is considered.

1. INTRODUCTION

THIS paper is devoted to an investigation of a physical effect in a system of three particles, consisting of the following. It turns out that if the pair forces are sufficiently resonant, then a whole family of bound levels must appear for the three particles. A case is even possible when the number of levels is infinite, i.e., there is a condensation in the spectrum. The effect is universal in the sense that it takes place regardless of the concrete form of the pair forces, and all that is required is that they be resonant.

Let us specify more precisely the resonant forces we have in mind. We assume that the resonance is due to the existence of a bound or virtual level of low energy (as in a two-nucleon system). The resonances connected with quasistationary levels are not considered. For the chosen interaction, the radius of the forces r_0 is much smaller than the scattering length a; the magnitude of the forces V is of the order of $1/r_0^2$. We confine ourselves in the present article to identical particles.

To understand the physics of the phenomenon, let us consider the simplest case, that of three spinless neutral particles of low energy ($Er_0^2 \ll 1$). They interact effectively in a volume whose maximum dimensions are of the order of a. If we modify somewhat the magnitude of the paired forces, then a changes radically, becoming infinite when the binding energy of two particles is equal to zero.

An idea concerning the character of the interaction within the limits of the volume can be obtained from the following considerations. We divide the pair forces into two parts: a resonant part (with characteristic dimension a) and a nonresonant part (of radius r_0). The latter is significant when the three particles come close together in a volume of the order of r_0^3. On the other hand, if some relative distances r_{ik} are much larger than r_0, then the nonresonant part can be neglected. If at the same time all the mutual distances are much smaller than a, then it is possible to put $a = \infty$ in the resonant part. This means that in the indicated region the interaction of the three particles does not depend on either r_0 or a, i.e., it is not characterized by any dimensional parameter. We shall show that it is of the form $1/R^2$ for identical particles ($3R^2/2 = r_{12}^2 + r_{23}^2 + r_{31}^2$). The magnitude and sign depend only on the symmetry of the state relative to permutations of the particles, the total angular momentum, and the parity.

From the side of smaller R, the $1/R^2$ interaction is cut off at $R \sim r_0$ by the nonresonant forces, and from the side of large R it is cut off by the value of a. The largest coupling between the three particles occurs in the symmetrical state with angular momentum L = 0. Here the interaction $1/R^2$ is attracted. When $a \to \infty$ it leads to a condensation of the level of the three-particle systems, in exactly the same manner as in the case of two particles with a $1/r^2$ interaction. This phenomenon does not depend on the nonresonant forces. For states with other quantum numbers, the centrifugal forces and the asymmetry of the wave function destroy the effect—the $1/R^2$ interaction is in this case repulsive. There are also states in which the resonant forces are not effective at all.

The foregoing arguments point directly to the condition for the existence of the phenomenon. The number of levels produced by the resonant forces is of the order of $\ln(|a|/r_0)$ (as many as there are levels in the potential $1/R^2$ cut off in the indicated manner). Therefore the effect is strongly pronounced when $\ln(|a|/r_0) \gg 1$.

The interaction of spinless (neutral and charged) particles is considered in Secs. 2–4. In Sec. 5 we investigate nucleon interaction. In Sec. 6 we assess the possibility of describing the phenomenon by zero-radius forces.

2. FUNDAMENTAL EQUATIONS

Let us proceed to prove the statements made in the introduction for spinless neutral particles. We shall show that in order for a family of levels to appear all that matters is the resonant part of the forces, and we shall obtain equations determining the wave functions and the energies of the levels. In the next section we determine the spectrum.

We first divide the pair forces into resonant and nonresonant parts. To this end we express the interaction of the three particles in terms of the pair amplitude. The pole term in the amplitude, corresponding to the bound or virtual state of two particles, will be called the resonant part; the remainder is the nonresonant part. It is important that a enters only in the resonant part.

The importance of each part is best analyzed by dividing the configuration space of the three particles into internal and external regions. In the internal region $R < R_0$ (R is defined in the introduction), where $R_0 \sim r_0$ and does not depend on a or E; $R_0/a \ll 1$, and $R_0|E|^{1/2} \ll 1$. We put $R_0 = Cr_0$, where C is a large number (the meaning of such a choice will be explained later). In the outer region $R > R_0$.

In view of the fact that the radius of the resonant forces is much larger than the dimensions of the internal region, the number of levels produced by the interaction of the three particles in this region depends little on a. Therefore the family of levels referred to in the introduction, if it does appear at all, is due only to the interaction in the internal region. But in the latter region the nonresonant forces are ineffective. Indeed, since they have a radius r_0, and the characteristic distance between the particles in the outer region is of the order of R_0 or larger, the probability of paired collisions is small (this is precisely why we chose $C \gg 1$). This can be verified also by another method, by considering the diagrams for the three-particle amplitude. Each succeeding diagram differs from the preceding one by a factor $\beta \sim t/\overline{R}$, where t is the characteristic value of the nonresonant amplitude and $\overline{R}$ is the characteristic value of R. Since $t \lesssim r_0$ and $\overline{R} \gtrsim R_0$, we get $\beta \lesssim r_0/R_0$.

Thus, it is sufficient to consider the resonant interaction in the external region. (We exclude the case when one of the levels produced by the interaction in the internal region has a low energy. The appearance of such a level has, in a certain sense, a random character, since it can be taken outside the region of small a^{-1} and E by a slight modification of the nonresonant interaction.) Of course, to determine the spectrum of the levels, the obtained wave functions must be joined together with the wave functions of the internal region.

In this section we consider a symmetrical state of the three particles. We write the wave function in the form[1)]

$$\Psi = \chi(\mathbf{r}_{12}, \rho_3) + \chi(\mathbf{r}_{23}, \rho_1) + \chi(\mathbf{r}_{31}, \rho_2) \equiv (1+Q)\chi(\mathbf{r}_{12}, \rho_3), \quad (1)$$

where Q is an operator representing the coordinates. χ is defined by the equation

$$(T-E)\chi(\mathbf{r}, \rho) = -V(\mathbf{r})(1+Q)\chi(\mathbf{r}, \rho), \quad T = -\Delta_r - \Delta_\rho. \quad (2)$$

The right-hand side vanishes when $r > r_0$. We shall therefore solve in the outer region, in lieu of (2), the equation

$$(T-E)\chi = 0 \quad (3)$$

with boundary conditions, which we shall now derive, at $r = r_0$ and $R = R_0$.

To derive the first condition it is convenient to start from the equation for $V(\mathbf{r})\Psi$. It can be readily obtained from (2). We have

$$V\Psi = VG_V QV\Psi, \quad G_V = -(T-E+V)^{-1}.$$

We introduce in it the paired amplitude $t = V + VG_0t$:

$$V\Psi = tG_0QV\Psi, \quad G_0 = -(T-E)^{-1}$$

and retain in the outer region only the resonant part $\langle r\rho | t_{res} | r'\rho' \rangle$, which has the following factorized form as a function of r and r':

$$t_{res} \sim V(\mathbf{r})\varphi_0(\mathbf{r})V(\mathbf{r}')\varphi_0(\mathbf{r}'),$$

where φ_0 is the wave function of the bound (virtual) state of two particles. It follows therefore that in the interaction region V(r) the function $\Psi(r, \rho)$ also factorizes[1,2]:

$$\Psi(\mathbf{r}, \rho) = \varphi_0(\mathbf{r})\Phi(\rho),$$

meaning that it satisfies at $r = r_0$ the same boundary condition as φ_0:

$$\frac{\partial r\Psi}{\partial r} = a^{-1}r\Psi.$$

Expressing Ψ in terms of χ by means of formula (1), we get the sought boundary condition on χ at $r = r_0$

$$\frac{\partial r\chi}{\partial r} + Q\chi = a^{-1}r\chi, \quad (4)$$

which can be regarded as imposed at $r = 0$ in order to solve Eq. (3) and with accuracy to r_0/R_0. Since the resonant forces in question act only in the s-state of the pair, χ does not depend on the angles of the vector r in the outer region. For $L = 0$, the function χ does not depend also on the angles of the vector ρ. Introducing for this case $\chi_0(r, \rho) = r\rho\chi(r, \rho)$, we ultimately obtain from (3) and (4)

$$(\Delta + E)\chi_0 = 0, \quad (5)$$

$$\left.\frac{\partial\chi_0(r,\rho)}{\partial r}\right|_{r=0} + \frac{8}{\sqrt{3}\rho}\chi_0\left(\frac{\sqrt{3}}{2}\rho, \frac{1}{2}\rho\right) = a^{-1}\chi_0(0,\rho), \quad (6)$$

where $\Delta = \partial^2/\partial r^2 + \partial^2/\partial\rho^2$. In addition, from the definition of χ_0 there follows the boundary condition at $\rho = 0$

$$\chi_0(r, 0) = 0. \quad (7)$$

As to the boundary condition at $R = R_0$, the quantity χ_0'/χ_0 should be joined here with the logarithmic derivative of the internal function. In the internal region, in view of $R_0/a \ll 1$ and $R_0|E|^{1/2} \ll 1$, we can put $a^{-1} = 0$ and $E = 0$. Therefore the logarithmic derivative, accurate to the indicated small quantities, does not depend on a or E.

3. LEVEL SPECTRUM

We shall show now that when $a = \infty$ there is a condensation of the levels. In this case the variables separate in the polar coordinate[3] $\rho = R\cos\alpha$ and $r = R\sin\alpha$, since the boundary condition (6) is formulated only for χ_0 as a function of the angle α:

$$\left.\frac{\partial\chi_0(R,\alpha)}{\partial\alpha}\right|_{\alpha=0} + \frac{8}{\sqrt{3}}\chi_0\left(R, \frac{\pi}{3}\right) = 0. \quad (8)$$

From this condition and from (7) we obtain the angle function

$$\varphi_{s_i}(\alpha) = \sin s_i\left(\frac{\pi}{2} - \alpha\right),$$

where s_i are the roots of the equation

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1) r and ρ are defined in the usual manner: $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, $\rho_3 = (2/\sqrt{3})(\mathbf{r}_3 - (\mathbf{r}_1 + \mathbf{r}_2)/2)$. In terms of these coordinates, $R^2 = r_{12}^2 + \rho_3^2$. The other sets of coordinates are obtained by permutation of the indices. We note here that we assume $\hbar = m = 1$, where m is the particle mass.

$$-s_i \cos s_i \frac{\pi}{2} + \frac{8}{\sqrt{3}} \sin s_i \frac{\pi}{6} = 0. \tag{9}$$

The equation for the radial function

$$\left(-\frac{d^2}{dR^2} - \frac{1}{R}\frac{d}{dR} + \frac{s_i^2}{R^2}\right) F_{s_i}(R) = EF_{s_i}(R)$$

has the form of the radial Schrödinger equation of the two-dimensional problem with potential s_i^2/R^2. If $s^2 < 0$, then, as is well known, in such a field there are bound states with arbitrarily low energy. Equation (9) was obtained by Danilov[4] in an investigation of the three-body problem with a zero forced radius (the case $r_0 \to 0$ is discussed in Sec. 6), and has one imaginary solution $s_0 \approx i$ and an infinite number of real solutions, the smallest of which is $s_1 \approx 4$. Therefore the spectrum of the three particles condenses to $E = 0$.

Let us determine the law governing the condensation. The radial function is a decreasing Bessel function $K_{s_i}(|E|^{1/2}R)$, and the complete solution is written in the form

$$\chi_0 = \sum_{s_i} K_{s_i}(KR) f_{s_i} \sin s_i \left(\frac{\pi}{2} - \alpha\right), \quad K = |E|^{1/2},$$

where f_{s_i} are coefficients determined from the conditions for joining together with the internal function χ_0^{in} at $R = R_0$. We note that the problem is formally analogous to the problem of the bound state of a particle in a non-central field. The role of the s-wave is played by the partial wave s_0 in the sense that there is no centrifugal barrier only for this function. Owing to the centrifugal barrier, the terms with $s_i \neq s_0$ decrease rapidly with increasing R, and we shall therefore assume that R_0 is chosen such as to be able to neglect them. Indeed, when $KR \ll 1$ the partial wave $s_i \neq s_0$ is of the form

$$\left(\frac{R_0}{R}\right)^{s_i} \sin s_i \left(\frac{\pi}{2} - \alpha\right) \chi_i^{\text{in}}(R_0),$$

where $\chi_{s_i}^{\text{in}}(R_0)$ is the partial component of the internal function. If it is assumed that for an initially chosen R_0 all the $\chi_{s_i}^{\text{in}}(R_0)$ are of the same order, then, by choosing a different joining radius $R_0' \gg R_0$ (but still $R_0' \ll 1/K$), we obtain

$$\chi_{s_i}^{\text{in}}(R_0') = (R_0/R_0')^{s_i} \chi_{s_i}^{\text{in}}(R_0) \ll \chi_i^{\text{in}}(R_0),$$

whereas for the partial wave s_0 we get $\chi_{s_0}^{\text{in}}(R_0') \sim \chi_{s_0}^{\text{in}}(R_0)$.

The asymptotic form of the Bessel function K_{s_0} is $\sin(|s_0| \ln(KR) + \Delta)$; $\Delta = -|s_0| \ln 2 - \arg\Gamma(s_0 + 1)$. The joining condition is

$$\frac{(\chi_{s_0}^{\text{in}})'(R_0)}{\chi_{s_0}^{\text{in}}(R_0)} \equiv \Lambda = \frac{|s_0|}{R_0} \operatorname{ctg}(|s_0| \ln KR_0 + \Delta),$$

from which we get the spectrum

$$E_N = -\frac{1}{R_0^2} e^{-2\pi N/|s_0|} \exp\frac{2}{|s_0|}\left[\operatorname{arcctg}\frac{\Lambda R_0}{|s_0|} - \Delta\right]. \tag{10}$$

Thus, the levels condense to zero exponentially with an exponent $E_N/E_{N+1} = e^{2\pi/|s_0|} \approx 500$.

We now proceed to $a \neq \infty$. We shall show that the condensation vanishes. To this end, we consider the interaction of three particles in the regions $R \ll a$ and $R \gg a$. If $R \ll a$ we have, as before, an attraction s^{20}/R^2. However, if $R \gg a$ the interaction is small.

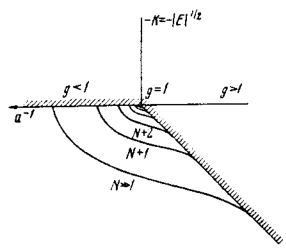

FIG. 1. Level spectrum of three spinless neutral particles. The cross hatching denotes the boundary of the continuous spectrum of the three particles. Neighboring level trajectories differ only in a scale transformation by an approximate factor of 22. For clarity, this ratio is not maintained in the figure.

Thus, the attraction is cut off at a distance $R \sim a$, and consequently the number of levels is finite. We can reach the same conclusion by considering the boundary condition (6). We rewrite it in the following form:

$$\left.\frac{\partial\chi_0(R,\alpha)}{\partial\alpha}\right|_{\alpha=0} + \frac{8}{\sqrt{3}}\chi_0\left(R, \frac{\pi}{3}\right) = \frac{R}{a}\chi_0(R, 0).$$

When $R \ll a$ we neglect the right-hand side and find the same three-particle interaction as when $a = \infty$. When $R \gg a$, conversely, we neglect $\chi_0(R, \pi/3)$. This is equivalent to neglecting the interaction between the pair and the third particle[2)], i.e., the weakness of the interaction of the three particles.

Thus, for any arbitrarily small but finite a^{-1}, the number of levels is finite. This means that when the pair interaction is increased (we separate in it the factor g: $V(r) \to gV(r)$; $g = 1$ corresponds to the appearance of the bound state) we obtain the following picture: when g increases, tending to unity, the number of levels of the three particles increases, so that when $g = 1$ it is infinite; with further increase of g, the number of levels decreases. This does not contradict the fact that the binding energy of each level increases with increasing g, inasmuch as the boundary of the continuous spectrum is determined for $g > 1$ by the threshold of disintegration into a bound pair and a third particle, and decreases like a^{-2} with increasing g (Fig. 1).

Let us determine the qualitative course of the level trajectories as a function of g. It turns out that it suffices to know only the trajectory of one level, while the remaining ones are obtained by a scale transformation. This follows from dimensional considerations.

To demonstrate this, we note that when $a^{-1} \neq 0$ the solution at $R > R_0$ contains prior to the joining two dimensional quantities, a^{-1} and K. The variables do not separate. When $R \ll a$, however, and the term $a^{-1}\chi_0(R, 0)$ in the boundary condition can be neglected, we can again separate the variables. The partial waves with $s_i \neq s_0$ attenuate rapidly with decreasing R (like $(KR)^{s_i}$) owing to the centrifugal barriers. Therefore when $R \sim R_0$ there remains only the partial wave s_0. It is given by $\sin(|s_0| \ln(KR) + \Delta(a^{-1}, K))$. The phase Δ, being dimensionless depends only on the ratio a^{-1}/K. Therefore the spectrum is determined, at a fixed ratio a^{-1}/K, by a formula of the same type as (10):

$$E_N = -\frac{1}{R_0^2} \exp\left(-\frac{2\pi N}{|s_0|}\right) \exp\frac{2}{|s_0|}\left[\operatorname{arcctg}\frac{\Lambda R_0}{|s_0|} - \Delta\left(\frac{a^{-1}}{K}\right)\right]. \tag{11}$$

Since the dependence on a^{-1}/K and N factors out,

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2)See the derivation of the boundary condition in Sec. 2. This neglect is equivalent to $Q \to 0$ in (2).

formula (11) describes a set of similar trajectories on the (a^{-1}, K) plane. Their dependence on the "angle" $\xi = \tan^{-1}(a^{-1}/K)$ is the same. Two neighboring trajectories differ only by a transformation of the "radius" $H = (K^2 + a^{-2})^{1/2}$ by a factor $e^{\pi/|s_0|} \approx 22$ (Fig. 1). In particular, the values of the pair amplitude at which the N-th and (N + 1)-st levels appear, when g increases and approaches unity, differ by the same factor, and for a given $a > 0$ the number of levels is (with logarithmic accuracy)

$$N(a) = \frac{|s_0|}{\pi} \ln \frac{a}{r_0}. \tag{12}$$

When g decreases and moves away from unity, the pair amplitudes at which the levels go off to the continuous spectrum have the same ratio, while the number of remaining levels is determined by formula (12).

A qualitative picture of the course of the individual trajectory follows from the following two properties, which are illustrated in Fig. 1.

1. When a trajectory goes off to the continuous spectrum, the binding energy of the third particle $\epsilon = a^{-2} - E$ is small compared with the pair binding energy a^{-2}. Therefore the third particles is at a distance $1/\sqrt{|\epsilon|}$, which is much larger than the pair dimension a. We are dealing in essence with a two-particle state (pair + particle) with a zero relative angular momentum and a low binding energy. In such a situation, as is well known[5], $\epsilon \sim (g - g_0)^2$, where g_0 is the value of g at which the bond breaks.

2. When the bound state is produced, the binding energy E is uniformly distributed among the three particles. It can be shown that as $E \to 0$ the radius of the bound state is $\sim a$, the binding energy is $E \sim (g - g_0)$; accordingly $K \sim (g - g_0)^{1/2}$.

The most unpleasant property of the levels, from the point of view of their possible existence, is the stringent requirement of the resonance of the forces. Assuming that the results of the present section are qualitatively valid also when $N \sim 1$, we find that the first level appears at $a_1 \sim 20r_0$ (although this number may be reduced by the nonresonant forces), the second at $a_2 \sim 20a_1$, etc. The low-energy excited level obtained by a number of authors[6] from a numerical solution of the Schrödinger equation with model paired interactions is apparently the first of this series of levels. Its properties are not yet fully identical with those indicated above, since this level, appearing at $a \gg r_0$, remains according to the calculation in the region of the discrete spectrum with increasing g.

How is the spectrum affected by the Coulomb forces? They introduce a new length—the Bohr radius a_C. If the dimension of the bound state is smaller than a_C, then the influence of the Coulomb forces is small, and if it is large the Coulomb forces break the bond. Therefore for charged particles there is no condensation; the maximum number of levels is $\sim \ln(a_C/r_0)$. The relation $a_C > r_0$ is satisfied for the interaction of only the lightest nuclei.

A possible level of this type is the level in C^{12} with excitation energy 7.65 MeV. It is assumed that it plays a cardinal role in the formation of elements in stars. In favor of the hypothesis concerning its structure (a similar hypothesis is advanced also in[7]), we present the following arguments: first, it lies near the threshold of the disintegration of C^{12} into three α particles (0.38 MeV above the threshold[8]); second, it has the required quantum numbers 0^+; third, the pair interaction of the α particles in the s-state is resonant (the resonance is the ground state of Be^8). On the other hand, from the point of view of the shell model of C^{12}, it is difficult to interpret this level, for a level with such quantum numbers and with a simple configuration should have a much higher excitation energy[9]. The possibility of quantitatively describing the properties of this level from the point of view developed above can be ascertained only by calculation, since r_0, a, and a_C do not differ greatly for α particles[10].

4. OTHER SYMMETRIES. NONZERO ANGULAR MOMENTA

We shall show that in other states of three particles there is no family of levels of low energy. We begin with antisymmetrical states (with arbitrary L). In this case parent forces are important only in odd states. Therefore the resonant interaction does not appear at all[2].

We proceed to symmetrical states with $L \neq 0$. We see immediately that for even L the resonant forces are effective only in even states of three particles $(2^+, 4^+, \ldots)$, whereas for odd L they are effective in odd states $(1^{-1}, 3^{-1}, \ldots)$. Indeed, the pair interacts resonantly in the s-state. The third particle should then have a relative angular momentum equal to L. The parity of such a three-particle state is $(-1)^L$.

Since the destructive influence of the centrifugal forces on the spectrum increases with increasing L, it suffices to verify the absence of a family of levels for the smallest L at each parity, i.e., for the 1^- and 2^+ states. Proceeding in the same manner as in Sec. 3, we obtain for s only real roots with $s_{min} \approx 3$ in both cases.

Finally, let us consider the states of three particles of mixed symmetry. In analogy with the symmetrical case, the pair of functions $\Psi^{(1)}$ and $\Psi^{(2)}$ forming the mixed representation can be expressed in terms of the function ξ:

$$\Psi^{(1)} = \xi(\mathbf{r}_{12}, \rho_3) - \frac{1}{2}\xi(\mathbf{r}_{23}, \rho_1) - \frac{1}{2}\xi(\mathbf{r}_{31}, \rho_2) = \left(1 - \frac{1}{2}Q\right)\xi(\mathbf{r}_{12}, \rho_3),$$
$$\Psi^{(2)} = \frac{\sqrt{3}}{2}\left(\xi(\mathbf{r}_{23}, \rho_1) - \xi(\mathbf{r}_{31}, \rho_2)\right),$$

which is a solution of the equation

$$(T - E)\xi = -V\left(1 - \frac{1}{2}Q\right)\xi. \tag{13}$$

Comparing (13) with (2), we see that the mixed case differs from the symmetrical only in the substitution $Q \to -Q/2$. Making this substitution in (4) and the succeeding equations, we find that for the 0^+ states all the roots are real, and $s_{min} \approx 2$. The same is obtained for the 1^- states.

Thus, in all the states except the symmetrical 0^+ the resonant forces are either ineffective or else produce repulsion $1/R^2$ at $r_0 \ll R \ll a$. In the latter case, the interaction in the internal region is immaterial, since the wave function is so small (compare with the analogous problem of the low-energy scattering of a particle by a field surrounded by a centrifugal barrier). In the first approximation in KR_0, the boundary condition at $R = R_0$ degenerates into $\chi_0 = 0$ or $\xi_0 = 0$ at $R = 0$.

5. RESONANT INTERACTION OF NUCLEONS

For particles with spin, the resonant forces are characterized by several parameters. In particular, nucleon-nucleon forces, which are considered in this section, are determined by singlet and triplet lengths a_s and a_t. The singlet length is quite large. We shall show, however, that the radius of the attractive $1/R^2$ interaction is determined by the smaller of the two lengths (i.e., by the deuteron dimension). It follows, in particular, that if the forces in any one state (singlet or triplet) are not resonant, then there is no attractive interaction at all. This is in sharp contrast with the spinless case.

We write the nucleon-nucleon interaction in the form

$$V = A_1 + A_2 \tau_1 \tau_2,$$

where each of the A_i is a sum of terms

$$A = V_1(\mathbf{r}) + V_2(\mathbf{r})\mathbf{s}_1\mathbf{s}_2 + V_{ls}(\mathbf{r})\mathbf{s} + V_{T\,mn}(\mathbf{r}) s_m s_n, \quad \mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2.$$

The influence of the tensor and spin-orbit forces on the resonant amplitude in the outer region reduces only to a change of a_t. Indeed, in the singlet channel these interactions are not effective at all. In the triplet channel, the tensor forces, besides changing a_t, produce an admixture of the d-state to the s-state. This admixture is of the order of $(kr_0)^2$ in amplitude, leading in the outer region to small corrections of the order of $(r_0/R_0)^2$. The correction due to the spin-orbit forces is of the same order. Thus, the structure of the resonant amplitude in the outer region can be assumed to be the same as when $V_{ls} = V_T = 0$. Since L and S are conserved in the latter case, it follows that L and S are good quantum numbers in the outer region.

Starting from the symmetry of the spin-isospin function, let us find the type of symmetry of the coordinate function and, using the results of Secs. 3 and 4, determine immediately the character of the three-nucleon $1/R^2$ interaction in states with different (T, S)[3] (see the table). Only in the state $(1/2, 1/2)$ with $L = 0$, which is not listed in the table, is attraction possible. This case will be considered in detail.

The spin-dependent forces mixed in this case the symmetrical and mixed functions. By the method developed in Sec. 2, we obtain the following equations:

$$(\Delta + E)\chi_0 = 0, \quad (\Delta + E)\xi_0 = 0; \tag{14}$$

$$\frac{\partial \chi_0}{\partial \alpha} + \frac{8}{\sqrt{3}}\chi_0\left(\frac{\pi}{3}\right) = R\,(a_+^{-1}\chi_0 + a_-^{-1}\xi_0) \quad \text{for} \quad \alpha = 0,$$

$$\frac{\partial \xi_0}{\partial \alpha} - \frac{4}{\sqrt{3}}\xi_0\left(\frac{\pi}{3}\right) = R\,(a_-^{-1}\chi_0 + a_+^{-1}\xi_0) \quad \text{for} \quad \alpha = 0,$$

where

$$a_\pm^{-1} = \frac{1}{2}(a_t^{-1} \pm a_s^{-1}).$$

[3]The results of Secs. 3 and 4 are not applicable directly to He^3 and Li^3, owing to the Coulomb forces.

T_3	T	S	Character of interaction
$-3/2$ trineutron	$3/2$	$1/2$	Repulsion
	$3/2$	$3/2$	—
$-1/2$ triton	$1/2$	$1/2$	Repulsion $(L \neq 0)$
	$1/2$	$3/2$	Repulsion
	$3/2$	$1/2$	Repulsion
	$3/2$	$3/2$	—

Let us investigate the character of the three-nucleon interaction at different ratios of a_s to a_t. If $a_s \sim a_t$, then qualitatively the situation is the same as for forces independent of the spin: when $r_0 \lesssim R \lesssim a$ we have attraction for χ_0 and repulsion for ξ_0 (the right-hand side of (14) can be neglected here); when $R \gtrsim a$ the interaction is small.

Let $a_t \gg a_s$.[4] The region $a_s \lesssim R \lesssim a_t$ is external with respect to the singlet resonant forces. According to the arguments of Sec. 2, these forces can be neglected here. By the same token, the resonant forces in this region are much weaker than when $R \lesssim a_s$, and the question arises whether these forces are sufficient to lead to attraction of the three particles. To clarify this, we turn to the conditions (14). Putting in these conditions $a^{-1} = 0$ and $R \gg a_s$, we have

$$\frac{\partial \chi_0}{\partial \alpha} + \frac{8}{\sqrt{3}}\chi_0\left(\frac{\pi}{3}\right) = \frac{R}{2a_s}(\chi_0 - \xi_0),$$

$$\frac{\partial \xi_0}{\partial \alpha} - \frac{4}{\sqrt{3}}\xi_0\left(\frac{\pi}{3}\right) = -\frac{R}{2a_s}(\chi_0 - \xi_0).$$

It is easy to conclude from these equations that the difference $\chi_0 - \xi_0$ is much smaller than each term separately:

$$\chi_0 - \xi_0 \sim \frac{a_s}{R}\chi_0 \sim \frac{a_s}{R}\xi_0,$$

After which we find that, accurate to a_s/R, we have $\chi_0 = \xi_0$ and

$$\frac{\partial \chi_0}{\partial \alpha} + \frac{1}{4}\frac{8}{\sqrt{3}}\chi_0\left(\frac{\pi}{3}\right) = 0.$$

This condition differs from the boundary condition (8) for a symmetrical function in the spinless case only by a factor $1/4$ ahead of the second term. After such a substitution, the transcendental equation (9) has no imaginary root corresponding to repulsion. Thus, the interaction of the three nucleons has the following form: attraction when $r_0 \lesssim R \lesssim a_s$ and repulsion when $a_s \lesssim R \lesssim a_t$.

Finally, let us consider the last case, $a_s \gg a_t$. In the region $a_t \lesssim R \lesssim a_s$ the triplet forces are ineffective; it is also possible to neglect the singlet forces, since the field $1/R^2$ produced by them is small compared with E (we recall that we are interested in the region of the discrete spectrum of the three nucleons, in which $E \leq -a_t^2$). This means that the attraction radius is in this case equal to a_t.

Combining all the cases, we conclude that the radius of the attractive $1/R^2$ interaction is determined, in order of magnitude, by the smaller of the scattering lengths.

We are now in a position to present a qualitative

[4]We assume, in accord with the experiment, that the triplet interaction is stronger than the singlet interaction, and that when the nuclear-force constant g increases a_t becomes infinite first and then a_s.

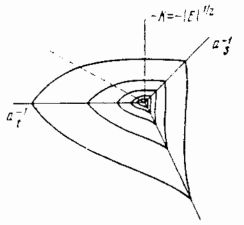

FIG. 2. Energy surfaces of three nucleons.

picture of the behavior of the levels as a function of g. When g increases and falls in the resonant region, $1/R^2$ attraction appears for the three nucleons; the radius of this attraction is of the order of the smaller of the scattering lengths, $a_t(g)$ or $a_s(g)$. The maximum attraction radius is determined by the maximum value of the smallest length $a_{min.max}$. Accordingly, the maximum number of levels is

$$N_{max} \approx \frac{|s_0|}{\pi} \ln \frac{|a_{min\ max}|}{r_0}.$$

N_{max} is infinite in the improbable case when the triplet and singlet bound states appears together. When g increases and goes out of the resonant region, both scattering lengths decrease and the levels go off to the continuous spectrum. An idea of the energy surfaces $K(a_t^{-1}, a_s^{-1})$ can be obtained from Fig. 2, which is the analog of Fig. 1 for spin-dependent forces. As before, it can be shown that all the energy surfaces differ from one another by transformation of the scale of the radius $H = (K^2 + a_t^{-2} + a_s^{-2})^{1/2}$ by a factor $\exp(\pi/|s_0|)$. The described picture of the behavior of the levels is obtained by passing through the surfaces the plane $a_t^{-1} = \gamma a_s^{-1} + \delta$, where γ and δ are constants. In particular, when $a_t^{-1} = a_s^{-1}$ (the forces are independent of the spin) we arrive at the trajectories of Fig. 1.

Let us use the results to estimate the extent of the resonant forces in the triton. The attraction radius is determined here by the triplet length 5.4 F, which is the dimension of the deuteron. It is possible to estimate the maximum attraction radius $a_{min.max}$ for the nuclear forces. It equals approximately 8 F. It is difficult to draw definite conclusions on the basis of this value. Notice should be taken only of the tendency to form a state with the same quantum numbers as the ground state of the triton. This tendency may be realized in the appearance of an excited bound level of the triton; this, however, is doubtful in view of the small difference between $a_{min.max}$ and r_0 (one should expect such a level at a deuteron binding energy $\lesssim 100$ keV). The nonresonant forces can hardly contribute to the appearance of the bound level, since the ground level of the triton has itself a small binding energy (compared with $1/r_0^2$). No bound levels of the triton have been observed experimentally as yet[11,12]. It is more probable that the indicated state is virtual. Then it might be manifest in a low-energy doublet with neutron-deuteron scattering. This may explain certain known anomalies in this process[11,13].

6. ZERO-RADIUS FORCES

As is well known, the interaction of two particles under the conditions assumed in the present paper can be described by the theory of the zero force radius (the Bethe-Peierls theory). In this section we shall ascertain the extent to which this approach is applicable in our case. Zero-radius forces have, of course, a formal character. Their advantage lies in the great mathematical simplification of the problem. Physically, however, their introduction is meaningful if the first approximation in the expansion in the parameters Kr_0 and r_0/a does not contain r_0. Obviously, it is then possible to obtain the expansion by putting $r_0 = 0$ from the very outset. This is precisely the situation in the case of two particles.

In our case, as $r_0 \to 0$ and at a = const, the coupling between three particles in a state with any finite number N becomes infinite. This agrees with the well known Thomas theorem[1] and represents the collapse of three particles in a $1/R^2$ attraction field. Therefore the zero-radius forces are not suitable for the description of the family of levels. Let us trace the character of the transition to $r_0 = 0$. A decrease of r_0 at a fixed a is simply a scale transformation of Figs. 1 and 2. Each trajectory, without changing shape, moves farther away from the origin[5)]. At $a^{-1} = 0$, the levels condense, as before, to zero.

Three particles with zero-radius forces were considered in a number of papers[2,4,14]. In particular, Minlos and Faddeev found that for the 0^+ states the level spectrum goes off to $-\infty$. We see that the correct limiting transition $r_0 \to 0$, being in this case purely formal, leads to the same result, and the spectrum obtained in[14] is a reflection of the physical fact that in the real case (at finite r_0) particles have a set of weakly bound 0^+ states.

For other three-particles states, the situation is entirely different. Here r_0 drops out completely (see Sec. 4). Therefore the case $Kr_0 \ll 1$, $r_0/a \ll 1$ is described by zero-radius forces.

In general for 0^+ states, as seen from Secs. 2 and 3, the radius of the forces enters only via the boundary condition at $R = R_0$. The interaction of the particles and the wave function in the outer region are determined by equations that do not contain r_0, and can therefore be found in the theory with the zero force radius[6)]. This explains why Eq. (9) was obtained by Danilov in the zero-radius theory. He also indicated that for other states s cannot be imaginary. This, as already established, corresponds to the absence of a family of levels in these states.

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[5)] In the formula (11), the quantity ΛR_0 may depend on r_0. Λ has the dimension of reciprocal length and is determined by the wave function in the internal region, where all the dimensional quantities except r_0 have been neglected. Therefore Λ is proportional to $1/r_0$ meaning that ΛR_0 does not depend on r_0.

[6)] Equation (5) with boundary conditions (6) and (7) reduces to the equation of Skornyakov and Ter-Martirosyan[2] if we write the solution (5) in the form $\chi_0(r,\rho) = \int \chi_0(\kappa) \sin \kappa\rho \exp(-\sqrt{\kappa^2 - E}) d\kappa$ and find the equation for $\chi_0(\kappa)$ from the condition (6).

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